A shorter proof that **wRRA** is a variety

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Introduction

Weakly representable relation algebras were introduced by Jónsson (1959).

Definition 1 *A relation algebra*

$$\mathcal{A} = (A, +, -, \cdot, 0, 1, 1', \breve{\ }, ;)$$

is said to be weakly representable *if it has a* weak representation *— a one-one map $h : A \to \wp(E)$, for some equivalence relation E on some set, that respects all the operations except perhaps $+, -$.*

This is weird! The operations $+, -$ are in the signature, and must satisfy the relation algebra axioms, but are ignored in weak representations.

wRRA denotes the class of weakly representable relation algebras.

Some (mostly grim) facts about wRRA

1. $\mathbf{RRA} \subseteq \mathbf{wRRA} \subseteq \mathbf{RA}$ (obvious)
2. $\mathbf{RRA}$ is not finitely axiomatisable over $\mathbf{wRRA}$ (Andréka 1994: answered Jónsson's Problem 3)
3. $\mathbf{wRRA}$ is not finitely axiomatisable (Haiman 1987, IH-Mikulás 2000)*
4. $\mathbf{RRA}$ and (co-)$\mathbf{wRRA}$ are recursively inseparable (Hirsch–IH 2002). Hence undecidable whether a finite algebra is in $\mathbf{wRRA}$.
5. $\mathbf{RA}_n \not\subseteq \mathbf{wRRA}$ for $3 \leq n < \omega$ (follows from recursive inseparability above). I think open whether $\mathbf{SRaCA}_n \subseteq \mathbf{wRRA}$ for some n.
6. $\mathbf{wRRA} \not\subseteq \mathbf{RA}_n$, and hence $\mathbf{wRRA} \not\subseteq \mathbf{SRaCA}_n$, for $5 \leq n < \omega$ (Hirsch-IH-Maddux 2011).
7. $\mathbf{wRRA}$ is a variety (Pécsi 2009; Alm 2007 under an assumption)*
8. $\mathbf{wRRA}$ is not canonical, hence not closed under completions (IH–Mikulás 2012).

*This answered part of Jónsson's Problem 1.

Pécsi's proof that wRRA is a variety

Jónsson asked (implicitly) in problem 1 of his 1959 paper whether **wRRA** is *equationally axiomatisable* (a variety).

50 years later, Pécsi (2009) answered this positively.

To show **wRRA** is a variety, it's enough to show that **wRRA** is closed under *subalgebras, products,* and *homomorphic images.*

The first two are easy and well known.
We (and Pécsi) ignore them and focus on showing that

wRRA *is closed under homomorphic images.*

Pécsi's proof of this took only about 5 pages.
He used reduced products ('ultraproducts' over a filter).

We can reduce his proof to 5 slides using first-order compactness.

First-order theory $T_{\mathcal{A}}$ defining weak representations

Fix a non-trivial relation algebra $\mathcal{A} = (A, +, -, \cdot, 0, 1, 1', \breve{}, ;)$ (so $|A| > 1$).

Regard each $a \in A$ as a binary relation symbol.
The theory $T_{\mathcal{A}}$ comprises the following sentences, for each $a, b \in A$:

- $\forall xy(a \cdot b(x, y) \leftrightarrow a(x, y) \wedge b(x, y))$
- $\forall xy \neg 0(x, y)$
- $\forall xy(1'(x, y) \leftrightarrow x = y)$
- $\forall xy(\breve{a}(x, y) \leftrightarrow a(y, x))$
- $\forall xy(a ; b(x, y) \leftrightarrow \exists z(a(x, z) \wedge b(z, y)))$

Also define $T^{1-1}_{\mathcal{A}} = T_{\mathcal{A}} \cup \{\exists xy\, a(x, y) : a \in A \setminus \{0\}\}$.

Then *the models of $T^{1-1}_{\mathcal{A}}$ 'are' the weak representations of $\mathcal{A}$.*

- If $M \models T^{1-1}_{\mathcal{A}}$ then $(a \mapsto a^M)$ is a weak representation of $\mathcal{A}$.
- And a weak representation of $\mathcal{A}$ yields a model of $T^{1-1}_{\mathcal{A}}$.

So *$\mathcal{A} \in \mathbf{wRRA}$ iff $T^{1-1}_{\mathcal{A}}$ is consistent (has a model).*

Theorem (Pécsi 2009): wRRA is closed under homomorphic images

Proof. Suppose $\mathcal{A} \in \mathbf{wRRA}$, and that $f : \mathcal{A} \to \mathcal{D}$ is a surjective homomorphism. Note: f preserves *all* relation algebra operations.

We show that $\mathcal{D} \in \mathbf{wRRA}$.
This is clear if $\mathcal{A}$ or $\mathcal{D}$ is trivial. So assume both $\mathcal{A}$, $\mathcal{D}$ are non-trivial.

We begin with a standard simplification (Pécsi did it too).

Fix any non-zero element $d_0 \in \mathcal{D}$.
It's enough to show that $T_{\mathcal{D}} \cup \{\exists xy\, d_0(x,y)\}$ has a model, say M_{d_0}.

For then, the disjoint union $\dot\bigcup_{d_0 \in \mathcal{D} \setminus \{0\}} M_{d_0}$ is a model of $T_{\mathcal{D}}^{1-1}$.
As $\mathcal{D}$ is non-trivial, this implies $\mathcal{D} \in \mathbf{wRRA}$ as required.

We will build M_{d_0} from a suitable model of $T_{\mathcal{A}}$.

Proof 2/4: a new theory U

Fix $a_0 \in A$ with $f(a_0) = d_0$. Let c_0, c_1 be new constants, and let

$$U \quad = \quad T_{\mathcal{A}} \cup \{a_0(c_0, c_1)\} \cup \{u(c_0, c_1) : u \in A,\ f(u) = 1\}.$$

Claim 1. U is consistent.

Proof of claim. If not, then by compactness there are
$u_0, \ldots, u_{n-1} \in f^{-1}(1)$ with $T_{\mathcal{A}} \models \neg\big(a_0(c_0, c_1) \wedge \bigwedge_{i<n} u_i(c_0, c_1)\big)$.
Since models of $T_{\mathcal{A}}$ respect $\cdot$, we have $T_{\mathcal{A}} \models \neg(a_0 \cdot \prod_{i<n} u_i)(c_0, c_1)$.
By the lemma on constants,

$$T_{\mathcal{A}} \models \forall xy \neg\Big(a_0 \cdot \prod_{i<n} u_i\Big)(x, y).$$

Since $\mathcal{A}$ has weak representations, it follows that $a_0 \cdot \prod_{i<n} u_i = 0$. So

$$0 = f(0) = f(a_0 \cdot \prod_{i<n} u_i) = f(a_0) \cdot \prod_{i<n} f(u_i) = f(a_0) = d_0.$$

This contradicts $d_0 \neq 0$ and proves the claim.

Proof 3/4: recall $U = T_{\mathcal{A}} \cup \{a_0(c_0, c_1)\} \cup \{u(c_0, c_1) : u \in A,\ f(u) = 1\}$

By claim 1, we can take $M \models U$.
Then $M \models T_{\mathcal{A}}$, so *$M$ respects $\cdot$, $\leq$, ;* (but maybe not $+, -$).
For $a \in A$, we write a^M for the interpretation of a in M.

Replace M by its substructure based on the 1^M-equivalence class of c_0, c_1. As U is $\forall_1$, we still have $M \models U$. We also have $M \models \forall xy\, 1(x, y)$.

Claim 2. $M \models \forall xy\, u(x, y)$ for each $u \in A$ with $f(u) = 1$.

Proof of claim. Trick: let $v = -(1 ; -u ; 1)$.
Then $-u \leq 1 ; -u ; 1 = -v$, so $v \leq u$. But M respects $\leq$, so $v^M \subseteq u^M$.
Now $f(v) = -(1 ; 0 ; 1) = 1$. So $M \models \forall xy(1(x, c_0) \wedge v(c_0, c_1) \wedge 1(c_1, y))$.
But M respects ;, so $M \models \forall xy(1 ; v ; 1)(x, y)$.

Now '$-v$ is an ideal element': $1 ; -v ; 1 = 1 ;(1 ; -u ; 1) ; 1 = 1 ; -u ; 1 = -v$.
So (eg Maddux's book, theorem 305) v is an ideal element: $1 ; v ; 1 = v$.

We arrive at $M \models \forall xy\, v(x, y)$. As $v^M \subseteq u^M$, the claim follows.

Proof 4/4: recall $U = T_{\mathcal{A}} \cup \{a_0(c_0, c_1)\} \cup \{u(c_0, c_1) : u \in A,\ f(u) = 1\}$

Claim 3. $a^M = b^M$ whenever $a, b \in A$ and $f(a) = f(b)$.

Proof of claim. If $f(a) = f(b)$ then $f(a \to b) = f(a \to a) = f(1) = 1$, so

$$a^M = a^M \cap \underbrace{(a \to b)^M}_{=M\times M \text{ by claim 2}} = (a \cdot (a \to b))^M \subseteq b^M.$$

Similarly, $b^M \subseteq a^M$. So $a^M = b^M$. This proves the claim.

Recall: we want to show that $T_{\mathcal{D}} \cup \{\exists xy\, d_0(x, y)\}$ has a model.
As f is a surjective relation algebra homomorphism and $f(a_0) = d_0$,

$$T_{\mathcal{D}} \cup \{\exists xy\, d_0(x, y)\} = f\left(T_{\mathcal{A}} \cup \{\exists xy\, a_0(x, y)\}\right)$$

(on the right, apply f to all symbols in the theory).

Now $M \models T_{\mathcal{A}} \cup \{\exists xy\, a_0(x, y)\}$, since $M \models U$.
So if we *interpret each $f(a)$ in M in the same way as a,* we get a model of $T_{\mathcal{D}} \cup \{\exists xy\, d_0(x, y)\}$. This is well defined, by claim 3. $\square$

Some references

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