Finite model property for guarded fragments

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Thanks to the organisers for the setting
and for inviting me.

Outline

I aim to prove that some guarded fragments of first-order logic have the finite model property.

This is not a survey of guarded fragments. I'll concentrate on proofs.

1. Introduction
2. Crash course in guarded fragments
3. Partial isomorphisms and automorphisms of structures; EPPA
4. Finite model property via EPPA
5. EPPA for relational structures
6. Clique-faithful EPPA for relational structures

1. Introduction, history

Guarded fragments are *modal fragments* of first-order logic.
Modal fragments should have the nice properties of modal logic.

GF introduced in 1998 by Andréka, van Benthem, Németi, following Németi et al in algebraic logic. They defined GF, proved decidability without =, stated decidability with = (and FMP — 'follow-up paper').

In two 1999 papers, Grädel:

- gave a proof of decidability with =, established complexity of guarded and loosely guarded fragments, and showed that the guarded fragment has the finite model property.
- Added fixed points, defined clique-guarded fragment, established decidability and complexity.

IH (2002): loosely guarded, packed, and, clique-guarded fragments have finite model property. IH-Otto (2003): simpler proof.

Andréka, van Benthem, Németi

Flier for ESSLLI guarded fragment workshop, Nancy, Aug 2004

It's been almost ten years since Andréka, van Benthem and Németi proved decidability of the guarded fragment of first order logic. Given how natural and expressive guarded quantification is, this result gave logicians a powerful tool of proving decidability of many formalisms arising in computer science applications, and generated much research into extensions of the guarded fragment to fixed point logic, transitive guards etc. A wealth of new proof techniques developed as a result.

Workshop programme committee

Natasha Alechina, Johan van Benthem, Erich Grädel, Maarten Marx, Hans de Nivelle, Martin Otto, Ulrike Sattler.

2. Crash course in guarded fragments

Idea: *relativise quantifiers to atomic formulas.*

Fix a first-order relational signature L.

An L-formula γ is said to *guard* an L-formula φ if every free variable of φ is free in γ.

Define the *guarded L-formulas:*

- Every atomic L-formula is guarded.
- Boolean combinations of guarded formulas are guarded.
- If φ is guarded, γ is *atomic* and guards φ, and $\bar{x}$ is any tuple of variables, then $\exists\bar{x}(\gamma \wedge \varphi)$ is guarded.

The *guarded fragment* of first-order logic (over L) consists of all guarded L-formulas.

Notes

1. We regard $\forall\bar{x}(\gamma \to \varphi)$ as guarded, since it is equivalent to $\neg\exists\bar{x}(\gamma \wedge \neg\varphi)$.

   Some people include both $\exists\bar{x}(\gamma \wedge \varphi)$ and $\forall\bar{x}(\gamma \to \varphi)$ in the definition of the guarded fragment.

2. ABN did not allow γ to be an equality. (We do!)

3. Can add constants to L, but it is a bit messier — and anyway, can eliminate them by adding relation symbols. Can't add function symbols and keep decidability.

Examples

1. $\forall x P(x)$ is equivalent to $\forall x(x = x \to P(x))$, which is guarded.
2. The 'standard translation' of any modal formula is guarded.
3. Statements such as 'R is reflexive/symmetric' can be written guardedly: $\forall x(x = x \to R(x, x))$ and $\forall xy(R(x, y) \to R(y, x))$.
4. 'R is transitive' is a classic unguarded statement:

$$\forall xy(R(x,y) \to \forall z(\underbrace{R(y,z)}_{\text{bad guard}} \to R(x,z)))$$

5. Temporal logic: e.g., the statement 'at time t, p holds until q':

$$\exists u\big(t < u \wedge Q(u) \wedge \forall v\big(\underbrace{t < v \wedge v < u}_{\text{not atomic}} \to P(v)\big)\big).$$

Loosely guarded ('pairwise guarded') fragment

Proposed by van Benthem (1997) in order to express Until.

- Every atomic L-formula is loosely guarded.
- Boolean combinations of loosely guarded formulas are l.g.
- $\exists\bar{x}(\gamma \wedge \varphi)$ is loosely guarded if
  - φ is loosely guarded,
  - γ is a conjunction of atomic L-formulas,
  - γ guards φ,
  - $\bar{x}$ is a tuple of variables in γ,
  - for each x in $\bar{x}$ and each free variable y of γ with $x \neq y$, there is a conjunct of γ in which x, y both occur.

$$\exists u\big(t < u \wedge Q(u) \wedge \forall v\big(\underbrace{t < v \wedge v < u}_{\text{loose guard}} \rightarrow P(v)\big)\big).$$

Packed fragment

Proposed by Marx (2000) as a variant of loosely guarded fragment characterisable by bisimulations. More natural for higher arities.

The most interesting and expressive first-order guarded fragment.

- Every atomic L-formula is packed.
- Boolean combinations of packed formulas are packed.
- Suppose φ is packed. Assume that
  - γ is a conjunction of atomic and existentially-quantified atomic formulas (possibly equalities),
  - γ guards φ,
  - if x, y are distinct free variables of γ, there is a conjunct of γ in which x, y both occur free.

  Then for any tuple $\bar{x}$ of variables, $\exists\bar{x}(\gamma \wedge \varphi)$ is packed.

Examples

- Any loosely guarded sentence can be equivalently rewritten as a packed sentence. E.g.,

$$\exists t, u\big(t < u \wedge Q(u) \wedge \forall v\big(t < v \wedge v < u \underbrace{\wedge\, t < u}_{\text{new}} \rightarrow P(v)\big)\big).$$

- $\forall xyz(\exists w R(x, y, w) \wedge \exists w R(x, z, w) \wedge \exists w R(z, y, w) \rightarrow R(x, y, z))$ (Marx) — not equivalent to a loosely guarded sentence. So for sentences, $LGF \subset PF$.

Clique-guarded fragment (Grädel, 1999)

Variant of packed fragment. All existential quantifiers in guards are moved to the front of the formula. E.g.,

$$\forall xyztuv(R(x, y, t) \wedge R(x, z, u) \wedge R(z, y, v) \rightarrow R(x, y, z)).$$

Semantic version of packed/clique-guarded fragment

Grädel (and Marx) also gave a semantic definition of clique-guarded fragment:

relativise quantifiers to cliques in the Gaifman graph.

Hence the name ‘clique-guarded fragment’.

Gaifman graph of a relational structure

Let L be a relational signature and let M be an L-structure.

Definition 1 A tuple $\bar{a} = (a_1, \ldots, a_n)$ of elements of M is said to be *guarded* if $M \models \alpha(a_1, \ldots, a_n)$ for some atomic L-formula $\alpha(x_1, \ldots, x_n)$ (possibly an equality).

Definition 2 Define the *Gaifman graph* $\mathcal{G}(M) = (\mathrm{dom}(M), E)$ of M by putting an edge between distinct $a, b \in M$ just when there is some guarded tuple $(a_1, \ldots, a_n)$ with $a, b \in \{a_1, \ldots, a_n\}$.

Packed fragment and Gaifman cliques

A *clique* in a graph $G = (V, E)$ is a set $C \subseteq V$ such that the subgraph of G on C is complete: $\forall x, y \in C(x \neq y \rightarrow E(x, y))$.

Packed fragment guards express cliques in Gaifman graph

If $\gamma(x_1, \ldots, x_n)$ is a guard in the sense of the packed fragment, then $M \models \gamma(a_1, \ldots, a_n)$ implies $\{a_1, \ldots, a_n\}$ is a clique in $\mathcal{G}(M)$.

Conversely, if L is finite, *'$\{x_1, \ldots, x_n\}$ is a clique in Gaifman graph'* is expressible by a disjunction $\bigvee_c \gamma_c$ of packed fragment guards γ_c. Each γ_c is a conjunction over all $i \neq j$ of formulas of form $x_i = x_j$ or

$$\exists z_1, \ldots, z_n R(z_1, \ldots, z_{k-1}, x_i, z_{k+1}, \ldots, z_{l-1}, x_j, z_{l+1}, \ldots, z_n)$$

for n-ary $R \in L$. We include a γ_c for each choice c of conjuncts.

Then $\exists \bar{x}\big((\bigvee_c \gamma_c) \wedge \varphi\big) \equiv \bigvee_c \exists \bar{x}(\gamma_c \wedge \varphi)$ — in packed fragment.

Nice properties of guarded fragments (cf. modal logic)

- Decidable. Complexity: 2EXPTIME-complete. Bounded-arity and finite-variable fragments EXPTIME-complete; some 2-variable fragments even in PSPACE [Grädel, 1999].
- Interpolation. . .
- Unravelling. Tree-like models.
- Bisimulation characterisation: any first-order formula is preserved under guarded (clique) bisimulations iff it's equivalent to a guarded (packed) formula.
- Grädel (1999): can add fixed points: μGF. Behaves like modal μ-calculus. Still decidable.

  Any 'guarded SO' sentence is equivalent to a μGF one iff it's preserved under guarded bisimulation [Grädel, Hirsch, Otto].

3. Partial isomorphisms and automorphisms of structures

For an L-structure A, a partial map $p : A \to A$ is said to be a *partial isomorphism* if it preserves all atomic L-formulas both ways:

$$A \models \alpha(\bar{a}) \leftrightarrow \alpha(p(\bar{a}))$$

for all atomic L-formulas $\alpha(\bar{x})$, and all $\bar{a} \in dom(p)$ (with $|\bar{a}| = |\bar{x}|$).

Here, if $\bar{a} = (a_1, \ldots, a_n)$ then $p(\bar{a}) = (p(a_1), \ldots, p(a_n))$.

Note: such a p is 1–1 (take α to be $x = y$).

An *automorphism* is a bijective partial isomorphism.

The set of automorphisms of A forms a *group* under composition of maps, denoted by $\mathrm{Aut}(A)$.

EPPA

Definition 3 Let $\mathcal{K}$ be a class of structures (in a common signature). $\mathcal{K}$ is said to have the *EPPA (extension property for partial automorphisms)* if for all $A \in \mathcal{K}$ there is $B \in \mathcal{K}$ with $B \supseteq A$ and such that every partial isomorphism of A extends to (is induced by) an automorphism of B.

Example: The class of graphs has EPPA (by classical model theory).

What about classes of *finite* structures?

Example: The class of all finite sets (structures in the empty signature) has EPPA.

Others. . . ?

Faithful EPPAs

EPPA says that given $A \in \mathcal{K}$ we can find $B \supseteq A$ in $\mathcal{K}$ such that all partial isomorphisms of A extend to automorphisms of B.

We would like some 'homogeneity' of B as well:

Point-faithful EPPA Any $b \in B$ can be mapped into A by an automorphism of B.

We can usually achieve this by manipulating B — see later.

Guarded-faithful EPPA Any guarded tuple of elements of B can be mapped into A by a (single) automorphism of B. Ditto.

K_n-faithful EPPA (graphs) Any clique of size n in B can be mapped into A by an automorphism of B.

Clique-faithful EPPA Any clique in the Gaifman graph of B can be mapped into A by an automorphism of B.

EPPA for classes of finite structures

1. Truss, 1992: Given a finite graph A and a partial isomorphism p of it, there is a finite graph $B \supseteq A$ and an automorphism of B extending p. Proof by explicit construction.

2. Hrushovski, 1992: *the class of finite (undirected loop-free) graphs has (guarded-faithful) EPPA.*
   Proof by permutation groups and combinatorics.

3. Herwig, 1995:

   (a) for any finite relational signature L, *the class of all finite L-structures has guarded-faithful EPPA.*

   (b) *the class of all finite graphs has K_3-faithful EPPA.*
   Hence, *the class of all finite triangle-free graphs has EPPA.*

   Proofs extended Hrushovski's.

EPPA for classes of finite structures (ctd.)

4. Herwig, 1998: *the class of all finite K_n-free graphs (any $n \geq 3$) has EPPA.* Also, for any class $\mathcal{T}$ of tournaments, the class of finite directed graphs omitting all $T \in \mathcal{T}$ has EPPA.

   Proof extended Herwig's 1995 proof.

5. Herwig–Lascar 2000:
   - new combinatorial proofs of (2) and (3a)
   - link to theory of free groups (Marshall Hall, Almeida, Delgado). E.g., HL proved EPPA for finite graphs from:

     Theorem 4 (Ribes, Zaleskiĭ, 1993) *If G is the free group on a finite set, and $H_1, \ldots, H_n \leq G$ are finitely generated, then the set $H_1 H_2 \ldots H_n$ is closed in the profinite topology on G.*

     HL also proved converse results.
   - new powerful extension of EPPA results.

EPPA for classes of finite structures (ctd.)

6. IH–Otto 2003, motivated by clique-guarded fragment/packed fragment:

   For any finite relational signature L, *the class of all finite L-structures has clique-faithful EPPA.*

   Combinatorial proof, using Herwig's (3a) as a lemma.

John Truss, Ehud Hrushovski, Bernhard Herwig

Daniel Lascar, Martin Otto

Remarks on motivation

1. EPPA originally proved in order to establish *small index property* for various ω-categorical structures in model theory (e.g., 'random graph'). [Hodges, IH, Lascar, Shelah, 1993].

2. Part of proof used by Grohe, e.g., to establish *arity hierarchies* in first-order logic with fixed points in finite model theory (1996): expressive power increases strictly as the arity of permitted fixed points increases. Answered question of Vardi (1982).

3. EPPA 'equivalent to' certain results in free groups.

4. EPPA $\vdash$ *'finite base property'* for $\mathsf{Crs}_n, \mathsf{D}_n, \mathsf{G}_n$ (finite n): Hirsch, IH, Marx, Mikulás, Reynolds, 1998; Andréka, IH, Németi, 1999.

   Grädel (1999): *guarded fragment has finite model property.*

   IH (2002), IH–Otto (2003): *packed fragment has fmp.*

4. Finite model property for guarded fragments via EPPA

Let σ be a guarded (or packed) sentence of relational signature L (wlog. finite). Assume that σ has a model, say M. We will construct a finite model of σ from M.

We take the guarded (and packed) cases together. (The packed case will be covered in parentheses.)

Plan: first do special case where *(all guards in σ are quantifier-free.)*

(i) Expand M by new relation symbols for subformulas of σ.

(ii) Take 'rich' finite substructure K of this expansion.

(iii) Take finite guarded- (clique-)faithful EPPA extension $H \supseteq K$.

(iv) Show $H \models \sigma$.

(v) Tackle general case where guards can have quantifiers (easy).

(i) The expansion M^+ of M

We want to regard subformulas of σ as atomic.

Let $\mathcal{S}(\sigma)$ denote the set of all subformulas of σ.

Let M^+ be an expansion of M by a new relation symbol for each $\varphi(\bar{x}) \in \mathcal{S}(\sigma)$, interpreted in the same way as φ.

(ii) The 'rich' substructure K

Let n be the maximum number of free variables of guards in σ.

Choose a finite substructure $K \subseteq M^+$ containing a representative of each isomorphism type of substructure of M^+ of size $\leq n$.

(iii) The EPPA extension H of K

Choose a finite structure $H \supseteq K$ (in the expanded language) such that

1. every partial isomorphism of K extends to an automorphism of H,
2. every guarded tuple (clique in $\mathcal{G}(H)$) is mapped by an automorphism of H into K.

This is possible by Herwig's 1995 (IH–Otto's 2003) EPPA result.

Note: if $\bar{a}$ is a tuple of elements of H, and $H \models \gamma(\bar{a})$ for some guard γ, then there is $g \in \mathrm{Aut}\, H$ such that all elements of $g(\bar{a})$ are in K.

Here and below, if $\bar{a} = (a_1, \ldots, a_n)$,
we write $g(\bar{a})$ for $(g(a_1), \ldots, g(a_n))$.

Automorphisms preserve formulas

Lemma 5 *Let $g \in \mathrm{Aut}\, H$, let $\varphi(\bar{x})$ be any first-order formula with $|\bar{x}| = n$, say, and let $\bar{a}$ be an n-tuple of elements of H. Then*

1. $H \models \varphi(\bar{a}) \iff H \models \varphi(g(\bar{a}))$,
2. *If $\varphi \in \mathcal{S}(\sigma)$ and all elements of $\bar{a}, g(\bar{a})$ are in K, then* $M \models \varphi(\bar{a}) \iff M \models \varphi(g(\bar{a}))$.

Proof.

1. Automorphisms preserve all first-order formulas.
2. Holds because $\varphi \in \mathcal{S}(\sigma)$ is represented by a new relation symbol, which means the same in M^+ and H, and is preserved by $g \in \mathrm{Aut}\, H$.

■

(iv) Preservation lemma

Lemma 6 *For each subformula* $\varphi(\bar{x})$ *of* σ*, and each tuple* $\bar{a}$ *of elements of* K*, we have* $H \models \varphi(\bar{a}) \iff M \models \varphi(\bar{a})$.

Proof. The lemma is shown by induction on φ.

For atomic φ it is clear.

The boolean cases are easy.

Consider the case $\varphi(\bar{x}) = \exists \bar{y}(\gamma(\bar{x}, \bar{y}) \wedge \psi(\bar{z}))$. Inductively assume the result for ψ.

Wlog. the free variables of γ are $\bar{x}\bar{y}$, and all variables in $\bar{z}$ occur in $\bar{x}\bar{y}$. For a tuple $\bar{c}$ of length $\bar{x}\bar{y}$, let $\bar{c}\dagger$ be to $\bar{c}$ as $\bar{z}$ is to $\bar{x}\bar{y}$.

Case $\varphi(\bar{x}) = \exists\bar{y}(\gamma(\bar{x},\bar{y}) \wedge \psi(\bar{z}))$**:** $\quad M \models \varphi(\bar{a}) \Rightarrow H \models \varphi(\bar{a})$

If $M \models \varphi(\bar{a})$, there is $\bar{b}$ in M with $M \models \gamma(\bar{a},\bar{b}) \wedge \psi(\bar{a}\bar{b}\dagger)$.

By choice of K, there are $\bar{a}', \bar{b}'$ in K with $\bar{a}'\bar{b}' \cong \bar{a}\bar{b}$.

So $M \models \gamma(\bar{a}',\bar{b}') \wedge \psi(\bar{a}'\bar{b}'\dagger)$.

So $H \models \gamma(\bar{a}',\bar{b}') \wedge \psi(\bar{a}'\bar{b}'\dagger)$ by inductive hypothesis (and γ q.f.).
Hence, $H \models \varphi(\bar{a}')$.

Now the map $\bar{a}' \mapsto \bar{a}$ is a partial isomorphism of K.
By EPPA, there is $g \in \mathrm{Aut}(H)$ with $g(\bar{a}') = \bar{a}$.

So by lemma 5, $H \models \varphi(\bar{a})$ as required.

Case $\varphi(\bar{x}) = \exists \bar{y}(\gamma(\bar{x}, \bar{y}) \wedge \psi(\bar{z}))$: $\quad H \models \varphi(\bar{a}) \Rightarrow M \models \varphi(\bar{a})$

Suppose that $H \models \varphi(\bar{a})$. So $H \models \gamma(\bar{a}, \bar{b}) \wedge \psi(\bar{a}\bar{b}\dagger)$ for some $\bar{b}$ in H.

$H \models \gamma(\bar{a}, \bar{b})$, so by faithfulness, *there is $g \in \mathrm{Aut}(H)$ with $g(\bar{a}\bar{b})$ in K.*

So:

$H \models \gamma(g(\bar{a}), g(\bar{b})) \wedge \psi(g(\bar{a}\bar{b})\dagger)$	as g preserves γ, ψ,
$M \models \gamma(g(\bar{a}), g(\bar{b})) \wedge \psi(g(\bar{a})g(\bar{b})\dagger)$	by inductive hypothesis (& γ q.f.)
$M \models \varphi(g(\bar{a}))$	by definition of φ,
$M \models \varphi(\bar{a})$	by lemma 5.

■

We know that $M \models \sigma$. By the lemma, $H \models \sigma$. So we have a finite model of σ.

(v) Guards with quantifiers (packed fragment)

Consider a conjunct χ of a guard γ in σ of the form $\chi = \exists \bar{y} \alpha(\bar{x}, \bar{y})$, where α is atomic.

- Introduce a new $|\bar{x}|$-ary relation symbol R_χ.
- Replace χ in γ by $R_\chi(\bar{x})$.
- Add to σ two GF conjuncts:
  - $\forall \bar{x} \bar{y} (\alpha(\bar{x}, \bar{y}) \rightarrow R_\chi(\bar{x}))$,
  - $\forall \bar{x} (R_\chi(\bar{x}) \rightarrow \exists \bar{y} \alpha(\bar{x}, \bar{y}))$.

Do this for all χ in all γ in σ. Call the result σ^*.

σ^* is in the packed fragment and all its guards are quantifier-free.

Assume σ has a model M.

M expands to a model $M^* \models \sigma^*$ (let $M^* \models R_\chi(\bar{a})$ iff $M \models \chi(\bar{a})$).

By the foregoing, there is a finite model H of σ^*.

Any model of σ^* is a model of σ. So we're done.

So the guarded and packed fragments have the finite model property (modulo EPPA). ■

5. EPPA for relational structures

I would have liked to prove

Theorem 7 (Herwig, 1995) *For any finite relational signature L, the class of all finite L-structures has the guarded-faithful EPPA.*

However, I only have time to prove

Theorem 8 (Hrushovski, 1992) *The class of all finite (undirected loop-free) graphs has the guarded-faithful EPPA. That is, any finite graph A is an induced subgraph of some finite graph B such that*

1. *any partial isomorphism of A extends to an automorphism of B,*
2. *for each $b \in B$, there is $g \in \mathrm{Aut}\, B$ with $g(b) \in A$,*
3. *for each edge xy of B, there is $g \in \mathrm{Aut}\, B$ with $g(x), g(y) \in A$.*

Hrushovski's proof used permutation groups and counting arguments. Herwig's proof (of theorem 7) extended this.

However...

. . . we use the simpler combinatorial method of Herwig–Lascar (2000).

Notation

For a set X, and an integer $k > 0$, write $[X]^k$ for the set of all subsets of X of size k.

Graphs will be written (V, E), where $V \neq \emptyset$ and $E \subseteq [V]^2$.

I will write an edge $\{x, y\} \in E$ simply as xy. Note: $xy = yx$.

The idea

Take a finite graph $A = (V, E)$.

(i) We can assume that all nodes of A have the same degree.

(ii) We embed A in a 'combinatorial' graph B.

(iii) We show that every partial isomorphism of A extends to an automorphism of B.

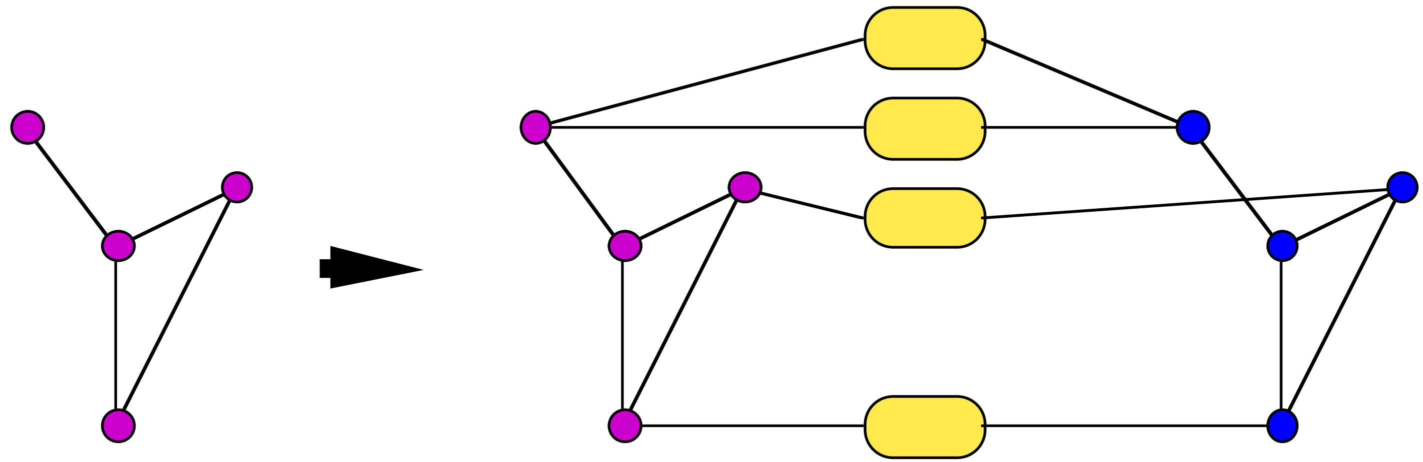
(iv) We arrange guarded-faithfulness by a trick at the end.

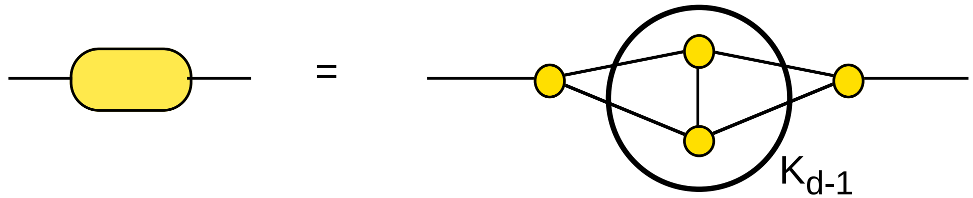
(i): Wlog. all nodes of A have the same degree

It is harmless to replace A by any finite graph A' of constant degree that contains (the original) A as an induced subgraph.

Any EPPA extension of A' is an EPPA extension of A.

E.g., to bring degree of the left-hand graph up to $d = 3$:



where 

(ii) What is B?

Assume then that every node of $A = (V, E)$ has degree d, for some $d \geq 2$.

We will let the extension graph B be $([E]^d, R)$, where

$$R = \big\{\{X, Y\} : X, Y \in [E]^d,\ |X \cap Y| = 1\big\}.$$

That is, for $X, Y \in [E]^d$, we let XY be an edge of B iff $X \cap Y$ has size 1.

Clearly, B is a finite (undirected loop-free) graph.
(R is irreflexive because for any $X \in [E]^d$, we have $|X \cap X| = d > 1$.)

Embedding A in B

Of course, A is not a subgraph of B.

But we can embed A into B, via: for each $x \in V$,

$\nu : x \mapsto \{xy : y \in V,\ xy$ an edge of $A\} = \{A$-edges incident with $x\}$.

$\nu(x) \in [E]^d$ because x has degree d.
ν is one-one because $d > 1$.

And for distinct nodes x, y of A:

$$\nu(x) \cap \nu(y) = \begin{cases} \{xy\}, & \text{if } xy \text{ is an edge of } A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

So xy is an edge of A iff $|\nu(x) \cap \nu(y)| = 1$,
iff $\nu(x)\nu(y)$ is an edge of B.

Hence, ν is a graph embedding.

(iii) Extending partial isomorphisms of A to automorphisms of B

Fix a partial isomorphism p of $A = (V, E)$.

p is a partial 1–1 map : $V \to V$ that preserves edges and non-edges.
So p induces a partial 1–1 map $\widehat{p} : E \to E$, via

$$\widehat{p}(xy) = p(x)p(y), \text{ for edges } xy \text{ of } A \text{ with } x, y \in \operatorname{dom} p.$$

We extend $\widehat{p}$ to a permutation of E, in two stages:

1. define $\widehat{p}$ on edges xy with $x \in \operatorname{dom} p$, $y \notin \operatorname{dom} p$, so that $\widehat{p}(xy) = p(x)y'$ for some $y' \notin \operatorname{rng} p$,
2. extend $\widehat{p}$ in any 1–1 way to the remaining edges.

Then for any $x \in \operatorname{dom} p$, $\widehat{p}$ *maps the A-edges incident with x onto the A-edges incident with $p(x)$.*

The automorphism extending p

$\widehat{p}$ induces an automorphism p^* of $B = ([E]^d, R)$, via

$$p^*(X) = \{\widehat{p}(e) : e \in X\}, \text{ for } X \in [E]^d.$$

Clearly, $|X \cap Y| = |p^*(X) \cap p^*(Y)|$. So $p^* \in \operatorname{Aut} B$.

We check that p^* *extends* p (if we identify A with its ν-image).
For any $x \in \operatorname{dom} p$, the following diagram commutes:

$$\begin{array}{ccc} & p & \\ x & \longrightarrow & p(x) \\ \nu \downarrow & & \downarrow \nu \\ \{A\text{-edges incident with } x\} & \longrightarrow & \{A\text{-edges incident with } p(x)\} \\ & p^* & \end{array}$$

(iv) Guarded-faithfulness

We now have an EPPA extension $B = ([E]^d, R) \supseteq A = (V, E)$.
We want to make B guarded-faithful.

1. Restrict the nodes of B to $\{g(a) : a \in V,\ g \in \mathrm{Aut}(B)\}$. This gives point-faithfulness.

2. Now replace R by

$$\{(g(x)g(y) : xy \in E,\ g \in \mathrm{Aut}(B)\}.$$

   This and (1) give guarded-faithfulness. ■

Remark 9 For a signature L with a single n-ary relation symbol, Herwig–Lascar got an EPPA extension of any L-structure M of size $\leq 2^{n! \cdot n \cdot |M|^n}$.

6. Clique-faithful EPPA for relational structures

Recall:

Theorem (Herwig, 1995) *For any finite relational signature L, the class of all finite L-structures has the guarded-faithful EPPA.*

I'd have liked to have proved this before. . .

We now build on it, to show:

Theorem 10 (IH–Otto, 2003) *Let L be a finite relational signature. The class of all finite L-structures has the clique-faithful EPPA.*

Martin Otto

Vague idea of proof

The problem with an ‘ordinary’ EPPA extension B of A is that some cliques in $\mathcal{G}(B)$ may not get mapped back into A by any automorphism of B.

Call these *‘false cliques’.*

Idea is to *break up false cliques.*

We’ll construct a ‘cover’ C of B, in which the ‘lifts’ of sets not mappable mapped back into A by any automorphism of B (e.g., false cliques in $\mathcal{G}(B)$) cannot be cliques in $\mathcal{G}(C)$.

Proof

Take finite A in finite relational signature L. We will build a finite clique-faithful EPPA extension of A.

By Herwig's 1995 theorem, there is a finite L-structure $B \supseteq A$ such that

1. each partial isomorphism of A extends to an automorphism of B,
2. each $x \in B$ is mapped into A by some $g \in \mathrm{Aut}\, B$.

If $B = A$, we are done. So assume that $B \supset A$.

Small sets

Definition 11 A subset $U \subseteq B$ is said to be

- *small* if there is some $g \in \mathrm{Aut}\, B$ with $g(U) \subseteq A$,
- *large* ($\sim$ *bad*) otherwise.

Remark 12

- B is large.
- If U is large then $|U| \geq 2$.
- If U is large and $g \in \mathrm{Aut}\, B$ then $g(U) \stackrel{\text{def}}{=} \{g(u) : u \in U\}$ is large.

The set $\mathcal{U}$

Write $\mathcal{U}$ for the set of large subsets of B.

Note that $\mathcal{U} = \{g(U) : U \in \mathcal{U}\}$ for any $g \in \mathrm{Aut}\, B$.

Domain of the new EPPA extension C

Definition 13 Let $b \in B$. A map $\chi : \mathcal{U} \to \omega$ is said to be a *b-valuation* if for all $U \in \mathcal{U}$:

- if $b \notin U$ then $\chi(U) = 0$,
- if $b \in U$ then $1 \leq \chi(U) < |U|$. (Note $|U| \geq 2$.)

Can view χ as a notion $[\![b \in U]\!] = \chi(U)$, for large U.
Value is 0 if $b \notin U$. Value is positive (many-valued logic!) if $b \in U$.

Definition 14 We let our EPPA extension C have domain

$$\{(b, \chi) : b \in B,\ \chi \text{ a } b\text{-valuation}\}.$$

We'll define the L-structure of C in a minute.

Definition 15 Also define the projection $\pi : C \to B$ by $\pi(b, \chi) = b$.

Generic sets

Definition 16 A set $S \subseteq C$ is said to be *generic* if for all distinct $(b, \chi), (c, \psi) \in S$, we have

1. $b \neq c$,

2. $\chi(U) \neq \psi(U)$ for all $U \in \mathcal{U}$ with $b, c \in U$.

Remark 17 A set is generic iff each two-element subset is generic.

Lemma 18 *If $S \subseteq C$ is generic, then $\pi(S) \stackrel{\text{def}}{=} \{\pi(s) : s \in S\}$ is small.*

Proof. Let $\pi(S) = U \subseteq B$. If $U \in \mathcal{U}$, then by genericity,

- $\pi \restriction S$ is 1–1, so $|U| = |S|$,
- the map $\theta : S \to \{1, 2, \ldots, |U| - 1\}$ given by $\theta(b, \chi) = \chi(U)$ is 1–1.

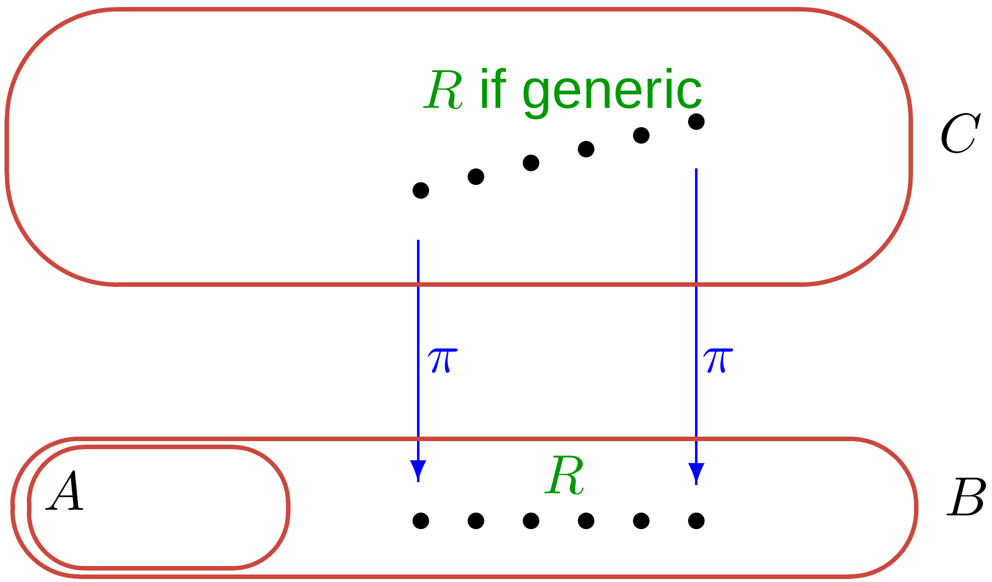
This is impossible. ■

The structure of C

Definition 19 Define C as an L-structure as follows.

If $R \in L$ is n-ary, and $(b_1, \chi_1), \ldots, (b_n, \chi_n) \in C$, then let $C \models R((b_1, \chi_1), \ldots, (b_n, \chi_n))$ iff

1. $\{(b_1, \chi_1), \ldots, (b_n, \chi_n)\}$ is generic,
2. $B \models R(b_1, \ldots, b_n)$.



Embedding A into C

Lemma 20 *A embeds into C.*

Proof. Let $U \in \mathcal{U}$. So there is no $g \in \mathrm{Aut}\, B$ with $g(U) \subseteq A$.
In particular, $U \not\subseteq A$. So $|U \cap A| < |U|$.

Enumerate $U \cap A$ as $\{a_1^U, \ldots, a_n^U\}$, with $n < |U|$. Do this for all $U \in \mathcal{U}$.
For $a \in A$, define an a-valuation $\chi_a : \mathcal{U} \to \omega$ by

$$\chi_a(U) = \begin{cases} 0, & \text{if } a \notin U, \\ \text{the } i \text{ such that } a = a_i^U, & \text{otherwise.} \end{cases}$$

Now define $\nu : A \to C$ by $\nu(a) = (a, \chi_a)$.

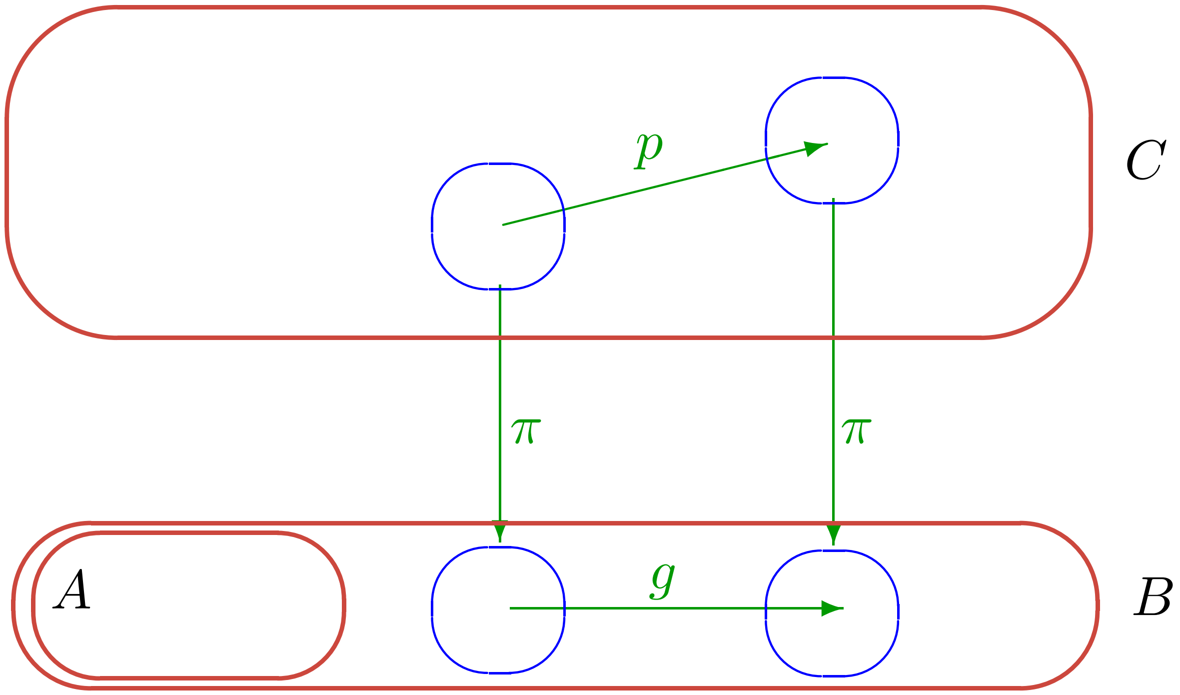
Note that $\nu(A) = \{\nu(a) : a \in A\}$ is generic. So $\nu : A \to C$ is an L-embedding. ∎

So we can *replace A by $\nu(A) \cong A$,* and prove the theorem for it. This will be easy after a definition and a rather technical lemma.

Compatible maps

Definition 21 Let $p : C \to C$ be a 1–1 partial map, and let $g \in \mathrm{Aut}\, B$. We say that p is *g-compatible* if for all $(b, \chi) \in \mathrm{dom}\, p$ we have

$$p(b, \chi) = (g(b), \chi') \text{ for some } \chi'.$$



Technical lemma

Lemma 22 *Let $p : C \to C$ be a 1–1 partial map with generic domain and range. Let $g \in \operatorname{Aut} B$, and suppose that p is g-compatible. Then p extends to some g-compatible $\widehat{p} \in \operatorname{Aut} C$.*

Proof. As $\operatorname{dom} p$ is generic, can write its elements in the form (b, χ_b).

We have

$$p(b, \chi_b) = (g(b), \chi'_{g(b)}),$$

say, for each $(b, \chi_b) \in \operatorname{dom} p$.

We need to extend this to a map of the form

$$\widehat{p} : (b, \chi) \mapsto (g(b), \chi'), \text{ for each } (b, \chi) \in C.$$

Fix a large set $U \in \mathcal{U}$. Then the set of pairs

$$\left\{ \left\langle \chi_b(U),\ \chi'_{g(b)}\big(g(U)\big) \right\rangle \ :\ (b, \chi_b) \in \operatorname{dom} p \right\}$$

is (the graph of) a 1-1 partial map on $|U| = \{0, 1, \ldots, |U| - 1\}$, fixing 0 if defined on it.

Extend it to a permutation θ_U of $|U|$, fixing 0. Do this for all $U \in \mathcal{U}$.

Now define $\widehat{p}(b, \chi) = (g(b), \chi')$, where

$$\chi'(g(U)) = \theta_U(\chi(U)), \text{ for each } U \in \mathcal{U}.$$

Then $\widehat{p}$

- is a well-defined permutation of C,
- extends p ($(b, \chi_b) \in \operatorname{dom} p \Rightarrow \chi'(g(U)) = \theta_U(\chi_b(U)) = \chi'_{g(b)}(g(U))$),
- is g-compatible,
- is an automorphism of C (because it is g-compatible and preserves generic sets). ■

Checking that C is as required

1. Certainly, $C \supseteq \nu(A)$ and C is finite.

2. Let p be a partial isomorphism of $\nu(A)$. We need to extend it to $\widehat{p} \in \operatorname{Aut} C$.

   Let $p{\downarrow} = \nu^{-1} \circ p \circ \nu$ be the corresponding partial isomorphism of A. By Herwig's theorem, we can extend $p{\downarrow}$ to some $g \in \operatorname{Aut} B$.

   Clearly, p is g-compatible.

   And $\operatorname{dom} p, \operatorname{rng} p$ are generic (as $\subseteq \nu(A)$).

   By lemma 22, p extends to some (g-compatible) $\widehat{p} \in \operatorname{Aut} C$.

Mapping cliques back into $\nu(A)$

3. Let $S \subseteq C$ be a clique in the Gaifman graph $\mathcal{G}(C)$ of C.

   We want $g \in \operatorname{Aut} C$ with $g(S) \subseteq \nu(A)$.

   S is generic. *So by lemma 18, $\pi(S)$ is small.*

   So there is $g \in \operatorname{Aut} B$ with $g(\pi(S)) \subseteq A$.

   The map

   $$p : x \mapsto \nu(g(\pi(x))) \in \nu(A) \quad \text{(for } x \in S\text{)}$$

   is 1–1, and has generic domain (S) and range $(\subseteq \nu(A))$, and is g-compatible.

   By lemma 22, p extends to $\widehat{p} \in \operatorname{Aut} C$, and $\widehat{p}(S) \subseteq \nu(A)$.

This completes the proof. ■

Remarks

- The theorem strengthens most previous EPPA results.
- Combined with the combinatorial proof of Herwig's 1995 result by Herwig–Lascar, it gives a purely combinatorial proof of them.
- The EPPA extension C is 'clique-homogeneous': any partial isomorphism of C whose domain and range are cliques in $\mathcal{G}(C)$ extends to an automorphism of C. (Exercise!)
- New and simple proof of finite model property for loosely guarded and packed (and clique-guarded) fragments.
- Otto has a variant argument to give this, using finite model property of the basic guarded fragment.

  This also shows (finite) satisfiability in the packed fragment reduces to (finite) satisfiability in the guarded fragment!

The end

We have

1. described the *guarded, loosely guarded, and packed fragments.*

2. discussed *EPPA,* and explained how to use it to establish the *finite model property for guarded fragments,*

3. proved *two EPPA results* along the way to showing that the guarded and packed fragments have the finite model property.

Decidability of these fragments follows (complexity is not optimal).

There are *applications* of the finite model property results in algebraic logic (e.g., in showing that certain relation algebras have 'nice' finite relativised representations). I hope there will be others too.

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