On the Priorean temporal logic with 'around now' over the real line

Ian Hodkinson*

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Abstract

We consider the temporal language with the Priorean operators G and H expressing that a formula is true at all future times and all past times, plus an operator $\Box$ expressing that a formula is true throughout some open interval containing the evaluation time (i.e., it is true 'around now'). We show that the logic of real numbers time in this language is finitely axiomatisable, answering an implicit question of Shehtman (1993). We also show that the logic has PSPACE-complete complexity, but is not Kripke complete and has no strongly complete axiomatisation.

Keywords Weak completeness, finite axiomatisation, filtration, lexicographic sum, Kripke-incomplete.

1 Introduction

Modal formulas can be given semantics in models based on topological spaces. In a topological model, the formula $\Box\varphi$ is true at a point if φ is true throughout some open neighbourhood of that point. So the set of points satisfying $\Box\varphi$ is the interior of the set of points satisfying φ. Topological semantics predates Kripke semantics and was first considered by McKinsey and Tarski [16], who proved that the logic of any separable dense-in-itself metric space, such as the rationals ($\mathbb{Q}$) and reals ($\mathbb{R}$) with the usual topology, is S4. Interest in this theorem is undergoing a renaissance and several simpler proofs have recently appeared [17, 18, 2, 1, 14, 7], either for $\mathbb{R}$ alone or for the general case. The theorem was extended by Kremer [9, 10] to a strong completeness result (for countable languages).

Additional connectives have also been considered. Shehtman added the universal modality $\forall$: a formula $\forall\varphi$ is true at an arbitrary point of a topological model if φ is true at every point. He showed [25] that the logic of any connected separable dense-in-itself metric space, such as $\mathbb{R}$, is S4UC, with S4 axioms for $\Box$, the usual axioms U for $\forall$, and a connectedness axiom C, namely $\forall(\Box p \vee \Box\neg p) \to \forall p \vee \forall\neg p$.

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Kudinov added the difference operator $[\neq]$: a formula $[\neq]\varphi$ is true at a point if φ is true at every *other* point. The difference operator is more expressive than $\forall$. In the language with $\Box, [\neq]$, Kudinov axiomatised the logic of all topological spaces, all dense-in-themselves topological spaces, and any zero-dimensional dense-in-itself metric space [11]. He also axiomatised the logic of $\mathbb{R}^n$ for $n \geq 2$ (unpublished), but proved (also unpublished: see [12]) that the logic of $\mathbb{R}$ is not finitely axiomatisable, and not even axiomatisable by formulas using finitely many variables in total.

In [24], Shehtman shifted attention to temporal logic by adding the Priorean temporal connectives G and H to the original $\Box$. This language is given semantics in *ordered topological models.* An ordered topological model is a topological model whose topology is the interval topology arising from an irreflexive linear order $<$ on the set of points. Examples include models based on $\mathbb{Q}$ and $\mathbb{R}$ with their usual orderings and topologies. Such models can be viewed temporally. We can regard the points as times and the order $<$ as the earlier-later relation, so that $x < y$ denotes that x is in the past of y and y in the future of x. A formula $G\varphi$ is true at a point or time x in such a model iff φ is true at all future times — all y satisfying $x < y$. A formula $H\varphi$ is true at x if φ is true at all past times $y < x$. Together, the connectives G, H are even more expressive than $[\neq]$. In the temporal context, $\Box\varphi$ can still be read topologically, but it also has a reasonable temporal reading as 'φ is true around now', and this view was promulgated by Scott. In [24], Shehtman gave a finite axiomatisation of the logic of $\mathbb{Q}$ in this language, observed that the logic of $\mathbb{R}$ in the same language is decidable, and implicitly posed [24, p.256] the problem of axiomatising it. Although the area of topological semantics of modal logic has recently attracted a good deal of attention, this problem has remained open.

Although it has no topological $\Box$-modality, the very expressive temporal language with U and S (Until and Since) is worth mentioning here. A formula $U(\varphi, \psi)$ is true at a time point x if there is a point $y > x$ at which φ is true and such that ψ is true at every z with $x < z < y$ — informally, ψ is true until φ becomes true. The meaning of S is obtained by swapping $<$ with $>$. The connectives U and S were introduced by Kamp [8] and they can easily express all the connectives we have considered so far. Indeed, over $\mathbb{R}$, they can express every connective whose meaning is definable in first-order logic [8]. Reynolds gave a finite axiomatisation of the logic of $\mathbb{R}$ with U, S in [20], and showed the logic to be PSPACE-complete in [21].

In the current paper we consider Shehtman's temporal language with G, H, and $\Box$, interpreted over $\mathbb{R}$. We answer Shehtman's implicit question [24] by showing that the logic of $\mathbb{R}$ in this language is finitely axiomatisable. Given Kudinov's result, this is perhaps surprising, but given Reynolds's, it is less so. It suggests that G, H, and $\Box$ are in some sense closer to Until and Since over $\mathbb{R}$ than to $[\neq]$ and $\Box$. We only obtain 'weak completeness', and we show that no strong completeness result can be proven. We also show that the logic is not Kripke compete. Shehtman observed in [24] that it is decidable, and we show that it is PSPACE-complete.

Our axiom system is similar to the one for $\mathbb{Q}$ given by Shehtman in [24] — the only difference is that we include an additional connectedness axiom $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\Diamond p \wedge \Diamond\neg p \wedge Fq)$ — and our completeness proof starts in the same way by a certain filtration of the canonical model. We then apply selective filtration and a closure technique designed to give a well behaved finite Kripke model, which we employ as a template to construct a model over $\mathbb{R}$, using lexicographic sums.

Layout of paper. Section 2 contains the basic definitions, and section 3 the system of axioms and inference rules for the logic of $\mathbb{R}$ in the language with G, H, and $\Box$. In section 4 we prove that the logic has no strong axiomatisation and is not Kripke complete, but is (decidable and) PSPACE-complete (decidability was known to Shehtman). Section 5 contains the material we need on lexicographic sums, and section 6 the completeness proof. We conclude in section 7 with some open problems.

Throughout, we use $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ to denote the ordered sets of natural numbers, integers, rationals, and real numbers (respectively).

2 Generalities

Here, we lay down the syntax and semantics of our logic, and define some basic terms.

2.1 Syntax

Let PV be a fixed countably infinite set of propositional atoms. We write $p, q, r, \ldots$ for atoms. We define the language $\mathcal{L}$ to consist of the following formulas:

1. $\top$ is a formula.
2. Every $p \in PV$ is a formula.
3. If φ, ψ are formulas then so are $\neg\varphi$, $\varphi \wedge \psi$, $G\varphi$, $H\varphi$, and $\Box\varphi$.

The *mirror image* of a formula φ is the formula obtained by replacing every G in φ by H, and every H in φ by G. As abbreviations we let $\bot = \neg\top$, $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi = \neg(\varphi \wedge \neg\psi)$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$, $F\varphi = \neg G\neg\varphi$, $P\varphi = \neg H\neg\varphi$, and $\Diamond\varphi = \neg\Box\neg\varphi$.

2.2 Semantics over $\mathbb{R}$

We define semantics for $\mathcal{L}$-formulas over $\mathbb{R}$ as follows. Let $h : PV \to \wp(\mathbb{R})$ be an assignment to atoms (where $\wp$ denotes power-set). Then for each $x \in \mathbb{R}$ and formula φ we define $(\mathbb{R}, h), x \models \varphi$ by induction:

1. $(\mathbb{R}, h), x \models \top$,
2. $(\mathbb{R}, h), x \models p$ iff $x \in h(p)$, for $p \in PV$,
3. $(\mathbb{R}, h), x \models \neg\varphi$ iff $(\mathbb{R}, h), x \not\models \varphi$,
4. $(\mathbb{R}, h), x \models \varphi \wedge \psi$ iff $(\mathbb{R}, h), x \models \varphi$ and $(\mathbb{R}, h), x \models \psi$,
5. $(\mathbb{R}, h), x \models G\varphi$ iff $(\mathbb{R}, h), y \models \varphi$ for all $y \in \mathbb{R}$ with $y > x$,
6. $(\mathbb{R}, h), x \models H\varphi$ iff $(\mathbb{R}, h), y \models \varphi$ for all $y \in \mathbb{R}$ with $y < x$,
7. $(\mathbb{R}, h), x \models \Box\varphi$ iff there exist $y, z \in \mathbb{R}$ with $y < x < z$ and $(\mathbb{R}, h), t \models \varphi$ for all $t \in \mathbb{R}$ with $y < t < z$.

A formula φ is said to be *satisfiable over* $\mathbb{R}$ if there exist an assignment $h : PV \to \wp(\mathbb{R})$ and $x \in \mathbb{R}$ with $(\mathbb{R}, h), x \models \varphi$. We say that φ is *valid over* $\mathbb{R}$ if $\neg\varphi$ is not satisfiable over $\mathbb{R}$. A set Σ of $\mathcal{L}$-formulas is said to be *satisfiable over* $\mathbb{R}$ if there exist an assignment $h : PV \to \wp(\mathbb{R})$ and $x \in \mathbb{R}$ with $(\mathbb{R}, h), x \models \varphi$ for every $\varphi \in \Sigma$.

The *$\mathcal{L}$-logic of* $\mathbb{R}$ is the set of all $\mathcal{L}$-formulas that are valid over $\mathbb{R}$.

2.3 Kripke semantics

Formulas have an alternative Kripke semantics. A *binary relation* on a set W is a subset of $W \times W$. A *Kripke frame* for $\mathcal{L}$ is a triple $(W, <, R)$, where W is a non-empty set and $<, R$ are binary relations on W. (In the main proof, $<$ will always be transitive but not always irreflexive.) Given an assignment $h : PV \to \wp(W)$, the tuple $\mathcal{M} = (W, <, R, h)$ is called a *Kripke model* for $\mathcal{L}$. For $w \in W$, we define $\mathcal{M}, w \models \varphi$ by induction on formulas φ:

1. $\mathcal{M}, w \models \top$,
2. $\mathcal{M}, w \models p$ iff $w \in h(p)$, for $p \in PV$,
3. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$,
4. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$,
5. $\mathcal{M}, w \models G\varphi$ iff $\mathcal{M}, u \models \varphi$ for all $u \in W$ with $w < u$,
6. $\mathcal{M}, w \models H\varphi$ iff $\mathcal{M}, u \models \varphi$ for all $u \in W$ with $u < w$,
7. $\mathcal{M}, w \models \Box\varphi$ iff $\mathcal{M}, u \models \varphi$ for all $u \in W$ with Rwu.

Let $\mathcal{F} = (W, <, R)$ be a Kripke frame, and $\mathcal{M} = (W, <, R, h)$ a Kripke model. A formula φ is said to be *satisfiable in* $\mathcal{M}$ if there is $w \in W$ with $\mathcal{M}, w \models \varphi$, and *satisfiable in* $\mathcal{F}$ if there are $h : PV \to \wp(W)$ and $w \in W$ with $(W, <, R, h), w \models \varphi$. A formula φ is said to be *valid in* $\mathcal{F}$ (resp. $\mathcal{M}$) if $\neg\varphi$ is not satisfiable in $\mathcal{F}$ (resp. $\mathcal{M}$).

2.4 General definitions

For a map $f : X \to Y$, and $X' \subseteq X$, we write $f(X')$ for $\{f(x) : x \in X'\}$. We write $\operatorname{dom} f$ for X and $\operatorname{rng} f$ for $f(X)$.

DEFINITION 2.1 Let W be a set, and R a binary relation on it.

1. For $w, u \in W$, we write any of Rwu, $w\,R\,u$, $R(w, u)$ to indicate that $(w, u) \in R$.
2. We let $R^\bullet$ denote the binary relation on W defined by $R^\bullet wu$ iff $Rwu \wedge \neg Ruw$.
3. For $w \in W$ we write $R(w) = \{u \in W : Rwu\}$.
4. A subset $X \subseteq W$ is said to be *R-generated* if $R(x) \subseteq X$ for every $x \in X$.
5. For $X \subseteq W$, we write $R \restriction X$ for the binary relation $R \cap (X \times X)$ on X.

DEFINITION 2.2 Let $\mathcal{M} = (W, <, R, h)$ be a Kripke model.

1. We write $u \leq w$ to abbreviate $u < w \vee u = w$.

2. An element $w \in W$ is said to be *$<$-reflexive* if $w < w$, and *$<$-irreflexive,* otherwise.
3. A *$<$-cluster* of/in $\mathcal{M}$ is a maximal non-empty subset $C \subseteq W$ such that $w < u$ for all $w, u \in C$.

   This usage of 'cluster' is slightly different from that in (e.g.) [24]. We remark that if $<$ is transitive, any $<$-reflexive point $w \in W$ lies in a unique $<$-cluster, namely, $\{u \in W : w < u < w\}$. We leave the proof as an exercise.
4. A *submodel of* $\mathcal{M}$ is a Kripke model $\mathcal{N} = (W', <', R', h')$, where $W' \subseteq W$, $<' = < \restriction W'$, $R' = R \restriction W'$, and $h'(p) = W' \cap h(p)$ for every atom $p \in PV$. For Kripke models $\mathcal{M}, \mathcal{N}$, we write $\mathcal{N} \subseteq \mathcal{M}$ to denote that $\mathcal{N}$ is a submodel of $\mathcal{M}$.
5. Such a submodel of $\mathcal{M}$ is said to be an *R-generated submodel* if W' is an R-generated subset of W, and a *$<$-generated submodel* if W' is both a $<$-generated and $>$-generated subset of W.
6. We say that $\mathcal{N}$ is a *generated submodel of* $\mathcal{M}$ if it is both an R-generated and a $<$-generated submodel of $\mathcal{M}$.

It is well known, and easily proved by induction on φ, that if $\mathcal{N}$ is a generated submodel of $\mathcal{M}$, and $w \in W'$, then $\mathcal{N}, w \models \varphi$ iff $\mathcal{M}, w \models \varphi$ for every $\mathcal{L}$-formula φ.

3 Axioms

We now present a Hilbert system that, as we will show, axiomatises the $\mathcal{L}$-logic of $\mathbb{R}$. It is based on a system of Shehtman [24, §2] that was shown to axiomatise the $\mathcal{L}$-logic of $\mathbb{Q}$. The only difference is that we have added a 'connectedness' axiom, axiom 5.

3.1 The system

The axioms are as follows. We assume familiarity with Sahlqvist formulas in temporal logic: see, e.g., [3]. The axioms 2–4 are Sahlqvist formulas and their first-order correspondents are reproduced below. (The correspondent of each normality axiom is equivalent to $\top$, and omitted.) Each correspondent is true in a Kripke frame iff the axiom is valid in the frame. Moreover, the correspondents are true in the frame of the canonical model of the logic axiomatised by the system.

1. all propositional tautologies
2. axioms for dense linear time without endpoints:

$G(p \to q) \to (Gp \to Gq)$	normality
$Gp \to GGp$	transitivity: $\forall xyz(x < y \land y < z \to x < z)$
$p \to GPp$	$\forall xy(x < y \to y > x)$
$GGp \to Gp$	density: $\forall xy(x < y \to \exists z(x < z \land z < y))$
$FPp \to p \lor Fp \lor Pp$	$\forall xyz(x < y \land y > z \to x = z \lor x < z \lor x > z)$

3. S4 axioms for $\Box$:

   $\Box(p \to q) \to (\Box p \to \Box q)$ normality

   $\Box p \to p$ reflexivity: $\forall x R(x, x)$

   $\Box p \to \Box\Box p$ transitivity: $\forall xyz(R(x, y) \wedge R(y, z) \to R(x, z))$

4. Shehtman's 'special axioms':
   (a) $Hp \wedge p \wedge Gp \to \Box p$ $\forall xy(Rxy \to x > y \vee x = y \vee x < y)$
   (b) $Gp \to G\Box p$ $\forall xyz(x < y \wedge Ryz \to x < z)$
   (c) $Gp \wedge \Box p \to \Box Gp$ $\forall xyz(Rxy \wedge y < z \to x < z \vee Rxz)$
   (d) $\Box p \to Fp$ $\forall x \exists y(x < y \wedge Rxy)$

5. $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\Diamond p \wedge \Diamond \neg p \wedge Fq)$ (connectedness)

6. all mirror images of the above axioms (swap G with H, and F with P; also swap $<$ with $>$ in the correspondents).

The rules of inference are the standard ones:

1. modus ponens: $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$

2. generalisation: $\dfrac{\varphi}{G\varphi}, \quad \dfrac{\varphi}{H\varphi}, \quad \dfrac{\varphi}{\Box\varphi}$

3. substitution: $\dfrac{\varphi(p)}{\varphi(\psi/p)}$

Some mirror images, such as $Hp \to HHp$, are redundant and can be omitted. We have not investigated the exact extent to which this can be done.

As usual, the *logic axiomatised by this system* is the smallest set S of $\mathcal{L}$-formulas that contains every axiom listed above and is closed under the rules of inference. We say that an $\mathcal{L}$-formula φ is *provable in the system* if $\varphi \in S$. A set Γ of $\mathcal{L}$-formulas is *consistent* if $\neg(\gamma_0 \wedge \ldots \wedge \gamma_{n-1}) \notin S$ for every $n < \omega$ and $\gamma_0, \ldots, \gamma_{n-1} \in \Gamma$, and *maximal consistent* if it is consistent but has no proper consistent extension.

We aim to show that S is the $\mathcal{L}$-logic of $\mathbb{R}$. The inclusion '$\subseteq$' (soundness) is straightforward:

THEOREM 3.1 *The system is sound over* $\mathbb{R}$.

Proof (sketch). All axioms other than axiom 5 are shown to be valid over any dense flow of time without endpoints in [24, lemma 2.2(2)]. Axiom 5 is valid over $\mathbb{R}$ because every interval of $\mathbb{R}$ is connected. Indeed, assume for contradiction that for some model $(\mathbb{R}, h)$ and $t \in \mathbb{R}$ we have

$$(\mathbb{R}, h), t \models F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \wedge G(Fq \to \Box p \vee \Box \neg p).$$

Let $v_1, v_2 > t$ satisfy $(\mathbb{R}, h), v_1 \models p \wedge Fq$ and $(\mathbb{R}, h), v_2 \models \neg p \wedge Fq$. We can find $u > \max(v_1, v_2)$ with $u \in h(q)$. Assume wlog. that $v_1 < v_2$. Let

$$s = \sup\{x \in \mathbb{R} : \forall y(v_1 \leq y < x \to y \in h(p))\}.$$

Then $s \leq v_2 < u$, so $(\mathbb{R}, h), s \models Fq$. As $s > t$, we have $(\mathbb{R}, h), s \models \Box p$ or $(\mathbb{R}, h), s \models \Box\neg p$. Hence, there is an open interval $O \ni s$ with $O \subseteq h(p)$ or $O \subseteq \mathbb{R} \setminus h(p)$. It is easily seen that each case contradicts the definition of s.

The inference rules obviously preserve validity. $\Box$

3.2 Simple theorems of the system

LEMMA 3.2 *$F\top$ and $P\top$ are provable in the system.*

Proof. As $\top$ is a tautology, it is provable, and we get $\Box\top$ by $\Box$-generalisation. By axiom 4d, we prove $F\top$. We prove $P\top$ similarly. $\Box$

LEMMA 3.3 *$G\neg p \wedge HFp \rightarrow \Diamond p$ is provable in the system.*

Proof. We can prove $G\neg p \wedge \Box\neg p \rightarrow \Box G\neg p$ by axiom 4c. By the mirror image of axiom 4d we have $\Box G\neg p \rightarrow PG\neg p$. Using propositional tautologies we deduce $G\neg p \wedge \Box\neg p \rightarrow \neg HFp$, and then the result. $\Box$

The connectedness axiom (5) has an important consequence: the well known Prior axiom

$$Fq \wedge FG\neg q \rightarrow F(G\neg q \wedge HFq). \tag{1}$$

We will prove this using the following lemma.

LEMMA 3.4 *$\Diamond Gq \rightarrow Gq$ and $\Diamond Hq \rightarrow Hq$ are provable in the system.*

Proof. The following are provable:

1. $\Diamond Gq \rightarrow GP\Diamond Gq$ — by axiom $p \rightarrow GPp$
2. $P\Diamond Gq \rightarrow PGq$ — Gq-instance of dual of axiom 4b ($Hp \rightarrow H\Box p$)
3. $GP\Diamond Gq \rightarrow GPGq$ — from previous by G-gen and normality
4. $PGq \rightarrow q$ — dual of temporal axiom $p \rightarrow HFp$
5. $GPGq \rightarrow Gq$ — from previous by G-gen and normality

The result now follows from lines 1, 3, and 5 by propositional tautologies. The second theorem is a mirror image. $\Box$

COROLLARY 3.5 *The Prior axiom $Fq \wedge FG\neg q \rightarrow F(G\neg q \wedge HFq)$ and its mirror image are provable in the system.*

Proof. We give a more informal proof along the lines of the preceding lemma. Assume $Fq \wedge FG\neg q$. Using the density axiom, this yields $FFq \wedge FG\neg q$. Taking $p = G\neg q$ and $q = \top$ in axiom 5 gives $F(G\neg q \wedge F\top) \wedge F(\neg G\neg q \wedge F\top) \rightarrow F(\Diamond G\neg q \wedge \Diamond\neg G\neg q \wedge F\top)$. By lemma 3.2, $F\top$ is equivalent to $\top$, so this reduces to

$$FG\neg q \wedge FFq \rightarrow F(\Diamond G\neg q \wedge \Diamond Fq).$$

So we obtain $F(\Diamond G\neg q \wedge \Diamond Fq)$. Now by standard temporal logic, we can prove $Fq \rightarrow HF(Fq)$ and $HF(Fq) \rightarrow HFq$. This gives us $F(\Diamond G\neg q \wedge \Diamond HFq)$. By lemma 3.4, we obtain $F(G\neg q \wedge HFq)$ as required. The mirror image can be derived similarly. $\Box$

4 Some facts about the $\mathcal{L}$-logic of $\mathbb{R}$

Here we prove some fairly straightforward results about the $\mathcal{L}$-logic of $\mathbb{R}$.

A Hilbert system (of axioms and rules) is said to be *sound* if all satisfiable formulas are consistent, and *strongly complete* if any consistent set of formulas using in all only countably many atoms is satisfiable.

THEOREM 4.1 *There is no sound and strongly complete Hilbert system for the $\mathcal{L}$-logic of $\mathbb{R}$.*

Proof. Let Σ be the following set of formulas written with atoms p, q, r:

$$\begin{array}{l} \Box p \\ F(r \wedge G\neg r) \\ G(r \vee Fr \to \Box p \vee \Box q) \\ F(\neg p \wedge F(\neg q \wedge F(\neg p \wedge F(\ \cdots \wedge Fr \underbrace{))\cdots)}_{n \text{ brackets}} \text{ for each integer } n \geq 1 \end{array}$$

It is easy to see that any finite subset of Σ is satisfiable over $\mathbb{R}$. However, Σ itself is not satisfiable over $\mathbb{R}$. For suppose that Σ is satisfied at 0 and $r \wedge G\neg r$ is true at 1, say. By the first and third formulas, each $x \in [0,1]$ belongs to some open interval $I_x \subseteq \mathbb{R}$ with $I_x \subseteq h(p)$ or $I_x \subseteq h(q)$. By the Heine–Borel theorem, $[0,1]$ is compact, so there are $n < \omega$ and $x_0 < \cdots < x_{n-1}$ in $[0,1]$ such that $[0,1] \subseteq \bigcup_{i<n} I_{x_i}$. By the final set of formulas, there are $0 < y_0 < y_1 < \cdots < y_n \leq 1$ with $y_j \notin h(p)$ if j is even and $y_j \notin h(q)$ if j is odd (each $j \leq n$). Now by the pigeonhole principle and convexity of the I_x, there are $i, j < n$ with $y_j, y_{j+1} \in I_{x_i}$. But $I_{x_i} \subseteq h(p)$ or $I_{x_i} \subseteq h(q)$, a contradiction.

If the $\mathcal{L}$-logic of $\mathbb{R}$ had a sound and strongly complete Hilbert system, then since Σ is finitely satisfiable, it would be consistent and so satisfiable, contradicting the above. $\Box$

We now consider Kripke completeness. I would like to thank Nick Bezhanishvili for helpful discussions on this material. We will consider the following formula φ, where a, b are atoms:

$$\varphi = H\neg a \wedge H\neg b \wedge \neg a \wedge \neg b \wedge \Diamond a \wedge \Diamond b \wedge G\neg(\Diamond a \wedge \Diamond b) \wedge FG\neg a. \tag{2}$$

LEMMA 4.2 *φ is satisfiable over $\mathbb{R}$, but is not satisfiable in any Kripke model whose frame validates the $\mathcal{L}$-logic of $\mathbb{R}$.*

Proof. Define $h : PV \to \wp(\mathbb{R})$ by $h(a) = \{1/2^n : n \in \mathbb{N}\}$ and $h(b) = \{1/3^n : n \in \mathbb{N},\ n > 0\}$. Evidently, $(\mathbb{R}, h), 0 \models \varphi$. So φ is satisfiable over $\mathbb{R}$.

Let $\mathcal{M} = (W, <, R, h)$ be a Kripke model such that every $\mathcal{L}$-formula that is valid over $\mathbb{R}$ is also valid in the frame $\mathcal{F} = (W, <, R)$. Let $w \in W$ and assume for contradiction that $\mathcal{M}, w \models \varphi$.

Plainly, $\mathcal{M}, w \models \Diamond a \wedge \Diamond b$, so there are $u, v \in R(w)$ with $\mathcal{M}, u \models a$ and $\mathcal{M}, v \models b$. By theorem 3.1, the axioms in §3.1 are all valid in $\mathcal{F}$. By axiom 4a, we have $u >$

$w \vee u = w \vee u < w$. As $\mathcal{M}, w \models H\neg a \wedge \neg a$, we cannot have $u \leq w$. So $u > w$. Similarly, $v > w$. By axiom 4b, $u < v < u < u$.

As $\mathcal{M}, w \models FG\neg a$, we can choose $x \in W$ with $w < x$ and $\mathcal{M}, x \models G\neg a$. Let $g : PV \to W$ be an assignment with $g(p) = \{u\}$, and let $\mathcal{N} = (W, <, R, g)$. Since $g(p) \subseteq h(a)$, we have $\mathcal{N}, x \models G\neg p$, so $\mathcal{N}, w \models Fp \wedge FG\neg p$. But by corollary 3.5 (or because it is valid in $\mathbb{R}$), the Prior axiom (1) is valid in $\mathcal{F}$, so $\mathcal{N}, w \models F(G\neg p \wedge HFp)$. Hence there is $y \in W$ with $w < y$ and

$$\mathcal{N}, y \models G\neg p \wedge HFp. \tag{3}$$

By lemma 3.3, $\mathcal{N}, y \models \Diamond p$. So Ryu. But also, $u < v$, so by axiom 4c we obtain $Ryv \vee y < v$.

If Ryv, then both $u, v \in R(y)$, so $\mathcal{M}, y \models \Diamond a \wedge \Diamond b$. Since $w < y$, we obtain $\mathcal{M}, w \models F(\Diamond a \wedge \Diamond b)$, contradicting that $\mathcal{M}, w \models \varphi$. If instead $y < v$, then since $v < u$ we have $\mathcal{N}, y \models Fp$, contradicting (3). Either way, our assumption that $\mathcal{M}, w \models \varphi$ has led to a contradiction. □

Recall that a modal logic L is said to be *Kripke complete* (respectively, to have the *finite model property*) if there exists a class $\mathcal{K}$ of (resp. finite) Kripke frames such that L is the set of all modal formulas that are valid in every frame in $\mathcal{K}$.

THEOREM 4.3 *The $\mathcal{L}$-logic of $\mathbb{R}$ is not Kripke complete and does not have the finite model property.*

Proof. If the $\mathcal{L}$-logic of $\mathbb{R}$ were the logic of a class $\mathcal{K}$ of Kripke frames of the form $(W, <, R)$, then as the formula φ of (2) is satisfiable over $\mathbb{R}$, it would be satisfiable over a frame in $\mathcal{K}$, which therefore (by the lemma) could not validate the logic of $\mathbb{R}$. □

As N. Bezhanishvili has observed, the $\mathcal{L}$-logic of $\mathbb{R}$ is a 'naturally occurring' example of a non-Kripke complete logic. As the finite model property is often used to show decidability, it may be surprising that the $\mathcal{L}$-logic of $\mathbb{R}$ is decidable [24, p. 256]. We end the section by establishing its complexity.

THEOREM 4.4 *The problem of deciding whether an $\mathcal{L}$-formula is valid over $\mathbb{R}$ is PSPACE-complete.*

Proof (sketch). We assume knowledge of temporal logic with Until and Since (U, S) as in [21], where it is proved that the problem of determining satisfiability over $\mathbb{R}$ of a formula written with U, S is PSPACE-complete. Given an $\mathcal{L}$-formula φ, introduce a new propositional atom q_ψ for each subformula ψ of φ, and define the formula $\widehat{\psi}$ as follows, where $\forall\psi$ abbreviates $\psi \wedge \neg U(\neg\psi, \top) \wedge \neg S(\neg\psi, \top)$:

- $\widehat{\top} = \forall q_\top$
- $\widehat{p} = \forall(p \leftrightarrow q_p)$ for $p \in PV$
- $\widehat{\neg\psi} = \forall(q_{\neg\psi} \leftrightarrow \neg q_\psi)$
- $\widehat{\psi \wedge \chi} = \forall(q_{\psi\wedge\chi} \leftrightarrow q_\psi \wedge q_\chi)$

- $\widehat{F\psi} = \forall(q_{F\psi} \leftrightarrow U(q_\psi, \top))$
- $\widehat{P\psi} = \forall(q_{P\psi} \leftrightarrow S(q_\psi, \top))$
- $\widehat{\Box\psi} = \forall(q_{\Box\psi} \leftrightarrow q_\psi \wedge U(\top, q_\psi) \wedge S(\top, q_\psi))$.

Let φ^* be the conjunction of all $\widehat{\psi}$ for subformulas ψ of φ, together with q_φ. It can be checked that φ is satisfiable over $\mathbb{R}$ iff φ^* is, and φ^* can be constructed from φ in polynomial time. Given φ, we may construct φ^* and then decide its satisfiability over $\mathbb{R}$ in PSPACE [21]. The combined procedure can be done in polynomial space, so the satisfiability, and hence the validity, of φ can be decided in PSPACE. The logic of $\mathbb{R}$ with $\Box$ alone is S4 [16], which is already PSPACE-hard [13]. $\Box$

5 Linear orders and maps

Here we recall from [4] a simple way to build maps defined on intervals of $\mathbb{R}$. It will be needed only at the end of the next section, but we present it now to avoid breaking the flow later.

A *linear order* is a structure $(I, <)$, where I is a non-empty set and $<$ is an irreflexive transitive binary relation on I that is also linear: that is, $(I, <) \models \forall xy(x < y \vee x = y \vee x > y)$. We often write $(I, <)$ simply as I. As usual we let $x \leq y$ abbreviate $x < y \vee x = y$. See, e.g., [23] for information about linear orders.

An *interval of* I is a non-empty convex subset of I. We use standard notation for intervals: if $x, y \in I$ and $x \leq y$ then $(x, y) = \{z \in I : x < z < y\}$, $[x, y) = \{z \in I : x \leq z < y\}$, $[x, y] = \{z \in I : x \leq z \leq y\}$, $(-\infty, x) = \{z \in I : z < x\}$, $[x, \infty) = \{z \in I : z \geq x\}$, etc. An *open interval* is one with no least or greatest element. In this section, we write ordered pairs in the form $\langle i, j\rangle$ where they might be confused with intervals.

5.1 Lexicographic sums of linear orders

Let $(J, <)$ be a linear order, and for each $j \in J$ let I_j be an interval of $\mathbb{R}$. (More generally, I_j can be any linear order, but we are only concerned with the case of intervals of $\mathbb{R}$.) We write

$$I = \sum_{j \in J} I_j = \{\langle i, j\rangle : j \in J,\ i \in I_j\},$$

and define an order $<$ on I lexicographically by $\langle i, j\rangle < \langle i', j'\rangle$ iff $j < j'$ or ($j = j'$ and $i < i'$). Clearly, $(I, <)$ is a linear order. If $(J, <) = (\{0, 1, \ldots, n\}, <)$ for some $n \in \mathbb{N}$ ($n \geq 0$), we can write I explicitly as $I_0 + \cdots + I_n$. It is plain that if I_0 has a greatest element and I_1 has no least element, or if I_0 has no greatest element and I_1 has a least element, then $I_0 + I_1$ is order-isomorphic to an interval of $\mathbb{R}$. More generally:

PROPOSITION 5.1 *Suppose that one of the following holds.*

1. *J is finite, say $(J,<) = (\{0,1,\ldots,n\},<)$ for some $n<\omega$. I_0 has no least or greatest element. Each I_j for $j>0$ has a least element but no greatest element.*
2. *$(J,<) = (\mathbb{Z},<)$ and each I_j has a least element and no greatest element.*
3. *$(J,<) = (\mathbb{R},<)$, each I_j has a least and a greatest element, and I_j is a singleton for every irrational j.*

Then $(\sum_{j\in J} I_j,<) \cong (\mathbb{R},<)$.

Proof. A linear order is isomorphic to $(\mathbb{R},<)$ iff it is dense, has no endpoints, has a countable dense subset, and is Dedekind complete. It is well known and easy to check that each of the three sum-orders above has these properties. Cf. [4, 23]. □

When J and the I_j meet one of the conditions in proposition 5.1, the linear order $(I,<)$ is isomorphic to $(\mathbb{R},<)$, and we will generally identify the two.

5.2 Functions on linear orders

We continue to let $(J,<)$ be a linear order and I_j $(j\in J)$ an interval of $\mathbb{R}$. Let S be an arbitrary non-empty set and for each $j\in J$ let $f_j : I_j \to S$ be a map. We define a map

$$f = \sum_{j\in J} f_j : I \to S$$

by $f(\langle i,j\rangle) = f_j(i)$. In the case where $(J,<) = (\{0,1,\ldots,n\},<)$, we may write the sum explicitly as $f_0+\cdots+f_n$. If I_j is a singleton $\{x\}$ and $f_j(x)=s$, say, we may write the map f_j simply as s.

EXAMPLE 5.2 If $s_0,\ldots,s_n\in S$ then, modulo a renaming of the elements of its domain, $s_0+\cdots+s_n$ is the map $f:\{0,\ldots,n\}\to S$ given by $f(i)=s_i$ for each $i\le n$.

For $j\in J$ we define $\mathrm{dom}_f(f_j) = I_j\times\{j\}\subseteq I$. We may sometimes regard f_j as a map $f_j : \mathrm{dom}_f(f_j)\to S$, via $\langle i,j\rangle\mapsto f_j(i)$ for each $\langle i,j\rangle\in\mathrm{dom}_f(f_j)$. In effect, we identify f_j with $f\restriction\mathrm{dom}_f(f_j)$.

5.3 Shuffles

There is an important special case known as the shuffle. Reynolds [21] described a shuffle as a 'thorough mixture' of its ingredients. Let $\mathcal{K}$ be a countable (possibly empty) set of intervals of $\mathbb{R}$ each of which has a least and a greatest element, and suppose that K_0 is a singleton interval of $\mathbb{R}$. For each $K\in\mathcal{K}\cup\{K_0\}$, let $g_K : K\to S$ be a map. Choose any $\theta:\mathbb{R}\to\mathcal{K}\cup\{K_0\}$ such that $\theta(j)=K_0$ for every irrational j and $\theta^{-1}(K)$ is a dense subset of $\mathbb{Q}$ for each $K\in\mathcal{K}$. This is not difficult to do. Then $\theta^{-1}(K)$ is dense in $\mathbb{R}$ for every $K\in\mathcal{K}\cup\{K_0\}$. Now define $I_j=\theta(j)$ and $f_j = g_{\theta(j)}$ for each $j\in\mathbb{R}$, so that $f_j : I_j\to S$, and let

$$\sigma = \sum_{j\in\mathbb{R}} f_j.$$

For each $K\in\mathcal{K}\cup\{K_0\}$, define $\mathrm{dom}_\sigma(g_K) = \{\langle i,j\rangle : j\in\mathbb{R},\ \theta(j)=K,\ i\in I_j\}$.

DEFINITION 5.3 Let $(I,<)$ be a linear order. A set $Y \subseteq I$ is *unbounded (in I)* if for all $x \in I$ there are $y, z \in Y$ with $y < x < z$.

LEMMA 5.4 *Let σ be as above. Then $\sigma^{-1}(s)$ is unbounded in $\operatorname{dom}\sigma$ for each $s \in \operatorname{rng}(\sigma)$.*

Proof. Let $s \in \operatorname{rng}\sigma$. Pick $K \in \mathcal{K} \cup \{K_0\}$ and $k \in K$ with $g_K(k) = s$. Let $\langle i, j\rangle \in \operatorname{dom}\sigma$ be arbitrary. As $\theta^{-1}(K)$ is dense in $\mathbb{R}$, there exist $j', j'' \in \theta^{-1}(K)$ with $j' < j < j''$. Then $\langle k, j'\rangle < \langle i, j\rangle < \langle k, j''\rangle$ and $\sigma(\langle k, j'\rangle) = \sigma(\langle k, j''\rangle) = s$. $\Box$

A point $x \in \operatorname{dom}(\sigma) = \sum_{j \in \mathbb{R}} I_j$ is said to be a *σ-endpoint* if it is of the form $\langle i, j\rangle$, where $j \in \mathbb{R}$ and i is the least or greatest element of I_j, and a *σ-irrational* if $x \in \operatorname{dom}_\sigma(g_{K_0})$. (Every σ-irrational is a σ-endpoint as well.)

LEMMA 5.5 *Let σ be as above, and let $x, y, z \in \operatorname{dom}\sigma$ with $y < x < z$.*

1. *If x is a σ-endpoint then $\sigma((y,z)) = \operatorname{rng}(\sigma)$.*
2. *If x is a σ-irrational, then $\sigma((y,x)) = \sigma((x,z)) = \operatorname{rng}(\sigma)$.*

Proof. Suppose $x = \langle i, j\rangle$, say, where i is the least element of I_j. If $y = \langle i', j'\rangle$, then we must have $j' < j$. Let $s \in \operatorname{rng}\sigma$ be given. Pick $K \in \mathcal{K} \cup \{K_0\}$ and $k \in K$ with $g_K(k) = s$. As $\theta^{-1}(K)$ is dense in $\mathbb{R}$, we may pick $j^* \in \theta^{-1}(K)$ with $j' < j^* < j$. Then $y < \langle k, j^*\rangle < x$ and $\sigma(\langle k, j^*\rangle) = s$. It follows that $\sigma((y,x)) = \operatorname{rng}\sigma$. A similar argument shows that if i is maximal in I_j then $\sigma((x,z)) = \operatorname{rng}\sigma$. The lemma is easily derived from this. $\Box$

By proposition 5.1(3), the linear order $(\sum_{j \in \mathbb{R}} I_j, <)$ is isomorphic to $(\mathbb{R}, <)$, so by choosing a suitable isomorphism we can regard σ as a map $\sigma : \mathbb{R} \to S$. This map depends on the choices of the isomorphism and θ, but any choices will do for us and in fact all choices lead to the same result modulo an automorphism (order-preserving permutation) of $(\mathbb{R}, <)$. So we let

$$\operatorname{Shuffle}\big(\{g_K : K \in \mathcal{K}\}\ ;\ g_{K_0}\big)$$

denote a map $\sigma : \mathbb{R} \to S$ as above, for arbitrary tacit choices of these items. The maps g_K (for $K \in \mathcal{K} \cup \{K_0\}$) are called the *ingredients* of the shuffle.

EXAMPLE 5.6 If $a, b, c \in S$ then $\operatorname{Shuffle}(\{a,b\}; c)$ can be taken to be a map $\sigma : \mathbb{R} \to \{a, b, c\}$ such that $\sigma^{-1}(c) = \mathbb{R} \setminus \mathbb{Q}$ and $\sigma^{-1}(a)$, $\sigma^{-1}(b)$ are dense sets of rationals.

5.4 S4 frames

We now use lexicographic sums to establish a relative of the McKinsey–Tarski theorem that the logic of $\mathbb{R}$ in the language with $\Box$ is S4 [16]. It will be needed in §6.7. A similar method is used in [7] to prove the McKinsey–Tarski theorem itself, and others.

An *S4-frame* is a pair (W, R), where R is a reflexive and transitive binary relation on the non-empty set W. Recall from definition 2.1 that $R(w) = \{u \in W : Rwu\}$ for $w \in W$.

DEFINITION 5.7 Let (W, R) be an S4-frame, $(I, <)$ a linear order, and $g : I \to W$ a map. We say that an element $x \in I$ is

- *g-fair* if there are $y, z \in I$ with $y < x < z$ and such that $g((y', z')) = R(g(x))$ for every $y', z' \in I$ with $y \leq y' < x < z' \leq z$,
- *g-good* if there are $y, z \in I$ with $y < x < z$ and such that $g((y', x)) = g((x, z')) = R(g(x))$ for every $y', z' \in I$ with $y \leq y' < x < z' \leq z$.

REMARK 5.8 Fairness and goodness are clearly 'local' properties depending only on arbitrarily small neighbourhoods of the point in question. So if $g = \sum_{j \in J} f_j$, $j \in J$, and $x \in \mathrm{dom}_g(f_j)$ is not an endpoint of $\mathrm{dom}_g(f_j)$, then x is g-fair iff it is f_j-fair, and g-good iff it is f_j-good. (Recall here that we identify f_j with $g \restriction \mathrm{dom}_g(f_j)$.)

DEFINITION 5.9 An S4-frame (W, R) is said to be *connected* if there do not exist non-empty disjoint R-generated subsets $X, Y \subseteq W$ with $W = X \cup Y$.

THEOREM 5.10 *Let $\mathcal{F} = (W, R)$ be a finite connected S4-frame. Then there is a map $g : \mathbb{R} \to W$ satisfying:*

1. *every $x \in \mathbb{R}$ is g-fair,*
2. *for each $w \in W$, the set $\{x \in \mathbb{R} : g(x) = w,\ x \text{ is } g\text{-good}\}$ is unbounded in $\mathbb{R}$. (Consequently, $g^{-1}(w)$ is unbounded in $\mathbb{R}$.)*

Proof. Recall that $R^\bullet wu$ means that $Rwu \wedge \neg Ruw$. As $\mathcal{F}$ is finite, we can define for each $w \in W$ a map $\nu_w : \mathbb{R} \to W$ by complete induction on $|R^\bullet(w)|$:

$$\nu_w = \mathrm{Shuffle}(\{w + \nu_u + w : u \in R^\bullet(w)\} \cup \{u : Rwu \wedge Ruw\}\ ;\ w).$$

This is well defined because $|R^\bullet(u)| < |R^\bullet(w)|$ for each $u \in R^\bullet(w)$, the domain of each map $w + \nu_u + w$ and of each map u can be taken to be an interval of $\mathbb{R}$ with a least and a greatest point (for u it is a singleton interval), and the map w can be taken to be defined on a singleton interval of $\mathbb{R}$.

Claim.

1. $\mathrm{rng}(\nu_w) = R(w)$.
2. Every $x \in \mathbb{R}$ is ν_w-fair.
3. Every ν_w-irrational is ν_w-good.

Proof of claim. The proof is by induction on $|R^\bullet(w)|$. Inductively assume the claim for ν_u, for all $u \in R^\bullet(w)$. For (1), let $u \in R(w)$ be given. If Ruw, the map u is an ingredient of the shuffle defining ν_w, and $\nu_w(x) = u$ for any $x \in \mathrm{dom}_{\nu_w}(u)$. So $u \in \mathrm{rng}(\nu_w)$. If instead $u \in R^\bullet(w)$, then inductively, $u \in R(u) = \mathrm{rng}(\nu_u)$. Since $w + \nu_u + w$ is another ingredient of ν_w, we have $\mathrm{rng}(\nu_u) \subseteq \mathrm{rng}(\nu_w)$. So again, $u \in \mathrm{rng}(\nu_w)$. Part 1 is proved.

Now take $x \in \mathbb{R}$ and suppose that $\nu_w(x) = u$, say. For part 2, we show that x is ν_w-fair. If x is a ν_w-endpoint, then the definition of ν_w tells us that either $u = w$, or

Rwu and Ruw. In both cases, $R(u) = R(w)$. By lemma 5.5 and part 1, whenever $y < x < z$ we have

$$\nu_w((y,z)) = \mathrm{rng}(\nu_w) = R(w) = R(u).$$

It follows that x is ν_w-fair. If x is not a ν_w-endpoint, then x is in the interior of the domain of ν_u for some $u \in R^\bullet(w)$. Inductively, x is ν_u-fair, and hence (see remark 5.8) it is ν_w-fair as well.

For part 3, suppose that x is a ν_w-irrational. Then x is a ν_w-endpoint, so by the above, $R(u) = \mathrm{rng}(\nu_w)$. By lemma 5.5, $\nu_w((y,x)) = \nu_w((x,z)) = R(u)$ whenever $y < x < z$, so plainly x is ν_w-good. This proves the claim.

Now $\mathcal{F}$ is connected and R is reflexive and transitive. It follows that $\mathcal{F}$ is path-connected in the sense that for each $u, v \in W$ there are $w_0, \ldots, w_n \in W$ (for some finite n) with $w_0 = u$, $w_n = v$, $R(w_i, w_{i+1})$ for each even $i < n$, and $R(w_{i+1}, w_i)$ for each odd $i < n$. Using this and the finiteness of $\mathcal{F}$, it is straightforward to find an infinite zigzag path through $\mathcal{F}$ that visits each point infinitely often. Formally, there are $u_i, d_i \in W$ $(i \in \mathbb{Z})$ satisfying:

- Ru_id_{i-1} and Ru_id_i for each $i \in \mathbb{Z}$,
- for each $w \in W$, the set $\{i \in \mathbb{Z} : u_i = w\}$ is unbounded in $\mathbb{Z}$.

We now define our desired map $g : \mathbb{R} \to W$ by

$$g = \sum_{i \in \mathbb{Z}} (u_i + \nu_{u_i} + u_i + \nu_{d_i}).$$

Since the domain of each $u_i + \nu_{u_i} + u_i + \nu_{d_i}$ has a least element and no greatest one, it follows from proposition 5.1 that $\mathrm{dom}(g)$ is order-isomorphic to $(\mathbb{R}, <)$, and as usual we identify the two.

Let $x \in \mathbb{R}$ be arbitrary. We show that it is g-fair. Fix the unique $i \in \mathbb{Z}$ such that

$$x \in \mathrm{dom}_g(u_i + \nu_{u_i} + u_i + \nu_{d_i}). \tag{4}$$

As in §5.2, we identify $u_i + \nu_{u_i} + u_i + \nu_{d_i}$ with $g \restriction \mathrm{dom}_g(u_i + \nu_{u_i} + u_i + \nu_{d_i})$. If $x \in \mathrm{dom}\,\nu_{u_i}$ then it is ν_{u_i}-fair by the claim, and hence (remark 5.8) g-fair. The case where $x \in \nu_{d_i}$ is similar. Suppose x is in the domain of the second u_i in (4). Take $y \in \mathrm{dom}\,\nu_{u_i}$ and $z \in \mathrm{dom}\,\nu_{d_i}$, so that $y < x < z$. By lemma 5.4 and part 1 of the claim, whenever $y \leq y' < x$ we have $g((y', x)) = \mathrm{rng}(\nu_{u_i}) = R(u_i)$, and whenever $x < z' \leq z$ we have $g((x, z')) = \mathrm{rng}(\nu_{d_i}) = R(d_i)$. Consequently, $g((y', z')) = R(u_i) \cup \{u_i\} \cup R(d_i)$. Since Ru_id_i, this is $R(u_i)$ — i.e., $R(g(x))$. It follows that x is g-fair as required. A similar argument covers the case where x is in the domain of the first u_i in (4). We simply note that the left and right neighbours of x are then $\nu_{d_{i-1}}$ and ν_{u_i}, respectively, and that Ru_id_{i-1}. So in all cases, x is g-fair.

Finally, let $w \in W$ and $r \in \mathbb{R}$. The set $\{i \in \mathbb{Z} : u_i = w\}$ is unbounded in $\mathbb{Z}$, so we can take $i \in \mathbb{Z}$ such that $u_i = w$ and $r < x$ for all $x \in \mathrm{dom}_g(u_i + \nu_{u_i} + u_i + \nu_{d_i})$. Take an ν_{u_i}-irrational x. By the last part of the claim, x is ν_{u_i}-good and hence (remark 5.8) g-good. Also, $g(x) = w$ and $x > r$. So $\{x \in \mathbb{R} : g(x) = w,\ x \text{ is } g\text{-good}\}$ has no upper bound, and a symmetrical argument shows that it has no lower bound either. $\square$

6 Completeness proof

In this section we will prove the main result of the paper:

THEOREM 6.1 *The system of §3.1 axiomatises the $\mathcal{L}$-logic of $\mathbb{R}$.*

Soundness was shown in theorem 3.1. The proof of completeness will occupy most of the rest of the paper. We will take a consistent formula φ_0 and successively construct models $\mathcal{M}_0, \ldots, \mathcal{M}_5$ satisfying it, the final one being a model over $\mathbb{R}$. The models $\mathcal{M}_0$–$\mathcal{M}_2$ are exactly as in Shehtman's axiomatisation of the logic of $F, P, \Box$ over $\mathbb{Q}$ in [24]. For $i < 5$, $\mathcal{M}_i$ will be a Kripke model $(W_i, <_i, R_i, h_i)$, but sometimes we drop the index i. Also, we sometimes identify (notationally) $\mathcal{M}_i$ with its domain W_i.

1. $\mathcal{M}_0$ is the canonical model.
2. $\mathcal{M}_1$ is a $<$-generated submodel of $\mathcal{M}_0$ satisfying φ_0. So it is linear and R_0-generated.
3. $\mathcal{M}_2$ is got by filtrating all $<$-clusters of $\mathcal{M}_1$, which consequently become finite.
4. $\mathcal{M}_3$ is a finite R-generated submodel of $\mathcal{M}_2$ got by selective filtration for $<$. We use linearity and the Prior axiom.
5. $\mathcal{M}_4$ satisfies $\mathcal{M}_3 \subseteq \mathcal{M}_4 \subseteq \mathcal{M}_2$ (any such model satisfies φ_0), and is obtained using the Prior axiom and induction. It has the property that every two consecutive $<$-clusters C, D in $\mathcal{M}_4$ either have an intervening irreflexive point or share a common configuration (a submodel with certain properties).
6. Finally $\mathcal{M}_5$ is a model of the form $(\mathbb{R}, h_5)$. Using theorem 5.10, we will construct a surjective map $g : \mathbb{R} \to \mathcal{M}_4$ such that if we define $h_5(p) = g^{-1}(h_4(p))$ for each atom p, then $g : (\mathbb{R}, h_5) \to \mathcal{M}_4$ preserves φ_0 both ways. So we have our model of φ_0 over $\mathbb{R}$.

Now to the details. Let φ_0 be an $\mathcal{L}$-formula consistent with the system defined in §3. Fix a finite set Ψ of formulas containing $\top$ and $P\varphi_0$ and closed under taking subformulas.

6.1 Model $\mathcal{M}_0$

This is just the canonical model of the system given in §3.1, over the set PV of atoms. So W_0 is the set of all maximal consistent sets of $\mathcal{L}$-formulas. We write $\Gamma, \Delta, \Xi, \Theta, \ldots$ for arbitrary members of W_0. The relations and assignment are defined by:

- $\Gamma <_0 \Delta$ iff $\varphi \in \Delta$ for every formula $G\varphi \in \Gamma$ (this is equivalent to each of the three statements $\varphi \in \Delta \Rightarrow F\varphi \in \Gamma$, $H\varphi \in \Delta \Rightarrow \varphi \in \Gamma$, and $\varphi \in \Gamma \Rightarrow P\varphi \in \Delta$),
- $\Gamma\, R_0\, \Delta$ iff $\varphi \in \Delta$ for every formula $\Box\varphi \in \Gamma$ (equivalently, $\varphi \in \Delta \Rightarrow \Diamond\varphi \in \Gamma$),
- $h_0(p) = \{\Gamma \in W_0 : p \in \Gamma\}$ for each atom $p \in PV$.

We assume familiarity with basic facts about canonical models — see, e.g., [5, 3] for details. The most important one is that $\mathcal{M}_0, \Gamma \models \varphi$ iff $\varphi \in \Gamma$, for each $\Gamma \in \mathcal{M}_0$ and each $\mathcal{L}$-formula φ.

LEMMA 6.2 *Every $<_0$-cluster in $\mathcal{M}_0$ is an R_0-generated subset of W_0.*

Proof. Let C be such a cluster. Let $\Gamma \in C$, so that $C = \{\Theta \in W_0 : \Gamma <_0 \Theta <_0 \Gamma\}$. Let $\Delta \in W_0$ and assume that $R_0\Gamma\Delta$. We have $\Gamma <_0 \Gamma\, R_0\, \Delta$, so by axiom 4b we obtain $\Gamma <_0 \Delta$. Similarly, $\Gamma >_0 \Gamma\, R_0\, \Delta$, so by the mirror image of axiom 4b we get $\Gamma >_0 \Delta$. Hence, $\Delta \in C$. □

6.2 Model $\mathcal{M}_1$

As φ_0 is consistent, we can take $\Gamma_0 \in \mathcal{M}_0$ containing φ_0. Let $\mathcal{M}_1$ be the submodel of $\mathcal{M}_0$ with domain $W_1 = \{\Delta \in \mathcal{M}_0 : \Delta = \Gamma_0 \vee \Delta <_0 \Gamma_0 \vee \Delta >_0 \Gamma_0\}$. It follows from the transitivity and linearity axioms for G, H that $\mathcal{M}_1$ is a $<_0$-generated and $>_0$-generated submodel of $\mathcal{M}_0$. By axiom 4a, it is also R_0-generated. Hence, $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$, so $\mathcal{M}_1, \Gamma_0 \models \varphi_0$ and φ_0 is satisfied in $\mathcal{M}_1$. Note that $\mathcal{M}_1$ is *linear* in that $\mathcal{M}_1 \models \forall xy(x = y \vee x < y \vee y < x)$; more than one disjunct can hold.

LEMMA 6.3 *Every $<_1$-cluster in $\mathcal{M}_1$ is an R_1-generated subset of W_1.*

Proof. Let C be such a cluster. Now $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$, so C is also a $<_0$-cluster in $\mathcal{M}_0$. By lemma 6.2, C is an R_0-generated subset of W_0, so since $\mathcal{M}_1 \subseteq \mathcal{M}_0$, it is an R_1-generated subset of W_1. □

6.3 Model $\mathcal{M}_2$

This is a certain filtration of $\mathcal{M}_1$ through Ψ, invented by Shehtman [24, §3].

6.3.1 Definition of $\mathcal{M}_2$

Let $\sim$ be the following binary relation on $\mathcal{M}_1$:

$$\Gamma \sim \Delta \iff \Gamma = \Delta \vee \big((\Gamma <_1 \Delta) \wedge (\Delta <_1 \Gamma) \wedge (\Gamma \cap \Psi = \Delta \cap \Psi)\big). \tag{5}$$

As $<_1$ is transitive, $\sim$ is an equivalence relation. For $\Gamma \in W_1$ we write $\Gamma/{\sim}$ for the equivalence class $\{\Delta \in W_1 : \Gamma \sim \Delta\}$, and for $X \subseteq W_1$ we write $X/{\sim}$ for the set $\{\Gamma/{\sim} : \Gamma \in X\}$ of equivalence classes having a non-empty intersection with X. The domain W_2 of $\mathcal{M}_2$ is now defined to be the set $W_1/{\sim}$ of $\sim$-equivalence classes in $\mathcal{M}_1$. The relations on $\mathcal{M}_2$ are: $<_2$ is induced existentially from $<_1$, and R_2 is the transitive closure of the relation induced existentially from R_1. Formally:

$$\begin{aligned}
&<_2 = \{(\Gamma/{\sim}, \Delta/{\sim}) : \Gamma, \Delta \in W_1,\ \Gamma <_1 \Delta\},\\
&R_2^0 = \{(\Gamma/{\sim}, \Delta/{\sim}) : \Gamma, \Delta \in W_1,\ \Gamma\, R_1\, \Delta\},\\
&R_2 \text{ is the transitive closure of } R_2^0.
\end{aligned}$$

We set $h_2(p) = \{\Gamma/{\sim} : \Gamma \in W_1,\ p \in \Gamma\}$ for each atom $p \in PV$. This defines the model $\mathcal{M}_2$.

LEMMA 6.4 (filtration) *For all $\Gamma \in \mathcal{M}_1$ and $\psi \in \Psi$ we have $\mathcal{M}_2, \Gamma/{\sim} \models \psi$ iff $\mathcal{M}_1, \Gamma \models \psi$ (iff $\psi \in \Gamma$). Hence, φ_0 is satisfied in $\mathcal{M}_2$.*

Proof. This is proved in [24, lemmas 3.2–3.3]. □

6.3.2 $<_2$-clusters in $\mathcal{M}_2$

LEMMA 6.5 *A subset $C \subseteq W_2$ is a $<_2$-cluster in $\mathcal{M}_2$ iff $\bigcup C$ is a $<_1$-cluster in $\mathcal{M}_1$. Every $<_2$-cluster in $\mathcal{M}_2$ is a finite R_2-generated subset of W_2.*

Proof. Let $\Gamma, \Delta \in W_1$. If $\Gamma <_1 \Delta$ then $\Gamma/{\sim} <_2 \Delta/{\sim}$ by definition of $<_2$. Conversely, if $\Gamma/{\sim} <_2 \Delta/{\sim}$ then by definition of $<_2$ there are $\Gamma', \Delta' \in W_1$ with $\Gamma \sim \Gamma'$, $\Delta \sim \Delta'$, and $\Gamma' <_1 \Delta'$. The definition of $\sim$ gives $\Gamma \leq_1 \Gamma'$ and $\Delta' \leq_1 \Delta$, so $\Gamma <_1 \Delta$ by transitivity of $<_1$. We conclude that $\Gamma/{\sim} <_2 \Delta/{\sim}$ iff $\Gamma <_1 \Delta$.

Let $C \subseteq W_2$. If C is a $<_2$-cluster in $\mathcal{M}_2$, let $\Gamma/{\sim} \in C$ be arbitrary. Then $C = \{w \in W_2 : \Gamma/{\sim} <_2 w <_2 \Gamma/{\sim}\}$. By the above,

$$\begin{aligned} \bigcup C &= \bigcup\{w \in W_2 : \Gamma/{\sim} <_2 w <_2 \Gamma/{\sim}\} \\ &= \{\Delta \in W_1 : \Gamma <_1 \Delta <_1 \Gamma\}, \end{aligned}$$

a $<_1$-cluster in $\mathcal{M}_1$. Conversely, if $\bigcup C$ is a $<_1$-cluster in $\mathcal{M}_1$, let $\Gamma \in \bigcup C$. Then $\bigcup C = \{\Delta \in W_1 : \Gamma <_1 \Delta <_1 \Gamma\}$. So

$$\begin{aligned} C &= \{\Delta/{\sim} : \Delta \in \bigcup C\} \\ &= \{\Delta/{\sim} : \Delta \in W_1,\ \Gamma <_1 \Delta <_1 \Gamma\} \\ &= \{w \in W_2 : \Gamma/{\sim} <_2 w <_2 \Gamma/{\sim}\}, \end{aligned}$$

a $<_2$-cluster in $\mathcal{M}_2$. The first part of the lemma follows.

Let C be a $<_2$-cluster. Then C has the form $D/{\sim}$ for a $<_1$-cluster $D = \bigcup C$, and by definition of $\sim$ we have $\Gamma \sim \Delta$ iff $\Gamma \cap \Psi = \Delta \cap \Psi$ for each $\Gamma, \Delta \in D$. Hence, the map $f : C \to \wp(\Psi)$ given by $f(\Gamma/{\sim}) = \Gamma \cap \Psi$ is well defined and one-one, so $|C| \leq |\wp(\Psi)|$ and C is finite (this is [24, lemma 3.4]).

Let $w \in C$ and $u \in W_2$, with $R_2^0 wu$. Pick $\Gamma \in w$ and $\Delta \in u$ with $R_1\Gamma\Delta$. By the above and lemma 6.3, $\bigcup C$ is R_1-generated, so $\Delta \in \bigcup C$, and therefore $u \in C$. It follows easily that C is an R_2-generated subset of W_2 (see also [24, lemma 3.6]). $\square$

By [24, lemma 3.3], the frame of $\mathcal{M}_2$ validates all axioms of the system of §3.1 except perhaps axiom 5. The following can now be obtained from [24, lemmas 3.5–3.6].

LEMMA 6.6 **(λ, ρ-lemma)** *Let $w \in \mathcal{M}_2$.*

1. *If w is $<_2$-reflexive then $R_2(w)$ is a subset of a $<_2$-cluster.*
2. *If w is $<_2$-irreflexive, then there are unique $<_2$-clusters $\lambda(w)$, $\rho(w)$ such that $R_2(w) = \lambda(w) \cup \{w\} \cup \rho(w)$. For every $t \in \mathcal{M}_2$, $u \in \lambda(w)$, and $v \in \rho(w)$, we have $t <_2 w$ iff $t <_2 u$, and $t >_2 w$ iff $t >_2 v$.*

Taking $t = u$ and $t = v$, we obtain $u <_2 w <_2 v$ for every $u \in \lambda(w)$ and $v \in \rho(w)$.

We now write down formulas to define individual elements within a $<_2$-cluster.

DEFINITION 6.7 For $w \in \mathcal{M}_2$ let $\chi_w = \bigwedge(\Psi \cap \Gamma) \wedge \neg \bigvee(\Psi \setminus \Gamma)$ for arbitrary $\Gamma \in w$. (By convention, $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.)

By definition of $\sim$ in (5), this definition is independent of the choice of Γ, and obviously $\chi_w \in \Gamma$. The set $\{\chi_w : w \in \mathcal{M}_2\}$ is finite, because Ψ is finite.

REMARK 6.8 If $w, u \in \mathcal{M}_2$, say $w = \Gamma/{\sim}$ and $u = \Delta/{\sim}$, and $\chi_w, \chi_u \in \Theta$ for some $\Theta \in \mathcal{M}_0$, then $\Psi \cap \Gamma = \Psi \cap \Delta$, and so χ_w is equivalent to χ_u.

These formulas are quite useful — firstly in the following lemma.

LEMMA 6.9 *Let C be a $<_2$-cluster in $\mathcal{M}_2$. Then $(C, R_2 \restriction C)$ is a connected frame.*

Proof. Suppose on the contrary that C is a $<_2$-cluster in $\mathcal{M}_2$ that is the union of disjoint non-empty R_2-generated sets X, Y. Let $\alpha = \bigvee_{w \in X} \chi_w$. Then for any $\Gamma \in \bigcup C$ we have

$$\alpha \in \Gamma \iff \Gamma/{\sim} \in X. \tag{6}$$

Choose any $\Gamma \in \bigcup C$ and let

$$\Delta_0 = \{\Diamond\alpha, \Diamond\neg\alpha\} \cup \{F\gamma, P\gamma : \gamma \in \Gamma\}.$$

We show that this is consistent. Since Γ is closed under conjunction, it suffices to take arbitrary $\gamma \in \Gamma$ and show that $\delta = \Diamond\alpha \wedge \Diamond\neg\alpha \wedge F\gamma \wedge P\gamma$ is consistent. Choose any $\Gamma_X \in \bigcup X$ and $\Gamma_Y \in \bigcup Y$. By lemma 6.5, $\bigcup C$ is a $<_1$-cluster in $\mathcal{M}_1$. So $\Gamma < \Gamma_X < \Gamma$ and similarly for Γ_Y. Now $\alpha \wedge F\gamma \in \Gamma_X$ and $\neg\alpha \wedge F\gamma \in \Gamma_Y$. So $F(\alpha \wedge F\gamma), F(\neg\alpha \wedge F\gamma) \in \Gamma$. By axiom 5, $F(\Diamond\alpha \wedge \Diamond\neg\alpha \wedge F\gamma) \in \Gamma$. By temporal axioms, $GP\gamma \in \Gamma$ as well, so $F\delta \in \Gamma$. If δ is inconsistent then $\neg\delta$ and hence $G\neg\delta$ are provable, so $G\neg\delta \in \Gamma$, contradicting its consistency. So δ is consistent.

So we may take $\Delta \in \mathcal{M}_0$ with $\Delta \supseteq \Delta_0$. By definition of Δ_0 we have $\Gamma <_0 \Delta <_0 \Gamma$, so $\Delta \in \mathcal{M}_1$ and $\Delta \in \bigcup C$. As $\Diamond\alpha, \Diamond\neg\alpha \in \Delta$, we may find $\Delta_X, \Delta_Y \in R_0(\Delta)$ with $\alpha \in \Delta_X$ and $\neg\alpha \in \Delta_Y$. Then $\Delta_X, \Delta_Y \in R_1(\Delta)$ as $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$. By lemma 6.3, $\bigcup C$ is an R_1-generated subset of $\mathcal{M}_1$, so $\Delta_X, \Delta_Y \in \bigcup C$ as well.

Let $w = \Delta/{\sim}$, $w_X = \Delta_X/{\sim}$, and $w_Y = \Delta_Y/{\sim}$. By (6), $w_X \in X$ and $w_Y \in Y$. By definition of R_2, we have $w_X, w_Y \in R_2(w)$. Since $w \in X \cup Y$, this contradicts that X and Y are disjoint and R_2-generated. $\square$

6.3.3 Hinterland

DEFINITION 6.10 We define the *hinterland* $H(\Gamma)$ of a set $\Gamma \in \mathcal{M}_1$ to be the set $R_1(\Gamma)/{\sim}$. It is the set of elements of $\mathcal{M}_2$ that Γ 'directly sees'.

Plainly, $H(\Gamma) \subseteq R_2(\Gamma/{\sim})$, and by lemma 6.6 the latter is contained in either a single $<_2$-cluster or the union of two such clusters with a singleton. So by lemma 6.5, $H(\Gamma)$ is finite.

The following is trivial but will be absolutely vital later. In a way, our entire proof hinges on it.

COROLLARY 6.11 *Let $\Gamma \in \mathcal{M}_1$, $w = \Gamma/{\sim}$, and $\Diamond\psi \in \Psi$, and suppose that $\mathcal{M}_2, w \models \Diamond\psi$. Then $\mathcal{M}_2, u \models \psi$ for some $u \in H(\Gamma)$.*

Proof. As $\mathcal{M}_2, w \models \Diamond\psi$, by lemma 6.4 we have $\mathcal{M}_1, \Gamma \models \Diamond\psi$. So there is $\Delta \in R_1(\Gamma)$ with $\mathcal{M}_1, \Delta \models \psi$. Let $u = \Delta/{\sim} \in H(\Gamma)$. By the lemma again, $\mathcal{M}_2, u \models \psi$. $\square$

This completes our study of $\mathcal{M}_2$.

6.4 Model $\mathcal{M}_3$

We now do a selective filtration of $\mathcal{M}_2$ to deliver our first verifiably finite model: an R_2-generated submodel $\mathcal{M}_3 \subseteq \mathcal{M}_2$ satisfying φ_0. We select the points of $\mathcal{M}_2$ to include in $\mathcal{M}_3$ in four steps.

1. Let $\Lambda_0 = \{P\varphi : \varphi \text{ is satisfied in } \mathcal{M}_1\}$. Then Λ_0 is consistent. For suppose that $\varphi_0, \ldots, \varphi_{n-1}$ are satisfied in $\mathcal{M}_1$ at $\Delta_0, \ldots, \Delta_{n-1}$, say. Choose $i < n$ such that $|\{j < n : \Delta_j \leq_1 \Delta_i\}|$ is maximal. It follows by linearity and transitivity of $<_1$ that for every $j < n$ we have $\Delta_j \leq_1 \Delta_i$. By lemma 3.2, $F\top \in \Delta_i$, so there is $\Gamma \in \mathcal{M}_1$ with $\Delta_i <_1 \Gamma$. Then $\Delta_0, \ldots, \Delta_{n-1} <_1 \Gamma$, so $P\varphi_0, \ldots, P\varphi_{n-1} \in \Gamma$, proving consistency of Λ_0.

   Let $\Lambda \supseteq \Lambda_0$ be maximal consistent. Then $\Lambda \in \mathcal{M}_0$. For every $\Gamma \in \mathcal{M}_1$ we have $\varphi \in \Gamma \Rightarrow P\varphi \in \Lambda_0 \subseteq \Lambda$, so $\Gamma <_0 \Lambda$. This means that $\Lambda \in \mathcal{M}_1$ and that Λ is a $<_1$-greatest point in $\mathcal{M}_1$. Hence also, $\Lambda <_1 \Lambda$, so Λ lies in a $<_1$-cluster C_∞ of $\mathcal{M}_1$. It is an R_1-generated subset of W_1. We select the whole of $C_\infty/{\sim}$, a finite $<_2$-cluster and an R_2-generated subset of W_2, and add it to $\mathcal{M}_3$.

2. Do mirror image of step 1, calling the cluster $C_{-\infty}$. Possibly, $C_{-\infty} = C_{+\infty}$.

3. Now consider in turn each $\psi \in \Psi$ (if any) such that ψ and $FG\neg\psi$ are both satisfied in $\mathcal{M}_1$. Then some (unique irreflexive) $\Delta \in \mathcal{M}_1$ contains $G\neg\psi \wedge HF\psi$. Such a Δ is plainly irreflexive, and by linearity it is unique. For existence, using linearity and (by lemma 3.2) lack of a least element of $<_1$ we may find $\Gamma \in \mathcal{M}_1$ containing $F\psi \wedge FG\neg\psi$. By the Prior axiom, which is provable by corollary 3.5, $F(G\neg\psi \wedge HF\psi) \in \Gamma$, so there is $\Delta >_1 \Gamma$ as required.

   Select the whole finite set $R_2(\Delta/{\sim})$ and add it to $\mathcal{M}_3$.

4. Also do mirror image of step 3.

Plainly, $\mathcal{M}_3$ is a non-empty finite R_2-generated submodel of $\mathcal{M}_2$.

DEFINITION 6.12 For models $\mathcal{M} \subseteq \mathcal{N}$, we write $\mathcal{M} \preceq_\Psi \mathcal{N}$ if $\mathcal{M}, w \models \psi$ iff $\mathcal{N}, w \models \psi$ for every $w \in \mathcal{M}$ and $\psi \in \Psi$.

The following is fairly standard.

LEMMA 6.13 *Let $\mathcal{M} = (W, <, R, h)$ be any R_2-generated submodel of $\mathcal{M}_2$ extending $\mathcal{M}_3$. Then $\mathcal{M} \preceq_\Psi \mathcal{M}_2$, and φ_0 is satisfied in $\mathcal{M}$. Moreover, if $w \in \mathcal{M}$, $F\psi \in \Psi$, and $\mathcal{M}, w \models F\psi$, then*

1. *there is $u \in C_\infty/{\sim}$ with $\mathcal{M}, u \models \psi$, or*
2. *there is irreflexive $u \in \mathcal{M}$ with $u > w$, and $v \in \mathcal{M}$ with $w < v \leq u \, R \, v$ and $\mathcal{M}, v \models \psi$.*

A mirror image holds for formulas $P\psi \in \Psi$.

Proof. We prove that $\mathcal{M} \preceq_\Psi \mathcal{M}_2$ by induction on ψ. The main cases are $F\psi$, $P\psi$, and $\Diamond\psi$. For the forward direction, if $\mathcal{M}, w \models F\psi$ then there is $u > w$ in $\mathcal{M}$ with $\mathcal{M}, u \models \psi$. Inductively, $\mathcal{M}_2, u \models \psi$, and plainly, $u >_2 w$ because $\mathcal{M} \subseteq \mathcal{M}_2$. Hence, $\mathcal{M}_2, w \models F\psi$. The cases $P\psi$ and $\Diamond\psi$ are similarly proved.

We consider the converse direction now. If $\mathcal{M}_2, w \models \Diamond\psi$ then $\mathcal{M}_2, u \models \psi$ for some $u \in R_2(w)$. As $\mathcal{M}$ is an R_2-generated submodel of $\mathcal{M}_2$, $u \in \mathcal{M}$ as well, and Rwu. Inductively, $\mathcal{M}, u \models \psi$, so $\mathcal{M}, w \models \Diamond\psi$.

Assume now that $\mathcal{M}_2, w \models F\psi$. Take any $\Gamma \in w$, so that $w = \Gamma/{\sim}$. By lemma 6.4, $F\psi \in \Gamma$.

There are now two cases. Suppose first that $GF\psi \in \Gamma$. Referring to the definition of $\mathcal{M}_3$, we have $F\psi \in \Lambda$ as $\Gamma <_1 \Lambda$, so there is $\Lambda' >_1 \Lambda$ in $\mathcal{M}_1$ containing ψ. Let $u = \Lambda'/{\sim}$. By lemma 6.4 again, $\mathcal{M}_2, u \models \psi$. But $\Lambda' <_1 \Lambda$ as well, because Λ is $<_1$-maximal, so $\Lambda' \in C_\infty$ and $u \in C_\infty/{\sim} \subseteq \mathcal{M}_3 \subseteq \mathcal{M}$. Inductively, $\mathcal{M}, u \models \psi$. Also, $\Lambda' >_1 \Lambda >_1 \Gamma$, so $u >_2 w$ and $u > w$ in $\mathcal{M}$. Hence, $\mathcal{M}, w \models F\psi$.

Alternatively, $FG\neg\psi \in \Gamma$. By definition of $\mathcal{M}_3$, there is $\Delta \in \mathcal{M}_1$ with $G\neg\psi \wedge HF\psi \in \Delta$ and $R_2(\Delta/{\sim}) \subseteq \mathcal{M}_3$. If $\Gamma \geq_1 \Delta$ then $G\neg\psi \in \Gamma$, contradicting its consistency. So by linearity, $\Gamma <_1 \Delta$.

Let $C = \{\Xi \in R_1(\Delta) : \Xi <_1 \Delta\}$. Then $C \neq \emptyset$ by the mirror image of axiom 4d. Let $\Xi \in C$. Then $F\psi \in \Xi$, so there is $\Theta >_1 \Xi$ containing ψ. Since $G\neg\psi \in \Delta$, by linearity we must have $\Theta \leq_1 \Delta$, so by axiom 4c, $\Theta \in C \cup \{\Delta\}$. By axiom 4b, $\Gamma <_1 \Theta$.

Let $v = \Theta/{\sim} \in \mathcal{M}_2$. Then $v \in R_2(\Delta/{\sim}) \subseteq \mathcal{M}_3 \subseteq \mathcal{M}$, and $w < v$. By lemma 6.4, $\mathcal{M}_2, v \models \psi$, and inductively, $\mathcal{M}, v \models \psi$. Hence, $\mathcal{M}, w \models F\psi$.

So in either case we have $\mathcal{M}, w \models F\psi$ as required. This also establishes the 'moreover' part: in (1) we have $u = \Lambda'/{\sim}$ and in (2) $u = \Delta/{\sim}$ and $v = \Theta/{\sim}$. The case of $P\psi$ is handled similarly. This completes the induction.

As φ_0 is satisfied in $\mathcal{M}_1$, we must have $P\varphi_0 \in \Lambda$, and by lemma 6.4, $\mathcal{M}_2, \Lambda/{\sim} \models P\varphi_0$. Recall that $P\varphi_0 \in \Psi$. By the above, $\mathcal{M}, \Lambda/{\sim} \models P\varphi_0$, and it follows that φ_0 is satisfied in $\mathcal{M}$. $\square$

6.5 Links

The model $\mathcal{M}_3$ can have consecutive $<_3$-clusters that are actually adjacent, with no intervening point. In our eventual model over $\mathbb{R}$, $<$-clusters correspond to open sets, and we have the problem of finding something suitable to put between them. So we would like to arrange, possibly in a larger model than $\mathcal{M}_3$, that any two such clusters have something in common that we can use as intervening material. This common something will be a certain small submodel, which we call a link.

Given $\Gamma \in \mathcal{M}_1$, consider two finite subsets of $\mathcal{M}_2$: $R_2(\Gamma/{\sim})$, and its subset $H(\Gamma) = R_1(\Gamma)/{\sim}$. The first is dependent on R_2 and the details of this are not available to Γ. So let us focus on the second, $H(\Gamma)$. We would like to find a formula $\gamma \in \Gamma$ that determines the isomorphism type of the frame $\mathcal{H}_\Gamma = (H(\Gamma), R_2 \restriction H(\Gamma))$, in the sense that $\mathcal{H}_\Delta \cong \mathcal{H}_\Gamma$ for any Δ containing γ. We would also like γ to determine some of the formulas true at points in $\mathcal{H}_\Gamma$. Then we will be able to use the Prior axiom to find faithful copies of $\mathcal{H}_\Gamma$ in adjacent clusters.

It is not clear how to write such a γ in general, because there may be $w, u \in H(\Gamma)$ with $R_2(w, u)$ but no 'witnesses' $\Delta \in w$, $\Theta \in u$ with $R_1\Delta\Theta$. However, if *for every* $\Delta \in R_1(\Gamma)$ *and* $u \in H(\Gamma)$ *there exists* $\Theta \in u$ *with* $R_1\Delta\Theta$ — a property that can be enforced by a formula — then $R_2 \restriction H(\Gamma)$ is forced to be the biggest possible relation, $H(\Gamma) \times H(\Gamma)$. The isomorphism type of $\mathcal{H}_\Gamma$ is then determined.

It may not be immediately obvious that there are any Γ like this, but in fact they are quite common. Any Γ has this property if $H(\Delta) = H(\Gamma)$ for every $\Delta \in R_1(\Gamma)$, a condition that holds if $R_1^\bullet(\Gamma) = \emptyset$, and more generally if $H(\Gamma)$ is of minimal cardinality in some suitable sense. We now formalise this and take it further.

DEFINITION 6.14 Let $\Gamma \in \mathcal{M}_1$.

1. Γ is said to be a *link* if $H(\Gamma) = H(\Delta)$ for every $\Delta \in R_1(\Gamma)$.
2. We write $\eta_\Gamma = \big(\Box \bigvee_{w \in H(\Gamma)} \chi_w\big) \wedge \big(\bigwedge_{w \in H(\Gamma)} \Box\Diamond\chi_w\big)$. (See definition 6.7 for χ_w.)
3. If $\Delta \in \mathcal{M}_1$, then Γ, Δ are said to be *similar* if η_Γ and η_Δ are equivalent (they lie in the same members of $\mathcal{M}_0$).

Because there are only finitely many formulas χ_w, there are only finitely many similarity types of elements of $\mathcal{M}_1$.

LEMMA 6.15 *For $\Gamma, \Delta \in \mathcal{M}_1$, suppose that $\eta_\Gamma, \eta_\Delta \in \Theta$ for some $\Theta \in \mathcal{M}_0$. Then η_Γ and η_Δ are equivalent.*

Proof. Let $w \in H(\Gamma)$ be arbitrary. As $\eta_\Gamma \in \Theta$, we have $\Box\Diamond\chi_w \in \Theta$, and reflexivity of R_0 yields $\Diamond\chi_w \in \Theta$. So there is $\Xi \in R_0(\Theta)$ containing χ_w. But $\eta_\Delta \in \Theta$ too, so $\Box \bigvee_{u \in H(\Delta)} \chi_u \in \Theta$ and $\bigvee_{u \in H(\Delta)} \chi_u \in \Xi$. Take $u \in H(\Delta)$ with $\chi_u \in \Xi$. By remark 6.8, χ_u is equivalent to χ_w.

We conclude that for every $w \in H(\Gamma)$ there is $u \in H(\Delta)$ such that χ_w is equivalent to χ_u, and (by symmetry) vice versa. It is now plain that η_Γ is equivalent to η_Δ. $\Box$

LEMMA 6.16 *1. Any R_1-generated subset of $\mathcal{M}_1$ contains a link.*

2. Any link is $<_1$-reflexive.

3. If $\Gamma \in \mathcal{M}_1$ is a link then $\eta_\Gamma \in \Gamma$.

4. If $\Gamma \in \mathcal{M}_1$ is arbitrary (not necessarily a link), $\Delta \in \mathcal{M}_1$ is $<_1$-reflexive, and $\eta_\Gamma \in \Delta$, then Δ is a link and Δ is similar to Γ.

Proof. 1. Let $X \subseteq \mathcal{M}_1$ be R_1-generated. Choose $\Gamma \in X$ such that $|H(\Gamma)|$ is least possible. By transitivity of R_1, if $\Delta \in R_1(\Gamma)$ then $R_1(\Delta) \subseteq R_1(\Gamma)$, so $H(\Delta) \subseteq H(\Gamma)$. As X is R_1-generated, $\Delta \in X$. So by minimality of $|H(\Gamma)|$ we have $H(\Delta) = H(\Gamma)$. As Δ was arbitrary, Γ is a link.

2. Let Γ be a link. Using axiom 4d, choose $\Delta \in R_1(\Gamma)$ with $\Gamma <_1 \Delta$. Then $\Gamma/{\sim} \in H(\Gamma) = H(\Delta)$, so there is $\Theta \in R_1(\Delta)$ with $\Theta \sim \Gamma$. By axiom 4b, $\Gamma <_1 \Theta$. If $\Theta = \Gamma$, then plainly $\Gamma <_1 \Gamma$ as required. Otherwise, as $\Theta \sim \Gamma$, by definition of $\sim$ we have $\Gamma <_1 \Theta <_1 \Gamma$, and transitivity gives $\Gamma <_1 \Gamma$ again.

3. Recall that
$$\eta_\Gamma = \Big(\Box \bigvee_{w \in H(\Gamma)} \chi_w\Big) \wedge \Big(\bigwedge_{w \in H(\Gamma)} \Box\Diamond\chi_w\Big).$$
Assume that Γ is a link. Take any $\Delta \in R_1(\Gamma)$. To prove that $\eta_\Gamma \in \Gamma$, it is enough to show that $\bigvee_{w \in H(\Gamma)} \chi_w \in \Delta$ and that $\Diamond\chi_w \in \Delta$ for every $w \in H(\Gamma)$. Let $u = \Delta/{\sim} \in H(\Gamma)$. First, $\chi_u \in \Delta$, so $\bigvee_{w \in H(\Gamma)} \chi_w \in \Delta$. Second, take arbitrary $w \in H(\Gamma)$. As Γ is a link, $H(\Gamma) = H(\Delta)$, so $w \in H(\Delta)$ and there is $\Theta \in R_1(\Delta) \cap w$. We have $\chi_w \in \Theta$, so $\Diamond\chi_w \in \Delta$ as required.

4. Take any $\Theta \in R_1(\Delta)$. Certainly, $H(\Theta) \subseteq H(\Delta)$. To show the converse, take any $u \in H(\Delta)$ and pick $\Xi \in R_1(\Delta) \cap u$. By assumption, $\eta_\Gamma \in \Delta$, so $\bigvee_{w \in H(\Gamma)} \chi_w \in \Xi$, and hence $\chi_w \in \Xi$ for some $w \in H(\Gamma)$. But $\Box\Diamond\chi_w \in \Delta$, so $\Diamond\chi_w \in \Theta$ and there is $\Xi' \in R_1(\Theta)$ containing χ_w.

   Now Δ is $<_1$-reflexive. It follows from axiom 4b that $R_1(\Delta)$ is contained in a $<_1$-cluster. Since $\Xi, \Xi' \in R_1(\Delta)$, we have $\Xi <_1 \Xi' <_1 \Xi$. Since $\chi_w \in \Xi \cap \Xi'$, we also have $\Xi \cap \Psi = \Xi' \cap \Psi$. We conclude that $\Xi \sim \Xi'$. Hence, $\Xi' \in u$, so $u \in H(\Theta)$. This proves that Δ is a link.

   By the preceding part, $\eta_\Delta \in \Delta$, and we were given that $\eta_\Gamma \in \Delta$. By lemma 6.15, η_Γ, η_Δ are equivalent. So Δ is similar to Γ.

$\Box$

LEMMA 6.17 *Let $\Gamma \in \mathcal{M}_1$ be a link. Then $R_2(w, u)$ for any $w, u \in H(\Gamma)$.*

Proof. Let $w, u \in H(\Gamma)$ be given. Choose $\Delta \in R_1(\Gamma) \cap w$. As Γ is a link, $u \in H(\Gamma) = H(\Delta)$, so there is $\Theta \in R_1(\Delta) \cap u$. Since $R_1\Delta\Theta$, we have R_2wu by definition of R_2. $\Box$

So the hinterland of a link has a fixed frame structure. Similar links also share some model properties. It can be shown that there is a bijection from $H(\Gamma)$ to $H(\Delta)$ that preserves the truth in $\mathcal{M}_2$ of all formulas in Ψ. However, the following more limited statement is sufficient for our later work.

LEMMA 6.18 *Let $\Gamma, \Delta \in \mathcal{M}_1$ be similar links, let $w \in H(\Gamma)$, and $u \in H(\Delta)$. Then $\mathcal{M}_2, w \models \Diamond\psi$ iff $\mathcal{M}_2, u \models \Diamond\psi$ for every formula $\Diamond\psi \in \Psi$.*

Proof. By lemma 6.17, we can suppose without loss of generality that $w = \Gamma/{\sim}$ and $u = \Delta/{\sim}$. We have $\eta_\Gamma, \eta_\Delta \in \Gamma \cap \Delta$ by the hypotheses and lemma 6.16(2).

Suppose that $\mathcal{M}_2, w \models \Diamond\psi$. By the filtration lemma (6.4), $\Diamond\psi \in \Gamma$. Take $\Gamma' \in R_1(\Gamma)$ with $\psi \in \Gamma'$ and let $v = \Gamma'/{\sim} \in H(\Gamma)$. Then $\Box\Diamond\chi_v \in \Delta$, so there is $\Delta' \in R_1(\Delta)$ containing χ_v. By definition of χ_v we have $\Psi \cap \Gamma' = \Psi \cap \Delta'$, so $\psi \in \Delta'$ and hence $\Diamond\psi \in \Delta$. The filtration lemma yields $\mathcal{M}_2, u \models \Diamond\psi$. The converse is similar. $\Box$

6.6 Model $\mathcal{M}_4$

Our next model arranges that any two consecutive clusters either have an irreflexive point between them, or contain similar links.

DEFINITION 6.19 1. For a model $\mathcal{M} = (W, <, R, h)$ and $X, Y \subseteq W$, we write $X < Y$ if $x < y$ for every $x \in X$ and $y \in Y$. We abbreviate $\{x\} < Y$ to $x < Y$, etc.

2. If $X, Y \subseteq \mathcal{M}_2$, we say that X *and* Y *contain similar links* if there is a link $\Gamma \in X$ and a link $\Delta \in Y$ that is similar to Γ.

3. For $<_2$-clusters $C, D \subseteq \mathcal{M}_2$ with $C <_2 D$ (possibly, $C = D$), let $\sharp(C, D)$ be the (finite) number of similarity types of links in the set

$$\{\Theta \in \mathcal{M}_1 : \Gamma <_1 \Theta <_1 \Delta\},$$

where $\Gamma \in \bigcup C$ and $\Delta \in \bigcup D$ are arbitrary. The value does not depend on the choice of Γ, Δ (since $\bigcup C$, $\bigcup D$ are $<_1$-clusters). Since $\bigcup C$ is R_1-generated by lemma 6.3, by lemma 6.16(1) it contains a link, so $\sharp(C, D) > 0$.

The following important technical lemma is needed just below.

LEMMA 6.20 *Let C be a $<_2$-cluster in $\mathcal{M}_2$, and let $w \in \mathcal{M}_2$ with $C <_2 w \notin C$. Then there is a $<_2$-irreflexive $u \in \mathcal{M}_2$ such that:*

1. *$C <_2 u \leq_2 w$,*
2. *$\bigcup C$ and $\bigcup \lambda(u)$ contain similar links,*
3. *if w is $<_2$-irreflexive and $u <_2 w$, then $\sharp(\rho(u), \lambda(w)) < \sharp(C, \lambda(w))$.*

(See lemma 6.6 for λ, ρ.) The mirror image also holds.

Proof. By lemma 6.5, $\bigcup C$ is a $<_1$-cluster in $\mathcal{M}_1$, so by lemma 6.3 it is an R_1-generated subset of $\mathcal{M}_1$. Using lemma 6.16(1), pick a link $\Gamma \in \bigcup C$. Also pick any $\Delta \in w$. Since $C <_2 w \notin C$, we have $\Gamma <_1 \Delta \not<_1 \Gamma$. So there is a formula δ with

$$\delta \in \Gamma \text{ and } G\neg\delta \in \Delta. \tag{7}$$

If w is $<_2$-irreflexive, then Δ is $<_1$-irreflexive, and in that case we can suppose that $H\delta \in \Delta$ as well.

As Γ is a link, lemma 6.16 yields $\eta_\Gamma \in \Gamma$. So $\eta_\Gamma \wedge \delta \in \Gamma$ and $G\neg(\eta_\Gamma \wedge \delta) \in \Delta$, and as $\Gamma <_1 \Gamma$, we obtain $F(\eta_\Gamma \wedge \delta) \wedge FG\neg(\eta_\Gamma \wedge \delta) \in \Gamma$. By the Prior axiom, $F(G\neg(\eta_\Gamma \wedge \delta) \wedge HF(\eta_\Gamma \wedge \delta)) \in \Gamma$, and this lets us take $\Theta >_1 \Gamma$ in $\mathcal{M}_1$ with

$$G\neg(\eta_\Gamma \wedge \delta) \wedge HF(\eta_\Gamma \wedge \delta) \in \Theta. \tag{8}$$

So Θ is $<_1$-irreflexive. Let $u = \Theta/\sim$. Then u is $<_2$-irreflexive. Since $\Gamma <_1 \Theta$, we have $C <_2 u$. Since $G\neg\delta \in \Delta$ and $HF\delta \in \Theta$, we see that $\Delta \not<_1 \Theta$. By linearity, $\Theta \leq_1 \Delta$, so $u \leq_2 w$.

As in lemma 6.13, it follows from (8) that some $\Theta' \in \{\Theta\} \cup \bigcup \lambda(u)$ contains $\eta_\Gamma \wedge \delta$. Bearing in mind that η_Γ implies $\Box\eta_\Gamma$, we see that some $\Xi \in \bigcup \lambda(u)$ contains η_Γ. As $\bigcup \lambda(u)$ is a $<_1$-cluster, $\Xi <_1 \Xi$. So by lemma 6.16(4), Ξ is a link similar to Γ.

Suppose that w is irreflexive and $u <_2 w$, so that $\rho(u) <_2 \lambda(w)$ by lemma 6.6 (possibly $\rho(u) = \lambda(w)$). We have $H\delta \in \Delta$ in this case, and by (8), $G\neg(\eta_\Gamma \wedge \delta) \in \Theta$. So $\eta_\Gamma \notin \Xi'$ for every $\Xi' \in \mathcal{M}_1$ with $\Theta <_1 \Xi' <_1 \Delta$. Therefore, there is no link Ξ' similar to Γ with $\Theta <_1 \Xi' <_1 \Delta$. Since there certainly is such a link with $\Gamma <_1 \Xi' <_1 \Delta$, namely $\Xi' = \Gamma$, we see that $\sharp(\rho(u), \lambda(w)) < \sharp(C, \lambda(w))$. $\Box$

DEFINITION 6.21 Let $\mathcal{M} = (W, <, R, h)$ be a submodel of $\mathcal{M}_2$.

1. We say that $<$-clusters C, D in $\mathcal{M}$ are

- *consecutive* if $C \neq D$ and $\{u \in \mathcal{M} : C < u < u < D\} = C \cup D$,
- *adjacent* if $C \neq D$ and $\{u \in \mathcal{M} : C < u < D\} = C \cup D$.

In each case, $C < D$. Consecutive clusters have no $<$-reflexive points between them, but may have $<$-irreflexive ones. Adjacent clusters have nothing between them.

2. We say that $\mathcal{M}$ is *good* if it is finite, R_2-generated, and for every $<_2$-cluster $C \subseteq \mathcal{M}_2$, if $C \cap \mathcal{M} \neq \emptyset$ then $C \subseteq \mathcal{M}$.
3. We say that $\mathcal{M}$ is *perfect* if it is good and $\bigcup C$ and $\bigcup D$ contain similar links whenever C, D are adjacent $<$-clusters in $\mathcal{M}$.

LEMMA 6.22 *Any good submodel of $\mathcal{M}_2$ extends to a perfect submodel of $\mathcal{M}_2$.*

Proof. Let $\mathcal{M} = (W, <, R, h)$ be a good submodel of $\mathcal{M}_2$. A *defect* in $\mathcal{M}$ is a pair (C, D) of adjacent $<$-clusters in $\mathcal{M}$ such that $\bigcup C$ and $\bigcup D$ do not contain similar links. Let

$$d(\mathcal{M}) = \sum \big\{\sharp(C, D) : (C, D) \text{ a defect of } \mathcal{M}\big\}.$$

We prove the lemma by induction on $d(\mathcal{M})$, which is finite because $\mathcal{M}$ is. If it is zero then $\mathcal{M}$ has no defects and is therefore perfect, and there is nothing to prove. So assume that $d(\mathcal{M}) > 0$ and assume the lemma for all good $\mathcal{M}' \subseteq \mathcal{M}_2$ with $d(\mathcal{M}') < d(\mathcal{M})$. Pick any defect (C, D) in $\mathcal{M}$, and pick any $w \in D$. Then $C <_2 w \not<_2 C$. Let $u \in \mathcal{M}_2$ be as provided by lemma 6.20, and let $\mathcal{N}$ be the submodel of $\mathcal{M}_2$ consisting of $\mathcal{M}$ together with $R_2(u)$. We let $<$ denote $<_2 \restriction \mathcal{N}$. So

$$\{v \in \mathcal{N} : C < v < D\} = C \cup \lambda(u) \cup \{u\} \cup \rho(u) \cup D.$$

Plainly, $\mathcal{N}$ is good. If $d(\mathcal{N}) < d(\mathcal{M})$, then inductively, $\mathcal{N}$, and hence $\mathcal{M}$, extends to a perfect submodel of $\mathcal{M}_2$, which completes the proof.

So suppose that $d(\mathcal{N}) \geq d(\mathcal{M})$. Outside the range C–D, all defects and their $\sharp$-values are the same in $\mathcal{M}$ and $\mathcal{N}$. So let us consider the remaining potential defects in $\mathcal{N}$. In $\mathcal{N}$, we have $C < u < D$. As u is $<_2$-irreflexive, $u \notin C \cup D$. So (C, D) is no longer a defect in $\mathcal{N}$. Nor is $(C, \lambda(u))$ a defect, because $\bigcup C$ and $\bigcup \lambda(u)$ contain similar links (possibly even $C = \lambda(u)$). So $(\rho(u), D)$ must be a defect in $\mathcal{N}$, and $\sharp(\rho(u), D) \geq \sharp(C, D)$. Since $C <_2 \rho(u)$, by definition of $\sharp$ we must have $\sharp(\rho(u), D) \leq \sharp(C, D)$, so $\sharp(\rho(u), D) = \sharp(C, D)$ and $d(\mathcal{N}) = d(\mathcal{M})$.

Now u is irreflexive, so $D \not<_2 u <_2 D$. Applying the mirror image of lemma 6.20, we obtain irreflexive $v \in \mathcal{M}_2$ with $u \leq_2 v <_2 D$, where $\bigcup \rho(v)$ and $\bigcup D$ contain similar links. Since $(\rho(u), D)$ is a defect in $\mathcal{N}$, we have $u \neq v$, so $u <_2 v$. By the lemma, $\sharp(\rho(u), \lambda(v)) < \sharp(\rho(u), D)$.

Let $\mathcal{N}'$ be obtained from $\mathcal{N}$ by adding $R_2(v)$, so that

$$\{v \in \mathcal{N}' : u \leq v < D\} = \{u\} \cup \rho(u) \cup \lambda(v) \cup \{v\} \cup \rho(v) \cup D,$$

where we write $<$ for $<_2 \restriction \mathcal{N}'$. If $(\rho(u), \lambda(v))$ is a defect in $\mathcal{N}'$ then $\sharp(\rho(u), \lambda(v)) < \sharp(\rho(u), D)$. Also, $(\rho(v), D)$ is not a defect in $\mathcal{N}'$. It now follows that $d(\mathcal{N}') < d(\mathcal{N}) = d(\mathcal{M})$. Inductively, there is perfect $\mathcal{P}$ with $\mathcal{N}' \subseteq \mathcal{P} \subseteq \mathcal{M}_2$. Since $\mathcal{M} \subseteq \mathcal{P}$, this completes the proof. $\square$

As $\mathcal{M}_3$ is plainly good, by the lemma we may choose a perfect $\mathcal{M}_4$ with $\mathcal{M}_3 \subseteq \mathcal{M}_4 \subseteq \mathcal{M}_2$. By lemma 6.13, $\mathcal{M}_4 \preceq_\Psi \mathcal{M}_2$ and φ_0 is satisfied in $\mathcal{M}_4$.

6.7 Model $\mathcal{M}_5$ over $\mathbb{R}$

Fix an enumeration without repetitions of the $<_4$-clusters in $\mathcal{M}_4$ as $C_0, \ldots, C_k$, with $k \geq 0$ and

$$C_0 <_4 C_1 <_4 \cdots <_4 C_k.$$

We have $C_0 = C_{-\infty}/{\sim}$ and $C_k = C_{+\infty}/{\sim}$. Possibly, $k = 0$ and $C_0 = C_k$.

DEFINITION 6.23 Let $i < k$. As $\mathcal{M}_4$ is perfect, (C_i, C_{i+1}) is not a defect, so there are only two possibilities.

1. There is $u \in \mathcal{M}_4$ with $C_i <_4 u <_4 C_{i+1}$ and $u \notin C_i \cup C_{i+1}$. Then u is not in any $<_4$-cluster, so is irreflexive. We must have $C_i = \lambda(u)$ and $C_{i+1} = \rho(u)$. It follows that u is unique. We define u_i to be this u, and we say that C_i is *right-open* and C_{i+1} is *left-open.*
2. $\bigcup C_i$ and $\bigcup C_{i+1}$ contain similar links. Let $\Delta_i \in \bigcup C_i$ and $\Sigma_{i+1} \in \bigcup C_{i+1}$ be similar links, and define

$$\begin{array}{rcl} d_i & = & \Delta_i/{\sim}, \\ s_{i+1} & = & \Sigma_{i+1}/{\sim}. \end{array} \tag{9}$$

   In this case, we say that C_i is *right-closed* and C_{i+1} is *left-closed.*

We also say that C_0 is left-open and C_k right-open.

DEFINITION 6.24 Let $i \leq k$.

1. Plainly, $(C_i, R_4 \restriction C_i)$ is a finite S4-frame, and by lemma 6.9 it is connected. Choose $f_i : \mathbb{R} \to C_i$ as per theorem 5.10. By the theorem, for each $w \in C_i$ we can pick $x_w \in \mathbb{R}$ with $f_i(x_w) = w$. As C_i is finite, there are bounds $l, r \in \mathbb{R}$ with $l < x_w < r$ for all $w \in C_i$. If C_i is left-closed, by the theorem we can assume further that l is f_i-good and $f_i(l) = s_i$. Similarly, if C_i is right-closed we can suppose that r is f_i-good and $f_i(r) = d_i$. To simplify notation, by some scaling we can assume that $l = 0$ and $r = 1$. In summary:
   (a) $f_i((0,1)) = C_i$,
   (b) if C_i is left-closed then $f_i(0) = s_i$ and 0 is f_i-good,
   (c) if C_i is right-closed then $f_i(1) = d_i$ and 1 is f_i-good.
2. If C_i is right-closed, then as Δ_i is a link, lemma 6.17 tells us that R_4 relates every two points of $H(\Delta_i)$, so $(H(\Delta_i), R_4 \restriction H(\Delta_i))$ is trivially a finite connected S4-frame. Let $f'_i : \mathbb{R} \to H(\Delta_i)$ be a map satisfying the conditions of theorem 5.10.
3. Now define a map g_i from an interval of $\mathbb{R}$ into $\mathcal{M}_4$ as follows. First suppose $i = 0$. Recall that C_0 is left-open.
   (a) If C_0 is right-open, define $g_0 = f_0$.
   (b) If C_0 is right-closed, define $g_0 = f_0 \restriction (-\infty, 1] + f'_0$.

   Note that $\operatorname{dom} g_0$ is an open interval of $\mathbb{R}$. We are not concerned about exactly which interval it is. Now suppose $1 \leq i \leq k$.
   (a) If C_i is left-open and right-open, define $g_i = u_{i-1} + f_i$.

(b) If C_i is left-open and right-closed, define $g_i = u_{i-1} + f_i \upharpoonright (-\infty, 1] + f'_i$.

(c) If C_i is left-closed and right-open, define $g_i = f_i \upharpoonright [0, \infty)$.

(d) If C_i is left-closed and right-closed, define $g_i = f_i \upharpoonright [0, 1] + f'_i$.

Note that $\operatorname{dom} g_i$ is an interval of $\mathbb{R}$ order-isomorphic to $[0, 1)$.

4. Finally let $g = \sum_{i \leq k} g_i$. By proposition 5.1(1), the domain of g can be identified with $\mathbb{R}$, and we will regard g as a map $g : \mathbb{R} \to \mathcal{M}_4$.

EXAMPLE 6.25 An example of the construction is shown in figure 1. In the figure, C_0 and C_4 are left- and right-open, C_1 is left-open and right-closed, C_2 is left- and right-closed, and C_3 is left-closed and right-open. The element d_2 is the small dot inside the square in C_2. The square itself is $H(\Delta_2)$. The square inside C_3 is $H(\Sigma_3)$ — we use another square because Σ_3 is similar to Δ_2 — and s_3 is the small dot inside it. Similarly, the big circles inside C_1 and C_2 are $H(\Delta_1)$ and $H(\Sigma_2)$, respectively. We can see that the circle $H(\Delta_1)$ is used in a sense as intervening material for g between C_1 and C_2 via f'_1, as intimated earlier. The points numbered 1–5 will be used in the lemma below. They exemplify all the different kinds of point as far as the definition of g is concerned. Points of types 1–3 are g-fair, but points of type 4 are not, and points of type 5 need not be.

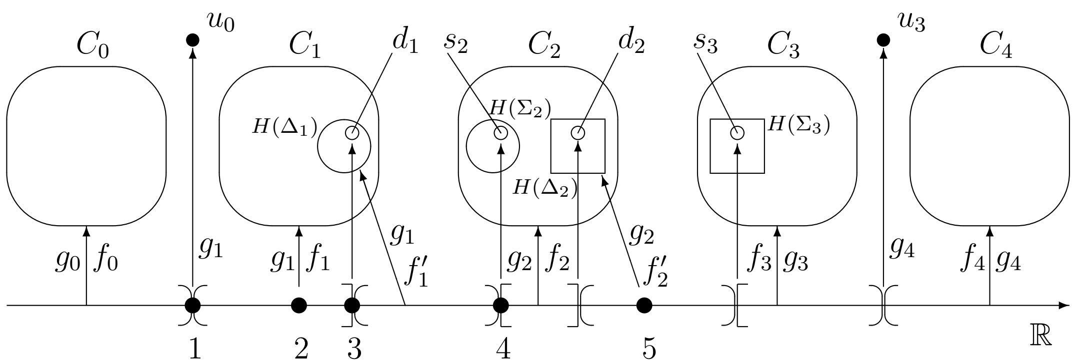

Figure 1: Example of parts of $g : \mathbb{R} \to \mathcal{M}_4$ when $k = 4$

Plainly, g is order preserving: if $x < y$ in $\mathbb{R}$ then $g(x) <_4 g(y)$. As in §5.2, for each $i \leq k$ we identify the map g_i with $g \upharpoonright \operatorname{dom}_g(g_i)$, so that g_i becomes a restriction of g, and $\operatorname{dom}(g_i) \subseteq \mathbb{R}$. We will further suppose in the same way that f_i, f'_i are restrictions of g_i, so their domains are subsets of $\mathbb{R}$ as well.

We define an assignment h_5 into $\mathbb{R}$ by $h_5(p) = g^{-1}(h_4(p))$, for each atom p. We let $\mathcal{M}_5 = (\mathbb{R}, h_5)$, our final model. We now prove the main 'truth lemma'.

LEMMA 6.26 *For every $\psi \in \Psi$ and $x \in \mathbb{R}$ we have $\mathcal{M}_5, x \models \psi$ iff $\mathcal{M}_4, g(x) \models \psi$.*

Proof. By induction on ψ. The lemma for atomic ψ is immediate from the definition of h_5, and the boolean cases are easy. The main cases are $F\psi$, $P\psi$, and $\Diamond\psi$. Since Ψ is closed under subformulas, $\psi \in \Psi$ as well. If $\mathcal{M}_5, x \models F\psi$ then there is $y \in \mathbb{R}$

with $y > x$ and $\mathcal{M}_5, y \models \psi$. Inductively, $\mathcal{M}_4, g(y) \models \psi$, and as g is order preserving we have $g(x) <_4 g(y)$, so $\mathcal{M}_4, g(x) \models F\psi$.

Conversely, suppose that $\mathcal{M}_4, g(x) \models F\psi$. By lemma 6.13, there are two possibilities. The first is that $\mathcal{M}_4, u \models \psi$ for some $u \in C_\infty/{\sim} = C_k$. By theorem 5.10, there are arbitrarily large $y \in \operatorname{dom} f_k$ with $f_k(y) = u$. As C_k is right-open, by definition of g_k there are arbitrarily large $y \in \operatorname{dom} g_k$ with $g_k(y) = u$. The final summand of g is g_k, so there are arbitrarily large $y \in \mathbb{R}$ with $g(y) = u$. Choose such a y with $y > x$. Inductively, $\mathcal{M}_5, y \models \psi$, and hence $\mathcal{M}_5, x \models F\psi$.

The second possibility according to the lemma is that there is some $<_4$-irreflexive $u \in \mathcal{M}_4$ with $u >_4 g(x)$, and $v \in \mathcal{M}_4$ with $g(x) <_4 v \leq_4 u \, R_4 \, v$ and $\mathcal{M}_4, v \models \psi$. The only irreflexive points in $\mathcal{M}_4$ are the u_i, so $u = u_i$ for some unique $i < k$. By definition of g, there is unique $y \in \mathbb{R}$ with $g(y) = u_i$. Then $x < y$ — for otherwise, $y \leq x$, so as g is order preserving, $u_i = g(y) \leq_4 g(x) <_4 u_i$, contradicting the irreflexivity of u_i. If $v = u$, we have $\mathcal{M}_5, y \models \psi$ by the inductive hypothesis, yielding $\mathcal{M}_5, x \models F\psi$ as required. If $v <_4 u$, then since R_4uv we have $v \in \lambda(u) = C_i$. Now by theorem 5.10, there are arbitrarily large $z \in \operatorname{dom} f_i$ with $f_i(z) = v$. As C_i is plainly right-open, there are arbitrarily large $z \in \operatorname{dom} g_i$ with $g_i(z) = v$. By considering the part $g_i + g_{i+1}$ of the sum defining g, we see that there are points $z < y$ arbitrarily close to y and with $g(z) = v$. So we may choose such a z with $z > x$. As before, $\mathcal{M}_5, z \models \psi$ by inductive hypothesis, so $\mathcal{M}_5, x \models F\psi$.

The case of $P\psi$ is similar. Finally we consider the case of $\Diamond\psi$. Let $x \in \mathbb{R}$.

Claim. If x is g-fair then $\mathcal{M}_5, x \models \Diamond\psi$ iff $\mathcal{M}_4, g(x) \models \Diamond\psi$.
Proof of claim. If x is g-fair, then $g((y, z)) = R_4(g(x))$ for all large enough $y < x$ and small enough $z > x$. If $\mathcal{M}_5, x \models \Diamond\psi$ then choose such y, z, and choose $t \in (y, z)$ with $\mathcal{M}_5, t \models \psi$. Then $g(t) \in R_4(g(x))$, and inductively, $\mathcal{M}_4, g(t) \models \psi$. So $\mathcal{M}_4, g(x) \models \Diamond\psi$.

Conversely, if $\mathcal{M}_4, g(x) \models \Diamond\psi$ then choose $w \in R_4(g(x))$ with $\mathcal{M}_4, w \models \psi$. Let $y < x < z$ in $\mathbb{R}$ be given. By g-fairness of x we have $R_4(g(x)) \subseteq g((y, z))$, so there is $t \in (y, z)$ with $g(t) = w$. Inductively, $\mathcal{M}_5, t \models \psi$. So $\mathcal{M}_5, x \models \Diamond\psi$ by semantics of $\Diamond$ in $\mathcal{M}_5$. This proves the claim.

We now divide into five subcases, according to how $g(x)$ is determined. The cases are illustrated by the points labelled 1–5 in figure 1. We leave it to the reader to confirm that all cases have been covered.

1. First suppose $g(x) = u_i$ for some $i < k$. So x is the least element of $\operatorname{dom} g_{i+1}$. We show that x is g-fair. As u_i is defined, C_i is right-open and C_{i+1} is left-open. So just before x, g is like the final part of f_i, and just after x it is like the initial part of f_{i+1}. It follows from the definition of f_i, f_{i+1} that $f_i^{-1}(w)$ is unbounded in $\operatorname{dom} f_i$ for each $w \in C_i$, and $f_{i+1}^{-1}(w)$ is unbounded in $\operatorname{dom} f_{i+1}$ for each $w \in C_{i+1}$. So for any $y \in \operatorname{dom} g_i$ and $z \in \operatorname{dom} g_{i+1} \setminus \{x\}$ we have $g((y, z)) = C_i \cup \{u_i\} \cup C_{i+1} = R_4(u_i)$. So indeed, x is g-fair. By the claim, $\mathcal{M}_5, x \models \Diamond\psi$ iff $\mathcal{M}_4, g(x) \models \Diamond\psi$.

2. Next suppose that x is in the interior of the part of $\operatorname{dom} f_i$ within $\operatorname{dom} g$, for some $i \leq k$. By theorem 5.10, x is f_i-fair, and by remark 5.8 it follows that x is g-fair. The claim now gives $\mathcal{M}_5, x \models \Diamond\psi$ iff $\mathcal{M}_4, g(x) \models \Diamond\psi$.

3. Suppose $x \in \operatorname{dom} f_i$ is an upper boundary point, so that C_i is right-closed and $g(x) = d_i$. As 1 is f_i-good, $g((y, x)) = f_i((y, x)) = R_4(d_i)$ for all large enough $y < x$. By definition of g, if $z > x$ is small enough then $z \in \operatorname{dom} f'_i$ and so $g((x, z)) = f'_i((x, z)) \subseteq H(\Delta_i) \subseteq R_4(d_i)$. Consequently, for all large enough y and small enough z with $y < x < z$, we have $g(y, z) = R_4(d_i) \cup \{d_i\} \cup H(\Delta_i) = R_4(d_i)$. Once again, x is g-fair, and the claim yields $\mathcal{M}_5, x \models \Diamond\psi$ iff $\mathcal{M}_4, g(x) \models \Diamond\psi$.
4. Suppose $x \in \operatorname{dom} f_i$ is a lower boundary point, so that C_i is left-closed and $g(x) = s_i$. This case is more intricate. Clearly, $i > 0$, C_{i-1} is right-closed, and Δ_{i-1} and Σ_i are similar links.

   Suppose first that $\mathcal{M}_4, g(x) \models \Diamond\psi$. So there is $w \in R_4(s_i)$ with $\mathcal{M}_4, w \models \psi$. As 0 is f_i-good, for all small enough $y > x$ we have $g((x, y)) = f_i((x, y)) = R_4(s_i)$. Hence, for all $y > x$ there is $z \in (x, y)$ with $g(z) = w$. For any such z we have $\mathcal{M}_5, z \models \psi$ by the inductive hypothesis. It follows that $\mathcal{M}_5, x \models \Diamond\psi$.

   Conversely suppose that $\mathcal{M}_5, x \models \Diamond\psi$. As above, we may choose $y > x$ with $g((x, y)) = R_4(s_i)$. If there is $z \in (x, y)$ with $\mathcal{M}_5, z \models \psi$, then inductively $\mathcal{M}_4, g(z) \models \psi$, and as $R_4(g(x), g(z))$ we have $\mathcal{M}_4, g(x) \models \Diamond\psi$ as required.

   Assume otherwise. By definition of g we may choose $y < x$ in $\operatorname{dom} f'_{i-1}$, and then $z \in (y, x)$ with $\mathcal{M}_5, z \models \psi$. Inductively, $\mathcal{M}_4, g(z) \models \psi$, so $\mathcal{M}_4, g(z) \models \Diamond\psi$. Recall from lemma 6.13 that $\mathcal{M}_4 \preceq_\Psi \mathcal{M}_2$. So $\mathcal{M}_2, g(z) \models \Diamond\psi$ as well. Now $g(z) \in H(\Delta_{i-1})$. As Δ_{i-1} and Σ_i are similar links, by lemma 6.18 we have $\mathcal{M}_2, s_i \models \Diamond\psi$. As $\mathcal{M}_4 \preceq_\Psi \mathcal{M}_2$ we obtain $\mathcal{M}_4, g(x) \models \Diamond\psi$ as required.
5. Finally suppose that $x \in O$ for some open $O \subseteq \operatorname{dom} f'_i$. This case has some intricacies too. We know that $f'_i : \operatorname{dom} f'_i \to H(\Delta_i)$ and x is f'_i-fair, so for all large enough $y < x$ and small enough $z > x$ we have

$$g((y, z)) = f'_i((y, z)) = (R_4 \restriction H(\Delta_i))(g(x)) = H(\Delta_i). \tag{10}$$

   If $\mathcal{M}_5, x \models \Diamond\psi$, then there are points $t \in \mathbb{R}$ arbitrarily close to x with $\mathcal{M}_5, t \models \psi$. By (10), there are such t with $g(t) \in H(\Delta_i) \subseteq R_4(g(x))$. Inductively, $\mathcal{M}_4, g(t) \models \psi$, and so $\mathcal{M}_4, g(x) \models \Diamond\psi$ by semantics.

   Conversely suppose $\mathcal{M}_4, g(x) \models \Diamond\psi$. Recalling again (lemma 6.13) that $\mathcal{M}_4 \preceq_\Psi \mathcal{M}_2$, we obtain $\mathcal{M}_2, g(x) \models \Diamond\psi$. By the critical corollary 6.11, we can find some $w \in H(\Delta_i)$ with $\mathcal{M}_2, w \models \psi$, and hence $\mathcal{M}_4, w \models \psi$. By (10) again, there are $z \in \mathbb{R}$ arbitrarily close to x and with $g(z) = w$, and so inductively, $\mathcal{M}_5, z \models \psi$. We conclude that $\mathcal{M}_5, x \models \Diamond\psi$ as required.

This completes the proof. □

As φ_0 is satisfied in $\mathcal{M}_4$, $\varphi_0 \in \Psi$, and plainly $g : \mathbb{R} \to \mathcal{M}_4$ is surjective, we conclude that φ_0 is satisfiable over $\mathbb{R}$, in the model $\mathcal{M}_5$. This finishes the proof of theorem 6.1.

7 Conclusion

We have shown that the logic of $\mathbb{R}$ in the temporal language $\mathcal{L}$ with modalities G, H, and $\Box$

- is finitely axiomatisable, answering an implicit problem of Shehtman [24],
- has PSPACE-complete complexity,
- has no strongly complete axiomatisation,
- is not Kripke complete.

We list some remaining open problems. First, some complexity problems.

PROBLEM 7.1 *For fixed $k \geq 0$, what is the complexity of the set of $\mathcal{L}$-formulas that are satisfiable over $\mathbb{R}$ and involve at most k $\Box$-operators?*

The methods of [19] may be helpful. If the answer is 'NP-complete', it might suggest that the language with F, P, and $\Box$ could be more tractable in practice than the more expressive language with Until and Since.

The operations of sum (+) and shuffle in §5, plus two more involving countable iterations, can be used to specify models over $\mathbb{R}$ in a finite way. By results in [4], any model over $\mathbb{R}$ can be specified up to any desired degree of first-order equivalence in such a way, so any satisfiable $\mathcal{L}$-formula has a model specified by these operations. This leads to the following problem.

PROBLEM 7.2 *Investigate the complexity of model checking for the language $\mathcal{L}$ for models over $\mathbb{R}$ specified by a finite sequence of operations of the above kinds.*

This problem was investigated in [6] for the language with Until and Since, and roughly speaking, exponential time upper bounds were obtained. One may also wish to develop alternative reasoning systems for $\mathcal{L}$ over $\mathbb{R}$, such as tableaux, and synthesis methods along the lines of [6, 22]. The end result of this research could justify the promotion of $\mathcal{L}$ as a viable language for specification and reasoning over the real line, possibly a more attractive one than the very expressive language with Until and Since.

It may be of interest to study the logic of $\mathbb{R}$ in the sublanguage of $\mathcal{L}$ without H: so the only non-boolean connectives are G and $\Box$. This logic is PSPACE-complete, by the same argument as in theorem 4.4. Theorem 4.1 survives: there is no strongly complete axiomatisation. The proof of theorem 4.3 can be adapted to show that it is not Kripke complete, using the formula

$$F(p \wedge G\neg p \wedge \neg a \wedge \neg b \wedge \Diamond a \wedge \Diamond b \wedge G\neg(\Diamond a \wedge \Diamond b) \wedge FG\neg a) \wedge G(Fp \to \neg a \wedge \neg b).$$

The Prior axiom is no longer expressible, but a variant $Fp \wedge FG\neg p \to F(G\neg p \wedge \Diamond p)$ can be used instead.

PROBLEM 7.3 (N. Bezhanishvili) Is the logic finitely axiomatisable?

An alternative and more expressive interpretation of $\Box$ is as 'derivative' $[d]$, so that $(\mathbb{R}, h), x \models [d]\varphi$ if there is an open neighbourhood O of x with $(\mathbb{R}, h), y \models \varphi$ for every $y \in O \setminus \{x\}$. Finite axiomatisations of the logic of $\mathbb{R}$ with $[d]$ alone (without G, H) and with $[d]$ and $\forall$ are given in [15] (see also Shehtman's habilitation thesis).

PROBLEM 7.4 Is the logic of $\mathbb{R}$ in the language with G, H, and $[d]$ finitely axiomatisable?

An even more expressive language consists of G, H and two modalities $\Box^{>}, \Box^{<}$, where $(\mathbb{R}, h), x \models \Box^{>}\varphi$ if there is $y > x$ such that $(\mathbb{R}, h), z \models \varphi$ for every $z \in (x, y)$, and $\Box^{<}$ is the mirror image.

PROBLEM 7.5 Find axiomatisations of the logic of $\mathbb{R}$ in this language and in sublanguages such as $\{G, \Box^{>}\}$.

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Department of Computing
Imperial College London
London SW7 2AZ, UK.
Email `i.hodkinson@imperial.ac.uk`
`http://www.doc.ic.ac.uk/~imh/`