Canonicity in power

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Introduction 1/3. Canonical models and frames

In this talk, we consider only *normal* modal logics.

For a (normal) modal logic L, let:

- $W = \{\text{maximal } L\text{-consistent sets}\}$
- $\Gamma R \Delta$ iff $\varphi \in \Delta \Rightarrow \Diamond\varphi \in \Gamma$, for $\Gamma, \Delta \in W$
- $h(p) = \{\Gamma \in W : p \in \Gamma\}$ for each propositional atom p.

Then (W, R, h) is 'the' *canonical model* for L.
And (W, R) is 'the' *canonical frame* for L.

But these are not unique! The domain W depends on how many propositional atoms we allow.

Introduction 2/3. Canonical logics

A modal logic is said to be:

- *canonical,* if it is valid in every canonical frame for the logic (made of maximal consistent sets using any set of propositional atoms),
- *κ-canonical,* for a cardinal κ, if it is valid in the canonical frame for the logic made using κ atoms.

κ-canonical $\Rightarrow$ λ-canonical for $\lambda < \kappa$.
n-canonical for all $n < \omega \not\Rightarrow \omega$-canonical [Goldblatt, 4, §6].

Fine's problem (second of three) [2, 1975]

Kit Fine asked (adapted to the definition above):

> '... can an ω-canonical [modal] logic fail to be α-canonical for some cardinal $\alpha > \omega$?'

Utterly basic question about canonicity. But still open ~50 years later.

Introduction 3/3. Resolved logics

We call logics that are either canonical or not ω-canonical *resolved.*
These are the ones satisfying ω-canonical $\Rightarrow$ canonical.
Fine was asking whether all modal logics are resolved.

No unresolved logics have been identified. The following are resolved:

- Sahlqvist-axiomatisable logics — all canonical
- logics of elementary classes of frames — all canonical [Fine]
- subframe logics [Fine (transitive), Wolter (arbitrary)]
- transitive cofinal subframe logics [Zakharyaschev]

We will sketch a proof that quite a few other modal logics are resolved:

1. extensions of '$K5_2$'
2. logics of 'achronal width 1'
3. (multimodal) logics of 'finite achronal width' — covers 1, 2, 4, 5
4. Fine's logics of finite width
5. linear temporal logics (with F, P)

We do it for case 1. If time, we look at cases 2–3.

Case 1: extensions of $K5_2$ (Goldblatt)

For $n < \omega$, the logic $K5_n$ is axiomatised by $5_n = \Diamond^n p \to \Box\Diamond p$.
Here, $\Diamond^n = \Diamond\Diamond \cdots \Diamond$ (n times).

Sahlqvist correspondent of 5_n is $\forall xyz(xR^n y \wedge xRz \to zRy)$.
Here, $R^n = R \mid R \mid \cdots \mid R \quad$ (n-fold relational composition).
So $K5_n$ is canonical.

5_0 is symmetry axiom 'B'.
5_1, aka. 5, is part of the axiomatisation of S5.

We look at $5_2 = \Diamond\Diamond p \to \Box\Diamond p$.
We will show that all normal modal logics containing 5_2 are resolved.
This should illustrate the general method.

Futures

Let $\mathcal{F} = (W, R)$ be a Kripke frame.
For a point $x \in W$, the *future of* x is the set $R(x) = \{y \in W : xRy\}$.

If $\mathcal{F} \models 5_2$, $(*)$ all points in the future of a point x *have the same future* — because the correspondent of 5_2 is $\forall xyzt(xRyRt \wedge xRz \to zRt)$.

LEMMA 1 *If $\mathcal{F} \models 5_2$, then for every $n < \omega$ and $x \in W$, all points in $R^n(x)$ have the same future.*

Proof. By induction on n. Trivial for $n = 0$ because $R^0(x) = \{x\}$.
Assume the lemma inductively for n. Suppose $xR^n yRy'$ and $xR^n zRz'$.
We show that y', z' have the same future.
Inductively, y, z have the same future. So $z' \in R(z) = R(y)$.
So both y', z' lie in the future of y.
Since $\mathcal{F} \models 5_2$, by $(*)$ they have the same future. $\square$

Varieties of BAOs

We work with boolean algebras with operators (BAOs) with a single unary operator $\Diamond$. Well known to be interchangeable with modal logic.

For a BAO $\mathcal{A}$, we let $\mathcal{A}_+$ denote its *canonical frame.*
So $\mathcal{A}_+ = (\{\text{ultrafilters of } \mathcal{A}\}, R)$, where $\mu R \nu$ iff $\forall a \in \mathcal{A}(a \in \nu \Rightarrow \Diamond a \in \mu)$.
$\mathcal{A}^\sigma$ denotes the *canonical extension* of $\mathcal{A}$ — based on $\wp(\mathcal{A}_+)$.

DEFINITION 2 Let V be a variety of BAOs.
Say V is *(totally) canonical* if $\mathcal{A}^\sigma \in V$ for every $\mathcal{A} \in V$.
Say V is *ω-canonical* if $\mathcal{A}^\sigma \in V$ for every countable $\mathcal{A} \in V$.
Say V is *resolved* if it's either canonical or not even ω-canonical.

If $\mathcal{A}$ is the free V-algebra on κ generators, then $\mathcal{A}_+$ 'is' the canonical frame with κ atoms for the logic of V.
Follows that definition 2 matches the earlier definitions for modal logics.

THEOREM 3 *Every ω-canonical variety $V \models 5_2$ is canonical. That is, every variety $V \models 5_2$ is resolved.*

Proof sketch. Take an arbitrary modal formula $\varphi_0(q_1, \ldots, q_n)$ satisfiable in $\mathcal{B}_+$ for some (large) $\mathcal{B} \in V$, under an assignment h of atoms into $\mathcal{B}_+$. It's enough to show φ_0 is satisfiable in $\mathcal{A}_+$ for some countable $\mathcal{A} \in V$.

Form a two-sorted structure $\mathfrak{B} = (\mathcal{B}, \mathcal{B}_+)$. Its signature $\mathcal{L}$ has

- the BAO operations $+, -, 0, 1, \diamond$ on $\mathcal{B}$
- $\underline{\in}$: $\mathcal{B} \times \mathcal{B}_+$ $\qquad$ $\mathfrak{B} \models x \,\underline{\in}\, y$ iff x is a member of the ultrafilter y
- $R : \mathcal{B}_+ \times \mathcal{B}_+$ $\qquad$ $\mathfrak{B} \models \forall xy(xRy \leftrightarrow \forall z(z \,\underline{\in}\, y \to \diamond z \,\underline{\in}\, x))$
- unary relation symbols $Q_0, Q_1, \ldots : \mathcal{B}_+$ $\qquad$ $Q_i^{\mathfrak{B}} = h(q_i)$ for $i < \omega$.

Sorts of variables are given by context!

$(\mathcal{B}_+, R^{\mathfrak{B}}, Q_i^{\mathfrak{B}})_{i<\omega}$ (we write just $\mathcal{B}_+$) is a Kripke model satisfying φ_0. Write $\psi^\dagger(x)$ (an $\mathcal{L}$-formula) for the standard translation of modal fmla ψ. So $\mathfrak{B} \models \exists x \varphi_0^\dagger(x)$.

Proof 2/5. Countable $\mathfrak{A} \preceq \mathfrak{B}$

Fix a countable elementary substructure $\mathfrak{A} = (\mathcal{A}, W) \preceq \mathfrak{B} = (\mathcal{B}, \mathcal{B}_+)$.
Then $\mathcal{A} \preceq \mathcal{B} \in V$, so $\mathcal{A} \in V$. And $\mathcal{A}$ is countable.
We aim to show that φ_0 is satisfiable in the canonical frame $\mathcal{A}_+$ of $\mathcal{A}$.

Warning: in general, $\mathcal{A}_+$ is not $(W, R^{\mathfrak{A}})$. (It's usually uncountable.)

To start, recall that $\mathfrak{B} \models \exists x \varphi_0^\dagger(x)$.
Since $\mathfrak{A} \preceq \mathfrak{B}$, we have $\mathfrak{B} \models \varphi_0^\dagger(w_0)$ for some $w_0 \in W$. Fix such a w_0.

Next, for $w \in W$ let $\widehat{w} = \{a \in \mathcal{A} : \mathfrak{A} \models a \in w\}$ — an ultrafilter, in $\mathcal{A}_+$.
Then $(w \mapsto \widehat{w})$ is a frame embedding : $(W, R^{\mathfrak{A}}) \to \mathcal{A}_+$.

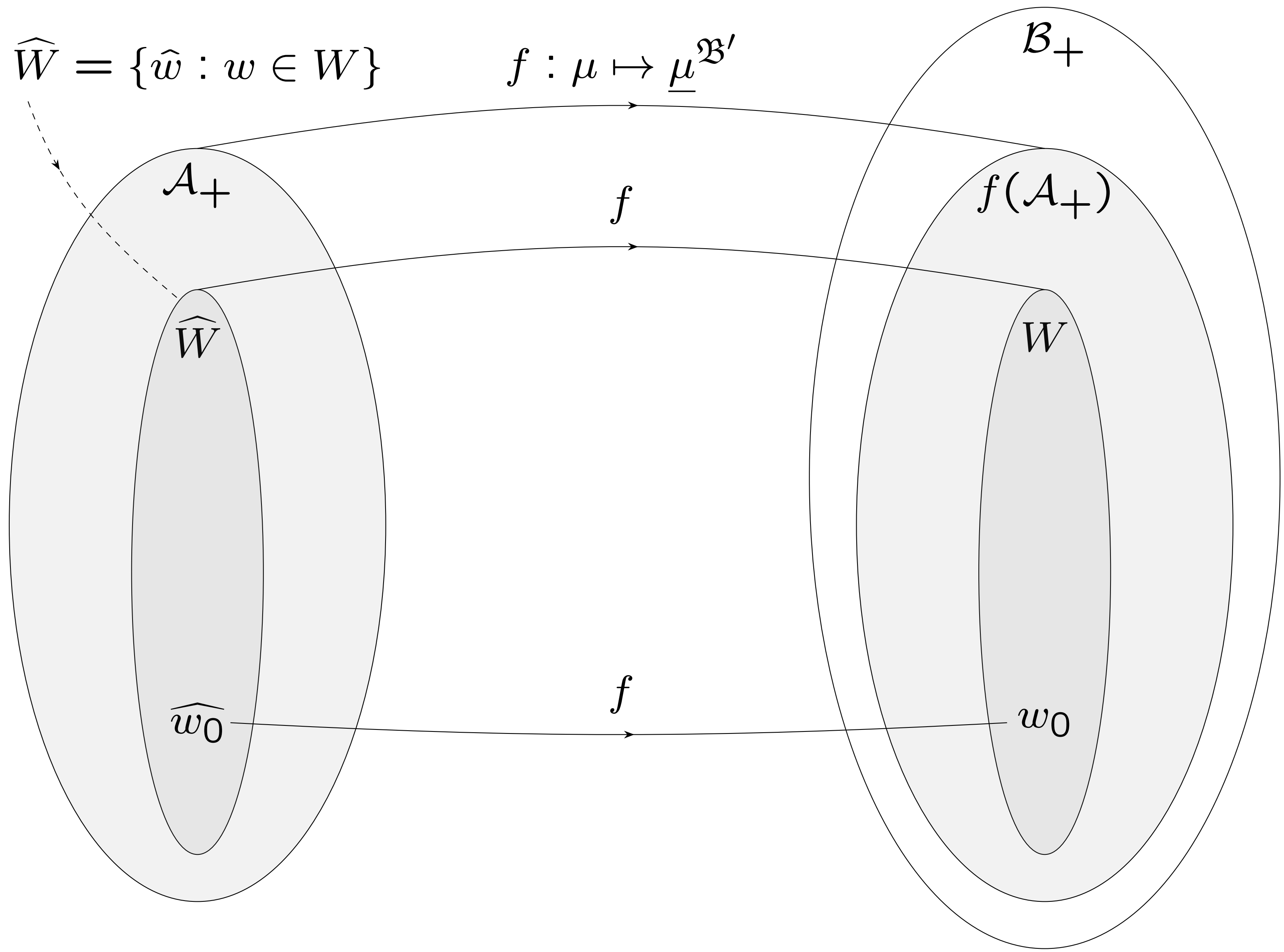
We find a frame embedding $f : \mathcal{A}_+ \to \mathcal{B}_+$ with $f(\widehat{w}) = w$ for $w \in W$.
(Recall $W \subseteq \mathcal{B}_+$.)

Proof 3/5. A frame embedding $f : \mathcal{A}_+ \to \mathcal{B}_+$

As $\mathfrak{A} \equiv \mathfrak{B}$, by Frayne's theorem (or the K–Sh theorem) there exist an ultrapower $\mathfrak{A}^*$ of $\mathfrak{A}$ and an $\mathcal{L}$-elementary embedding $\sigma : \mathfrak{B} \to \mathfrak{A}^*$.

1. Expand $\mathcal{L}$ to $\mathcal{L}'$ by adding a unary relation symbol $\underline{\mu}$ of sort $\mathcal{A}$ for each ultrafilter $\mu \in \mathcal{A}_+$.
2. Expand $\mathfrak{A}$ to an $\mathcal{L}'$-structure $\mathfrak{A}'$ by interpreting each $\underline{\mu}$ as μ: that is, $\mathfrak{A}' \models \underline{\mu}(a)$ iff $a \in \mu$, for each $a \in \mathcal{A}$.
3. Let $\mathfrak{A}'^*$ be the ultrapower of $\mathfrak{A}'$ using the same ultrafilter as before. It is an $\mathcal{L}'$-expansion of $\mathfrak{A}^*$.
4. Expand $\mathfrak{B}$ to an $\mathcal{L}'$-structure $\mathfrak{B}'$ by: $\mathfrak{B}' \models \underline{\mu}(b)$ iff $\mathfrak{A}'^* \models \underline{\mu}(\sigma(b))$ for each $\mu \in \mathcal{A}_+$ and $b \in \mathcal{B}$.
   Then $\underline{\mu}^{\mathfrak{B}'}$ is an ultrafilter of $\mathcal{B}$, so a member of $\mathcal{B}_+$.

Our frame embedding $f : \mathcal{A}_+ \to \mathcal{B}_+$ is given by $f(\mu) = \underline{\mu}^{\mathfrak{B}'}$.
It can be made to satisfy $f(\widehat{w}) = w$ for each $w \in W$.


$\widehat{W} = \{\hat{w} : w \in W\}$
$f : \mu \mapsto \underline{\mu}^{\mathfrak{B}'}$
$\mathcal{B}_+$
$\mathcal{A}_+$
$f(\mathcal{A}_+)$
f
$\widehat{W}$
W
f
$\widehat{w_0}$
w_0

Pause for remarks

$\mathfrak{A}' \subseteq \mathfrak{B}'$ ($\mathcal{L}'$-substructure), but $\mathfrak{A}' \not\preceq \mathfrak{B}'$ and $\mathfrak{A}' \not\equiv \mathfrak{B}'$ in general.

$f : \mathcal{A}_+ \to \mathcal{B}_+$ is not in general a bounded morphism (aka. p-morphism).

Open whether $f : \mathcal{A}_+ \to \mathcal{B}_+$ is elementary, though it does preserve more than just $\neq$ and $\pm R$.

Open whether $\mathcal{A}_+ \equiv \mathcal{B}_+$ as frames.

(Answers would be very interesting. Surendonk worked on this a lot — eg [6].)

Proof 4/5. Heart of proof: 'f preserves modal formulas'

Let $\mathcal{G}$ be the subframe of $\mathcal{A}_+$ generated by $\widehat{w_0}$.

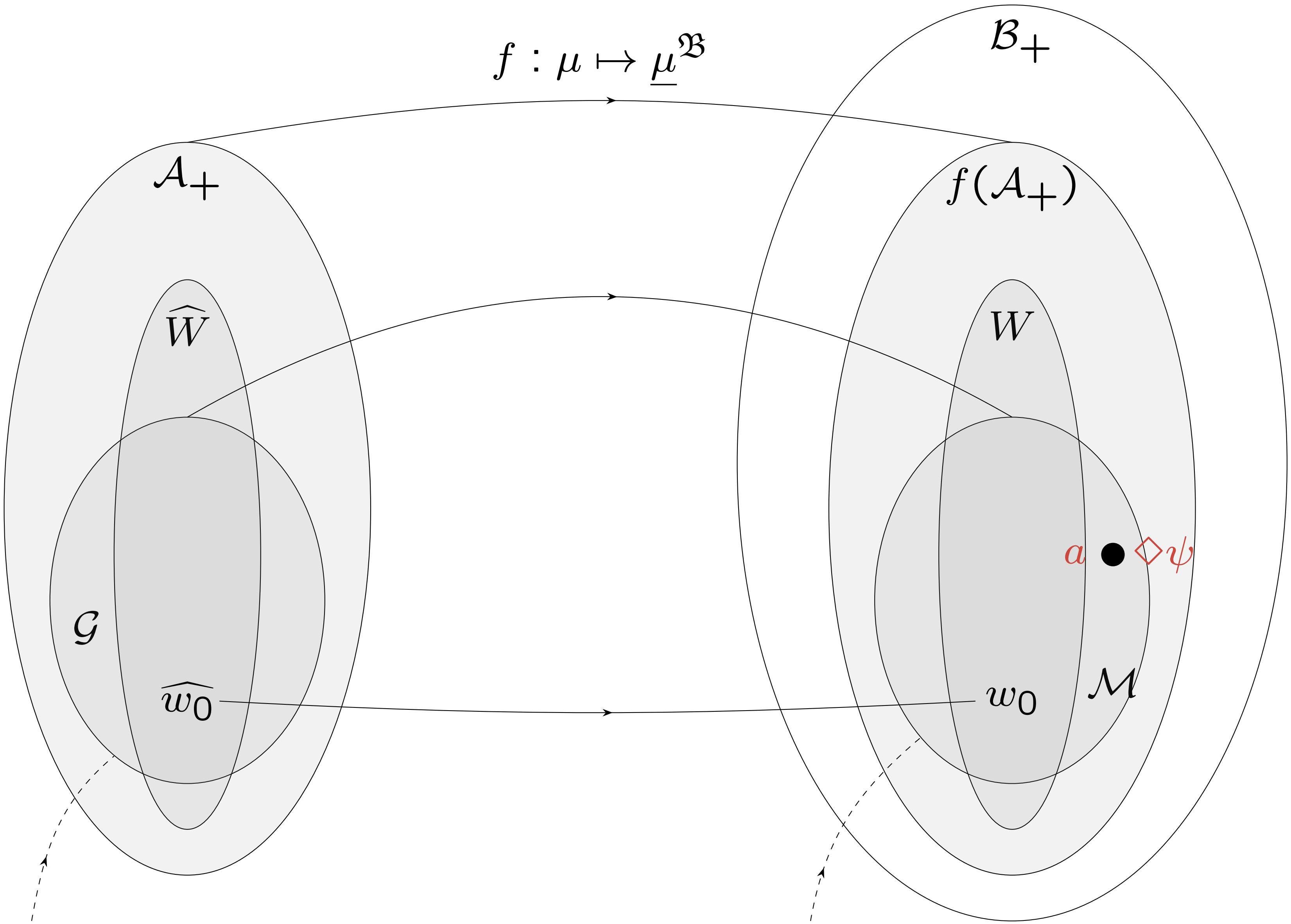
Let $\mathcal{M}$ be the submodel of $\mathcal{B}_+ = (\mathcal{B}_+, R^{\mathfrak{B}}, Q_i^{\mathfrak{B}})_{i<\omega}$ with domain $f(\mathcal{G})$.

We show that for each $a \in \mathcal{M}$ and modal formula ψ, we have

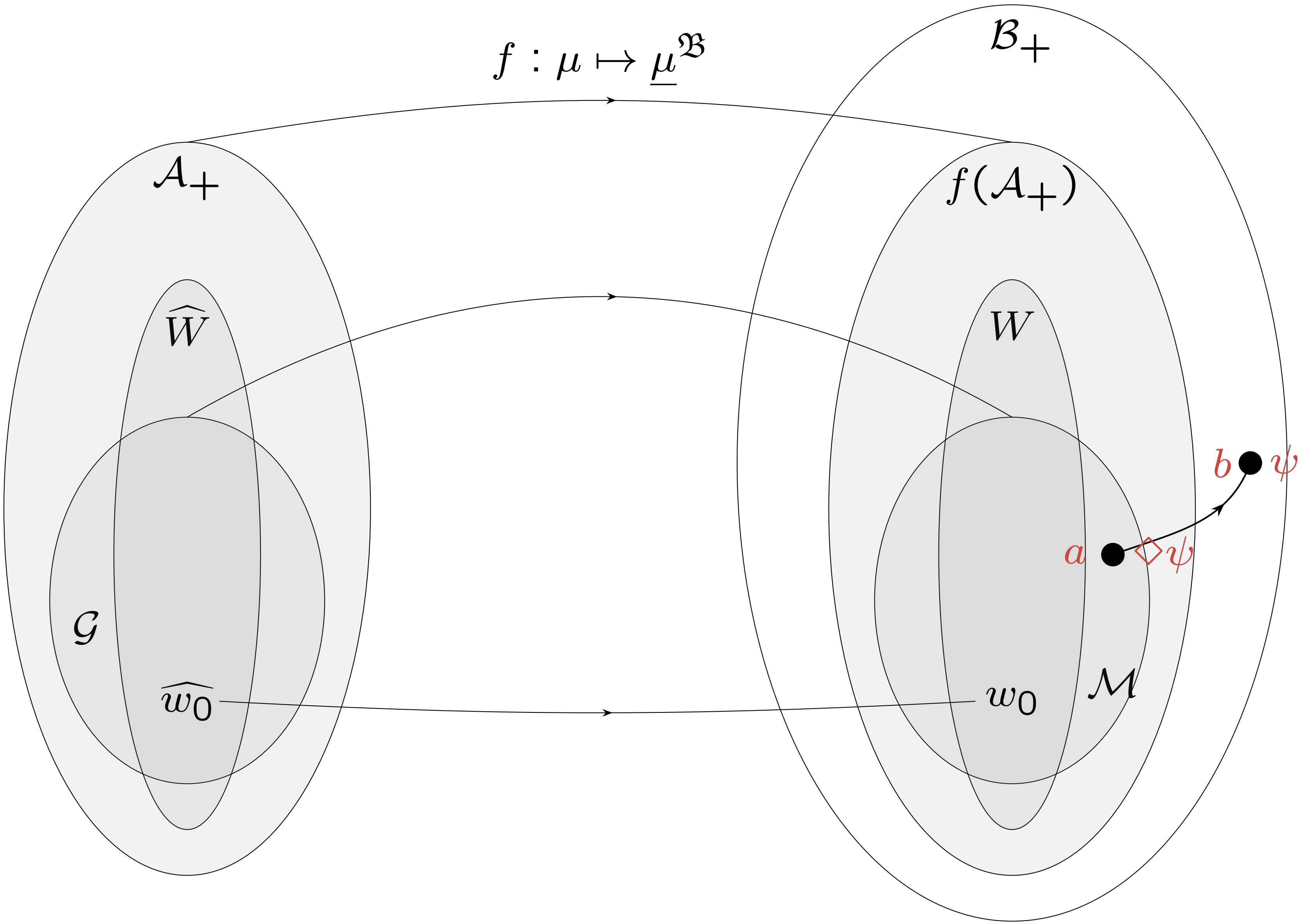
$$\mathcal{B}_+, a \models \psi \text{ iff } \mathcal{M}, a \models \psi.$$

Proof: Induction on ψ.

Big case: assume $\mathcal{B}_+, a \models \Diamond\psi$; show $\mathcal{M}, a \models \Diamond\psi$.



$\mathcal{G}$ = subframe of $\mathcal{A}_+$ generated by $\widehat{w_0}$. $\mathcal{M} = f(\mathcal{G})$.



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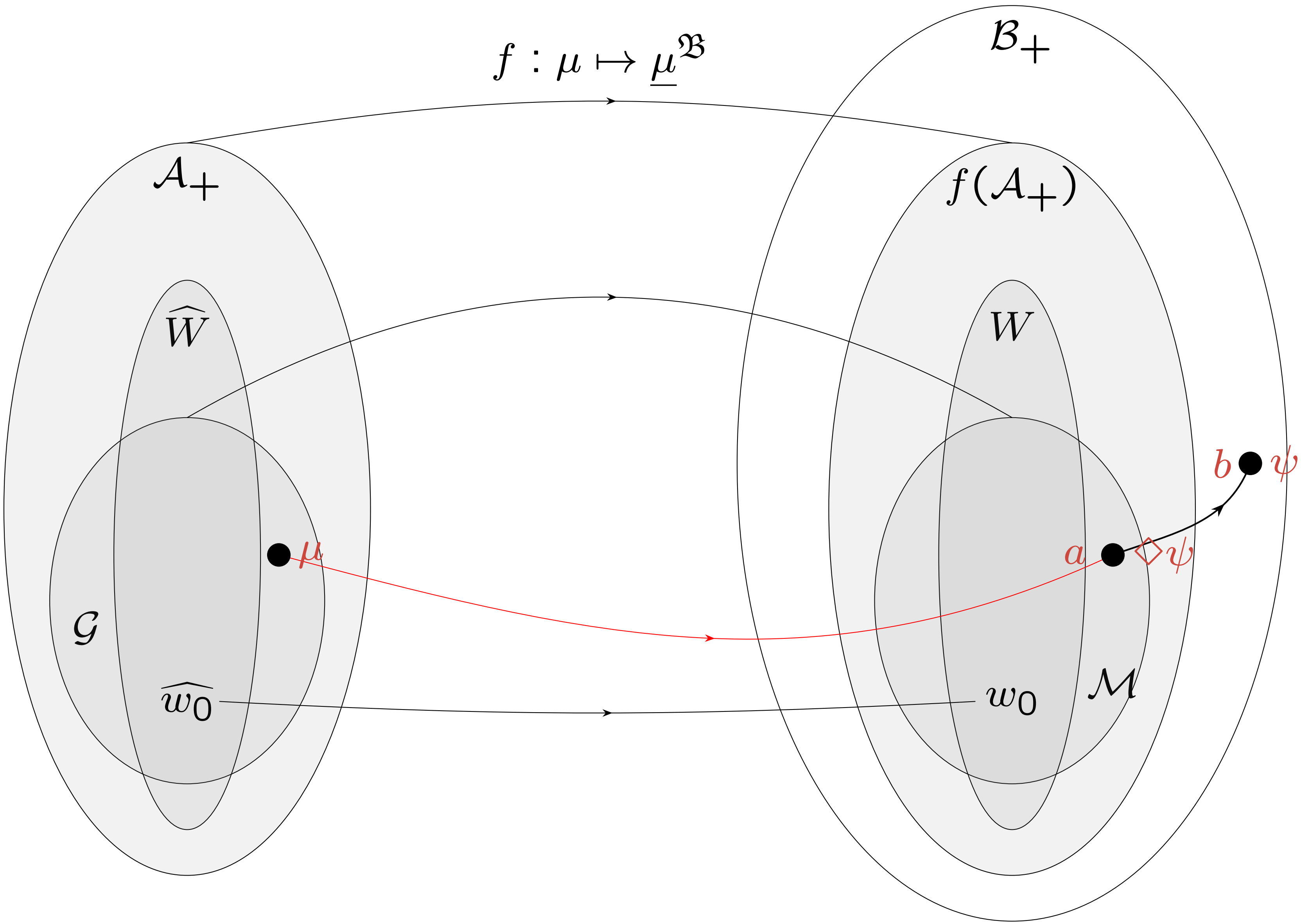

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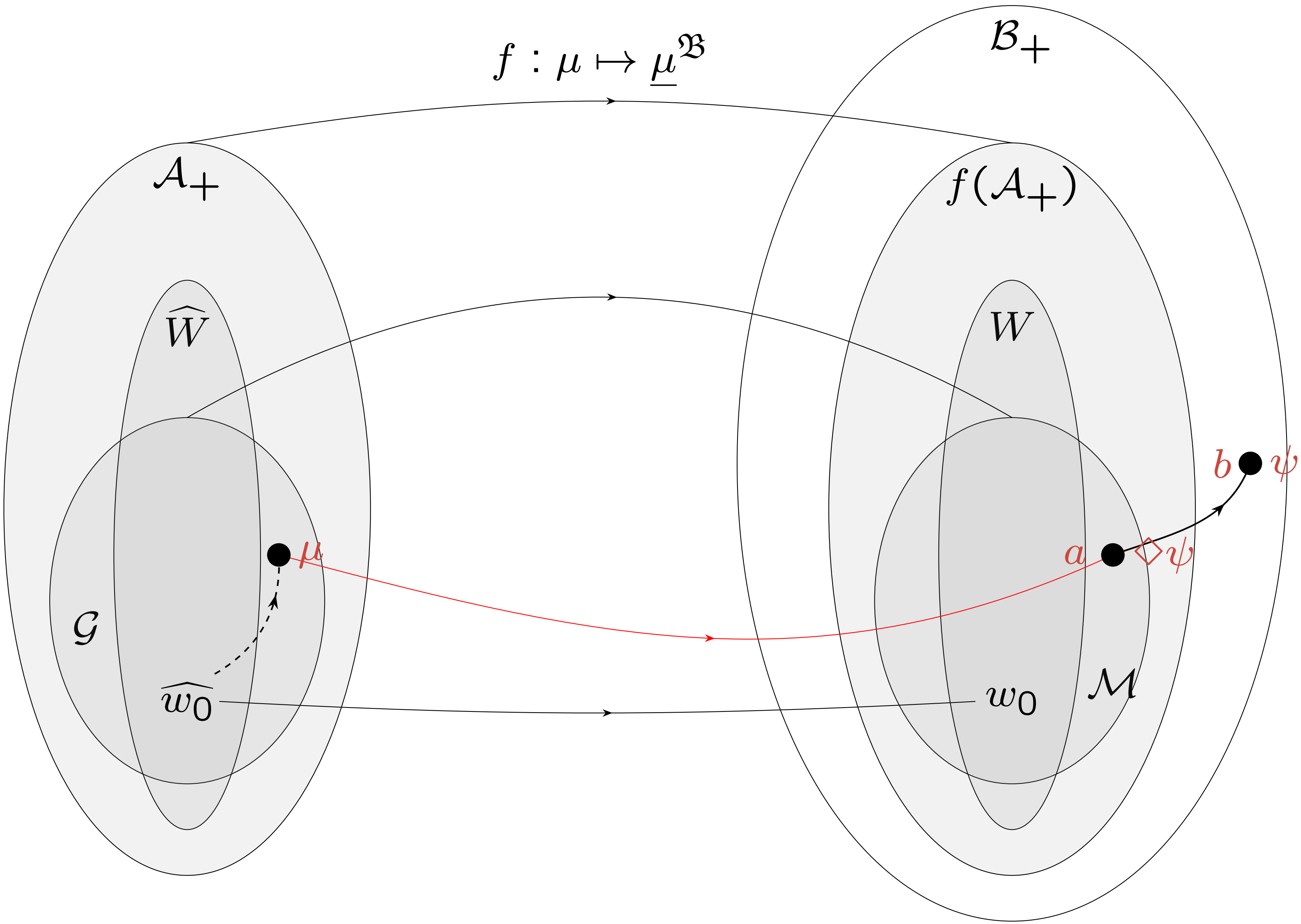

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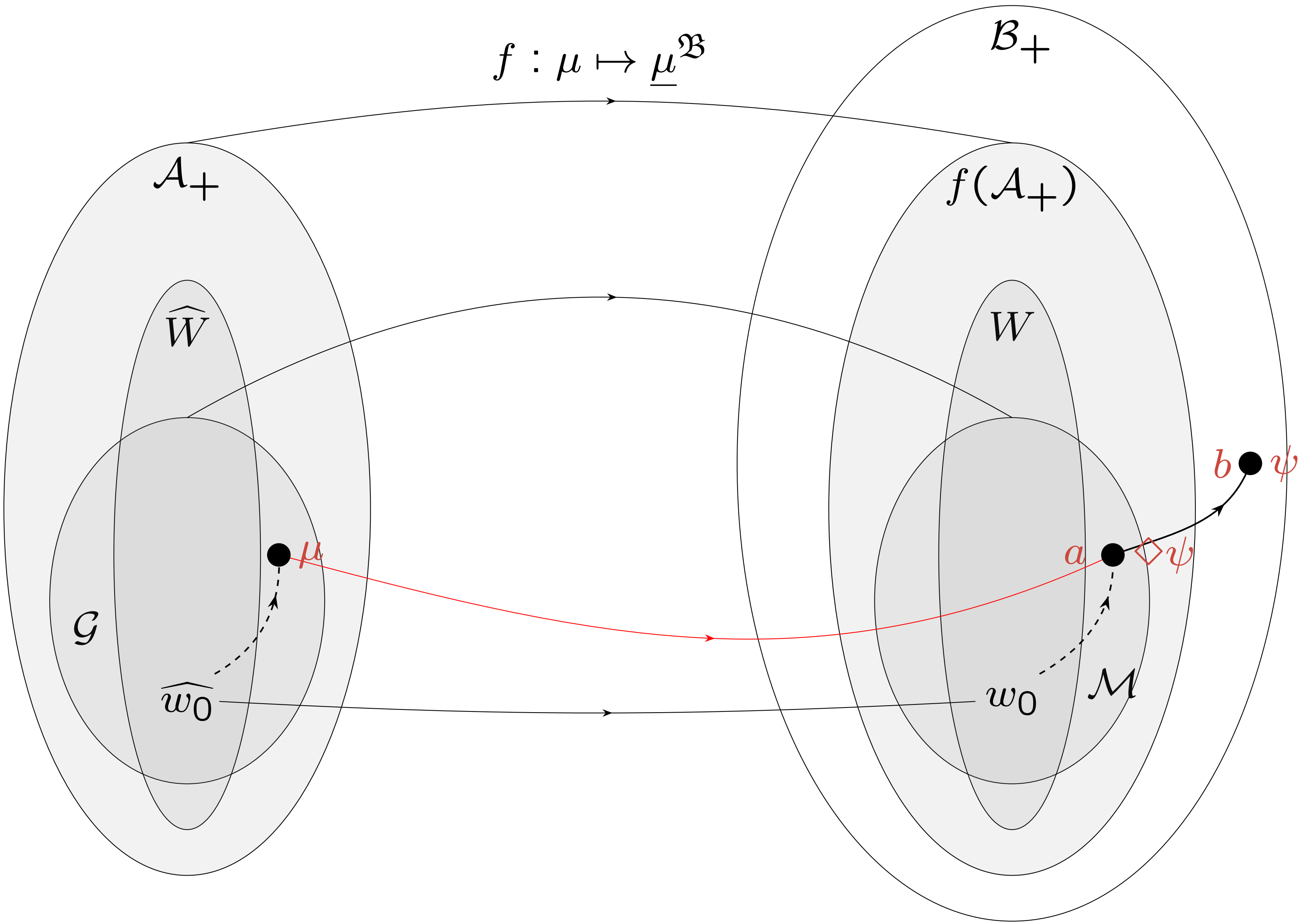

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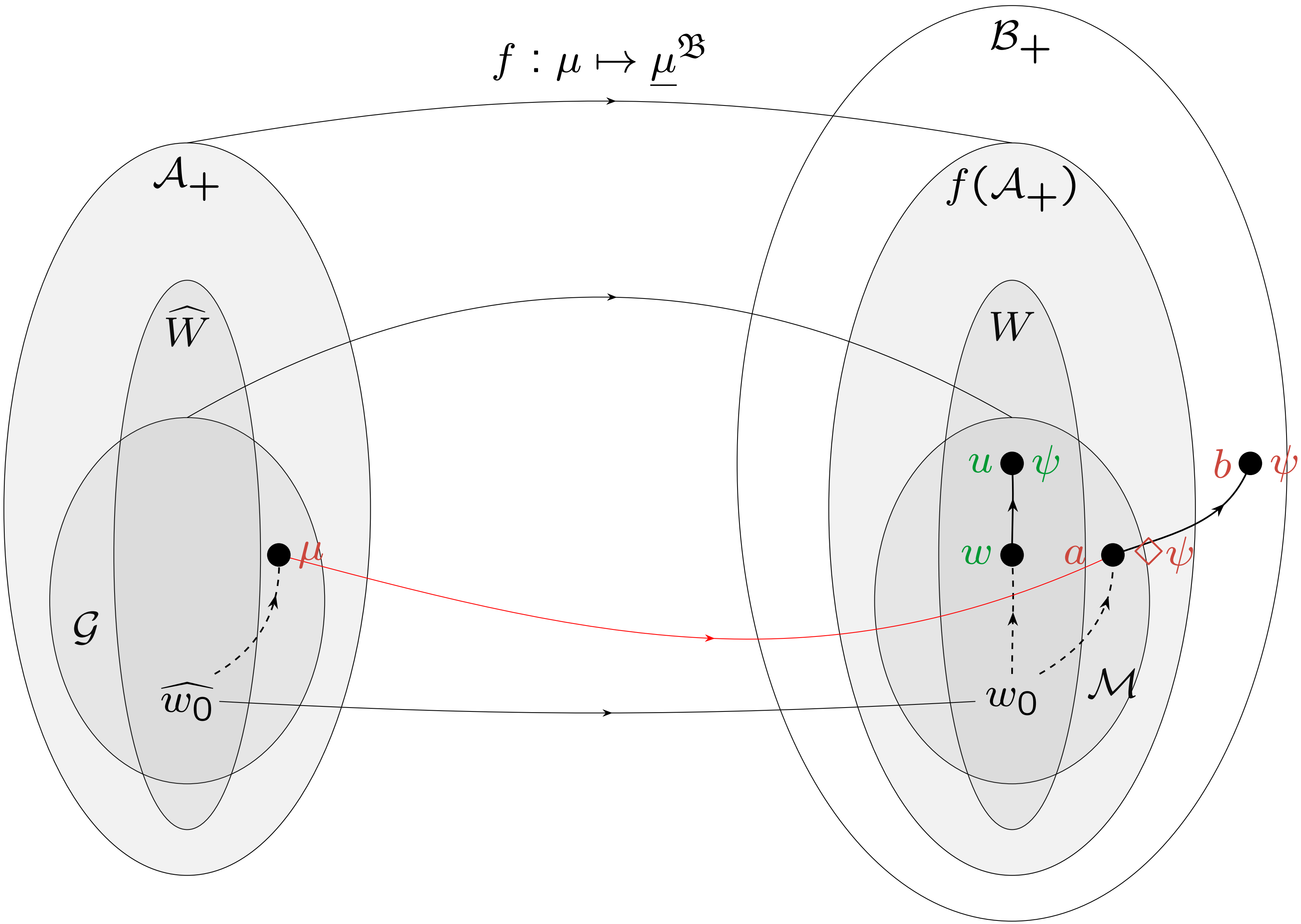

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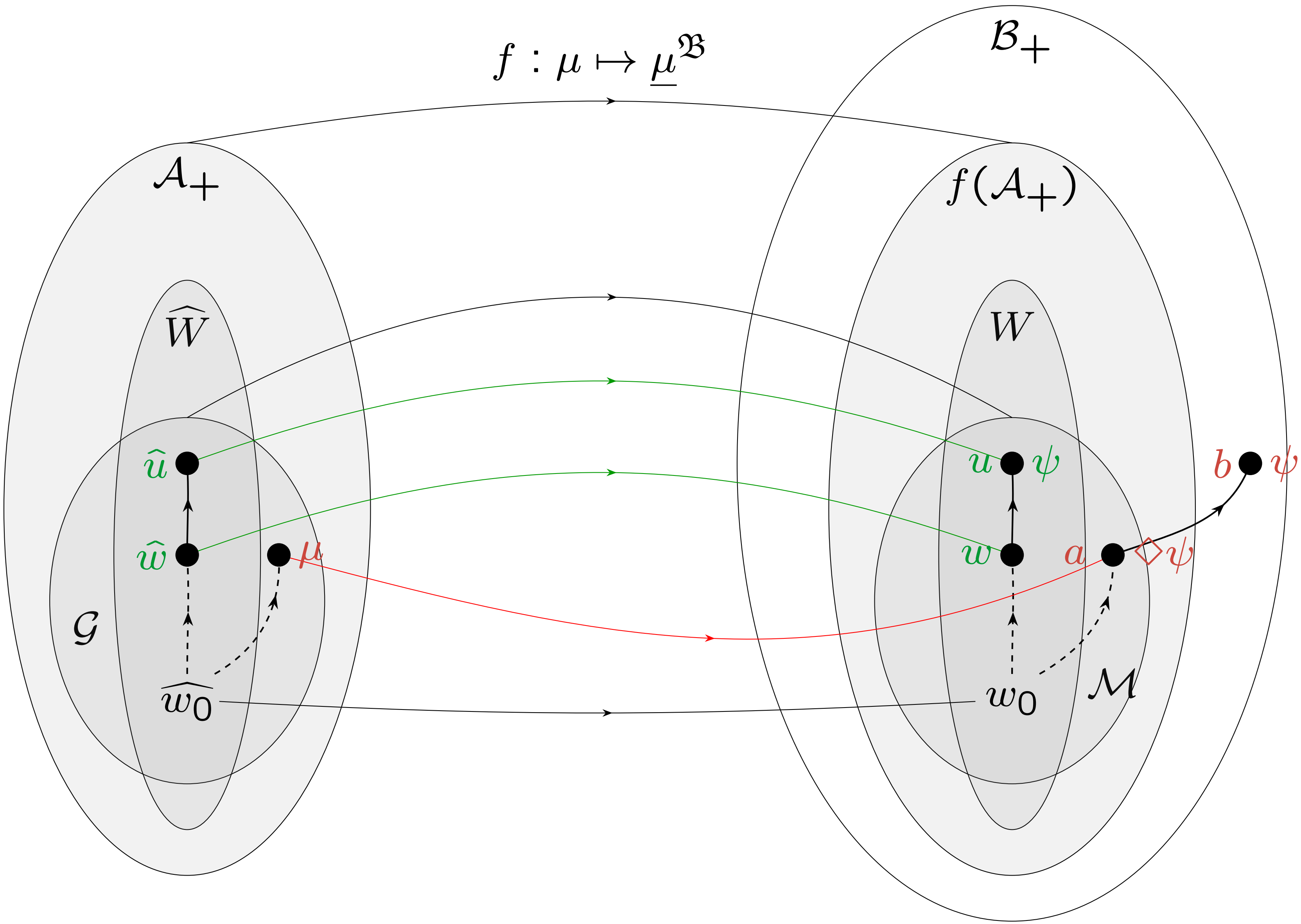

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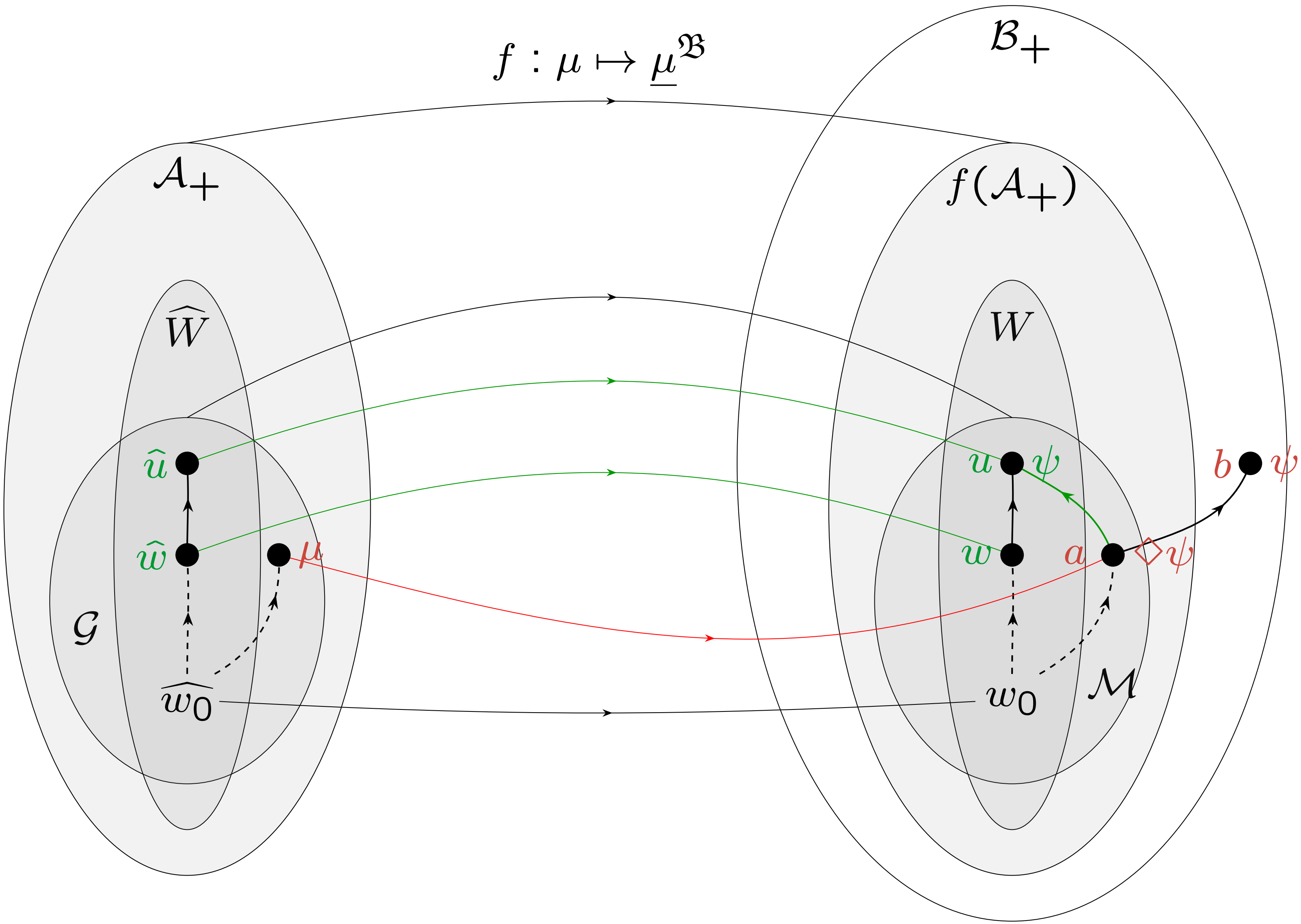

$\mathcal{G}$ = subframe of $\mathcal{A}_+$ generated by $\widehat{w_0}$. $\mathcal{M} = f(\mathcal{G})$.

Proof 5/5. Conclusion

By choice of w_0 we had $\mathcal{B}_+, w_0 \models \varphi_0$.

By the induction just done, $\mathcal{M}, w_0 \models \varphi_0$.

But as frames, $\mathcal{M} \cong \mathcal{G}$ (by f^{-1}).

So φ_0 is satisfiable in $\mathcal{G}$ — a generated subframe of $\mathcal{A}_+$.

Hence, φ_0 is satisfiable in $\mathcal{A}_+$, as required. Theorem 3 is proved. $\square$

Breathe

So all modal logics extending $K5_2$ are resolved.

But the method can do more.

Case 2: logics of ‘achronal width 1’

Proof above used $R(w) = R(a)$. But it needed only $R(w) \subseteq R(a)$!

Let

$$U_1 = \Box(\Box q_0 \to \Box q_1) \vee \Box(\Box q_1 \to \Box q_0).$$

Sahlqvist correspondent:

$$\forall x y_0 y_1 (xRy_0 \wedge xRy_1 \to [R(y_0) \subseteq R(y_1)] \vee [R(y_1) \subseteq R(y_0)]).$$

Any two points in any $R(x)$ have $\subseteq$-comparable futures.
True in linear frames, and others too.

Lemma 1 generalises (with similar proof):

LEMMA 4 *if $\mathcal{F} \models U_1$ then any two points in any $R^n(x)$ have $\subseteq$-comparable futures.*

We now extend the proof for $K5_2$ to KU_1:

THEOREM 5 *Every variety $V \models U_1$ is resolved.*

Proof idea. All is as before, except that in the induction, we have $n < \omega$ with $\mathfrak{B} \models w_0 R^n a \wedge (\Diamond\psi)^\dagger(a)$, and we need $w \in W$ with
$\mathfrak{B} \models \underbrace{w_0 R^n w \wedge (\Diamond\psi)^\dagger(w)}_{\text{call this } \delta(w)} \wedge R(w) \subseteq R(a)$.

Assume for contradiction that there's no such w.
Since $V \models U_1$, by lemma 4 we get
$\mathfrak{B} \models \delta(w) \to R(a) \subseteq R(w)$ for all $w \in W$.

Even though $a \notin W$ in general, some model theory via $\mathfrak{A}'^*$ gives
$\mathfrak{B} \models \forall y(\delta(y) \to R(a) \subseteq R(y))$.

Now $\mathfrak{B} \models \delta(a)$.
So $\mathfrak{B} \models \exists x[\delta(x) \wedge \forall y(\delta(y) \to R(x) \subseteq R(y))]$ — a witnesses it.
Since $\mathfrak{A} \preceq \mathfrak{B}$, there is $w \in W$ with
$\mathfrak{B} \models \delta(w) \wedge \forall y(\delta(y) \to R(w) \subseteq R(y))$.

Taking $y = a$ gives $\mathfrak{B} \models \delta(w) \wedge R(w) \subseteq R(a)$, contradiction. $\square$

Case 3: logics of finite achronal width

For $1 \leq n < \omega$ let

$$U_n = \bigvee_{0 \leq i \leq n} \Box\Big(\Box q_i \to \bigvee_{j \leq n, j \neq i} \Box q_j\Big).$$

Sahlqvist correspondent:

$$\forall x\, y_0 \ldots y_n\Big(\big(\bigwedge_{i \leq n} xRy_i\big) \to \bigvee_{i \neq j} R(y_i) \subseteq R(y_j)\Big).$$

Say $S \subseteq \mathcal{F}$ is *achronal* if no distinct $x, y \in S$ have $\subseteq$-comparable futures.

Then $\mathcal{F} \models U_n$ iff $\forall x \in \mathcal{F}$ (every achronal subset of $R(x)$ has size $\leq n$).

We say that a logic has *finite achronal width* if it contains some U_n.

Logics of finite achronal width are resolved

Lemma 1 generalises again (using Ramsey's theorem):
if $\mathcal{F} \models U_n$ then any achronal subset of any $R^m(x)$ is *finite.*

The proof for U_1 extends to show:

THEOREM 6 *For each $n \geq 1$, every $V \models U_n$ is resolved.*

We 'convert' from n to 1 by more model theory (adding parameters).

Can formulate a *multimodal* version of the U_n, covering (eg.) linear temporal logics.
Theorem 6 applies to them too. This is our most general result.

Facts about the U_n

$$
\begin{array}{ccccccccccc}
K4 & \subset & \cdots & \subset & K4I_2 & \subset & K4I_1 & & & & \\
\cup & & & & \cup & & \cup & & & & \\
K & \subset & \cdots & \subset & KU_2 & \subset & KU_1 & \subset & K5_2 & \subset & K5
\end{array}
$$

Here,

4 = $\Diamond\Diamond p \to \Diamond p$ — transitivity

$I_n = (\bigwedge_{i \le n} \Diamond q_i) \to \bigvee_{i,j \le n,\, i \ne j} \Diamond(q_i \wedge (q_j \vee \Diamond q_j))$ — Fine [3, 1974]

Correspondent: each R-antichain contained in some $R(x)$ has size $\le n$.

The chains in the figure intersect in $K4$ and K.

All other inclusions follow by transitivity from the ones shown.

All gaps are proper and contain 2^ω logics.

Kripke-complete extensions of the U_n and Fine's I_n

- All extensions of any $K4I_n$ are Kripke complete — Fine [3, 1974].

- KTU_1 has a Kripke-incomplete extension, where $T = p \to \Diamond p$ (reflexivity) — van Benthem [1, 1978].

- $K4U_2$ has a Kripke-incomplete extension — Goldblatt–IH.

- Open whether $K4U_1$ has a Kripke-incomplete extension.

Conclusion

We have seen that modal logics of finite achronal width are *resolved* — that is, *either canonical or not even ω-canonical.*

But many logics are not of finite achronal width, and are not covered.

Fine's original question of whether all modal logics are resolved remains wide open.

We also don't know whether all canonical frames for a logic are elementarily equivalent.

So lots to do...

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