# Monodic fragments of first-order temporal logics

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## Outline of talk

Most propositional temporal logics are decidable.

But the decision problem in predicate (first-order) temporal logics has seemed near-hopeless.

I will report on some recent work on this problem.

I will consider **monodic fragments** of the first-order temporal language, in which formulas beginning with a temporal operator have at most one free variable. The first-order part is also restricted.

Validity of formulas in these fragments can be decided by combining:

- an algorithm to decide the first-order part of the formula,
- an algorithm deciding monadic second-order logic over the given flow of time.

Works for linear and (with additional restrictions) for branching time.



## Organisation

1. First-order temporal logic

   Syntax, semantics, validity.

   Some known results.

2. Monodic first-order temporal logic

   Definitions, examples.

3. Monodic fragments over linear time

   Decidability, complexity, reasoning.

4. Monodic fragments over branching time

   Undecidability and decidability.

5. Applications

6. Conclusion

7. References



## 1. First-order temporal logic

We want to combine temporal and first-order logic, to gain expressive power.

We use a first-order extension of the branching-time temporal logic $\mathcal{CTL}^*$ [Emerson–Halpern, 1986].

$L$ — first-order relational signature *without equality*.
We fix a set $\mathcal{V}$ of first-order variables.

### Syntax

- Any atomic $L$-formula is a temporal formula.
- If $\varphi, \psi$ are (temporal) formulas, so are

| | |
|---|---|
| $\neg\varphi$ | |
| $\varphi \wedge \psi$ | |
| $\exists x \varphi$ | (where $x$ is any variable in $\mathcal{V}$) |
| $\varphi \,\mathsf{U}\, \psi$ | (until) |
| $\varphi \,\mathsf{S}\, \psi$ | (since) |
| $\bigcirc\varphi$ | (tomorrow) |
| $\mathsf{E}\varphi$ | (for some history) |

## Semantics

A *flow of time* is an irreflexive partial order $T = (T, <)$.
$<$ is called the *earlier-later relation.*

$T$ is *linear* if any two distinct times are $<$-related.
It's a *tree* if for all $t \in T$, $\{u \in T : u < t\}$ (the past of $t$) is linearly ordered by $<$.

A *history* (or *branch*) of $T$ is a maximal linearly-ordered subset of $T$.
We often consider *ω-trees*: trees with all branches isomorphic to $\mathbb{N} = \{0, 1, 2, \ldots\}$.

For non-linear flows, we often have an idea of the 'intended history'.
Semantics — evaluate formulas at a time point but relative to a history.



## Evaluation

Models have the form $\mathfrak{M} = (T, (D_t : t \in T))$, for a flow of time $T = (T, <)$ and first-order $L$-structures $D_t$ with fixed domain $D$ (say).

For a time $t \in T$, a history $h$ of $T$ containing $t$, and an assignment $\mathfrak{a}$ of variables to elements of $D$, define $\mathfrak{M}, h, t \models^{\mathfrak{a}} \varphi$ by induction on $\varphi$:

- for atomic $\varphi$, we let $\mathfrak{M}, h, t \models^{\mathfrak{a}} \varphi$ iff $D_t \models^{\mathfrak{a}} \varphi$
- booleans as usual
- $\mathfrak{M}, h, t \models^{\mathfrak{a}} \exists x \varphi$ iff $\mathfrak{M}, h, t \models^{\mathfrak{b}} \varphi$ for some assignment $\mathfrak{b}$ that agrees with $\mathfrak{a}$ on all variables other than $x$
- $\mathfrak{M}, h, t \models^{\mathfrak{a}} \bigcirc\varphi$ iff there is an immediate successor $t^+$ of $t$ in $h$ with $\mathfrak{M}, h, t^+ \models^{\mathfrak{a}} \varphi$
- $\mathfrak{M}, h, t \models^{\mathfrak{a}} \varphi \,\mathsf{U}\, \psi$ iff there is $u \in h$ with $t < u$, $\mathfrak{M}, h, u \models^{\mathfrak{a}} \psi$, and $\mathfrak{M}, h, v \models^{\mathfrak{a}} \varphi$ for all $v$ with $t < v < u$ *(strict interpretation!)*
- $\varphi \,\mathsf{S}\, \psi$ — mirror image
- $\mathfrak{M}, h, t \models^{\mathfrak{a}} \mathsf{E}\varphi$ iff $\mathfrak{M}, h', t \models^{\mathfrak{a}} \varphi$ for some history $h'$ of $T$ containing $t$.



## Remarks

- We use *constant domains.* (Expanding domains can be represented in this setting.)
- $\bigcirc\varphi \equiv \bot \,\mathsf{U}\, \varphi$, but it helps later when defining fragments to have $\bigcirc$ as primitive.

## Fragments

This is a powerful logic. Sometimes it's too much. So we define some fragments:

- propositional temporal logic ($\mathcal{CTL}^* +$ Since): require that all $L$-relations are nullary; throw out $\exists$. The $D_t$ are essentially just propositional valuations.
- linear fragment: over linear time, the history $h$ is unique so can be dropped from the notation. And $\mathsf{E}\varphi \equiv \varphi$, so we throw out E.
- more to come…



## Some abbreviations

$\vee, \to, \forall$: as usual

$$G\varphi = \varphi \wedge \neg(\top \,\mathsf{U}\, \neg\varphi)$$

$$A\varphi = \neg E \neg\varphi.$$

## Some formulas

$$A\forall x(researches(x) \,\mathsf{U}\, \exists y\, admin\_job\_for(y,x))$$

$$AG\forall x(\exists y\, admin\_job\_for(y,x) \to \neg researches(x))$$

$$AG\forall x(researches(x) \to E \bigcirc researches(x))$$

## Validity

A formula φ is *valid over a flow of time $T$* if it is true at all times and histories under all assignments in all models with flow of time $T$.

φ is *valid in linear time* if it is valid in all linear flows.

Define *valid over trees, ω-trees,* etc., similarly.

A formula φ is *satisfiable* if ¬φ is not valid.

### Other semantic choices

- *Bundled semantics:* we can restrict $h$ to come from a given set $\mathcal{B}$ — a *bundle* — of histories. $\mathcal{B}$ is included as part of $\mathfrak{M}$. We require $\bigcup \mathcal{B} = T$. This gives a different notion of validity (weaker — fewer validities).
- May restrict to models with *finite domain* $D$. Again, this gives a different notion of validity (stronger — more validities).



## Propositional temporal logic: good news

Most *propositional* temporal logics are quite well behaved:

Over linear time:

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and linear-time validity of propositional temporal formulas is decidable and finitely axiomatisable.
- Validity problem PSPACE-complete [Sistla & Clarke 1985; Reynolds 200?].

Over ω-trees:

- Decidable.
- Finitely axiomatisable. If drop Since (i.e., $\mathcal{CTL}^*$), recursively axiomatisable. [Reynolds]
- Without Since, validity 2EXPTIME-complete [lower bound: Vardi–Stockmeyer 1985; upper bound: Emerson–Jutla 1988].



## Predicate temporal logic: bad news

Of course, predicate logic is undecidable.
But temporal predicate logic is worse.

Negative results began with unpublished work of Scott and Lindström in 1960s.

### Example

Over $\mathbb{N}$, even with only 2 variables and unary relation symbols, predicate temporal logic is

- undecidable,
- not recursively axiomatisable.

Can be proved by reduction of tiling problem (or halting problem).

Note: 2-variable and monadic fragments of first-order logic are decidable.



## What did we do to deserve this?

We took a '3-dimensional product':

- two dimensions for the 2 first-order variables,
- one more dimension for time.

3-dimensional products are often badly behaved (undecidable, not finitely axiomatisable).

### How can we recover?

*Limit the interaction between dimensions.*

- When we quantify over domain elements ($\exists x \varphi$), we do it at a single time point.
- But when we move through time, (φ U ψ etc), there may be many free variables in the formula.
  This is not fair! The negative results retreat if we ban it from happening.

## 2. Monodic first-order temporal logic

A formula $\varphi$ is *monodic* if every subformula of $\varphi$ of the form $\psi \,\mathsf{U}\, \chi$ or $\psi \,\mathsf{S}\, \chi$ or $\bigcirc\psi$ or $\mathsf{E}\psi$ has *at most one free variable.*

### Examples

Barcan formula, $\forall x G\varphi(x) \leftrightarrow G\forall x\varphi(x)$

"List all persons who have been unemployed between jobs" [Chomicki–Toman]:

$$\begin{array}{ll} & \top \,\mathsf{S}\, \exists y\, \mathit{WorksFor}(x,y) \\ \wedge & \neg\exists y\, \mathit{WorksFor}(x,y) \\ \wedge & \top \,\mathsf{U}\, \exists y\, \mathit{WorksFor}(x,y) \end{array}$$

### Non-example

Rigidity of a binary relation (over $\mathbb{N}$):

$$\mathsf{G}\forall xy(R(x,y) \leftrightarrow \bigcirc R(x,y)).$$



## 3. Monodic fragments over linear time

Any first-order $L$-formula is monodic. So validity for monodic formulas is undecidable.

But given an oracle for first-order $L$-satisfiability, we can decide monodic satisfiability over *linear time.*

We can get rid of the oracle by restricting to a decidable fragment of first-order logic.

- If we restrict $\exists$ in formulas to *guarded form,*

  $\exists\bar{x}(R(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{y}))$ for atomic $R(\bar{x},\bar{y})$,

  we get the *temporal guarded fragment, $TGF$.*

  The monodic guarded fragment (the monodic fragment of $TGF$) is decidable over linear time, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, etc., and for finite domains (the general case is open), $\mathbb{R}$.
- So are the monodic loosely guarded and monodic packed/clique-guarded fragments.
- So is the monodic 2-variable fragment.
- So is the monodic monadic fragment, etc.



## Proof of decidability: quasimodels

Let $\mathcal{F}$ be the monodic guarded, packed, 2-variable, or monadic fragment.

A *quasimodel* for a sentence $\sigma \in \mathcal{F}$ is obtained from a model of $\sigma$ by

- regarding formulas beginning with a temporal operator as atomic (i.e. unary or nullary relation symbols)
- replacing each $L$-structure $D_t$ by its '$\sigma$-theory' $\Sigma_t$ (theory w.r.t. subformulas of $\sigma$ – a finite object)
- axiomatising the links between the $\Sigma_t$ $(t \in T)$ that are enforced by their origin in a model.

**Lemma 1.** Let $T$ be linear. Then $\sigma$ has a model over $T$ iff it has a quasimodel over $T$.



## Quasimodels and decidability

A quasimodel boils down to a monadic expansion of the flow of time, with properties expressible in *monadic second-order logic.*

**Lemma 2.** We can construct from $\sigma$ a monadic second-order sentence $\widehat{\sigma}$ such that for any linear $T$,

$$T \models \widehat{\sigma} \quad \text{iff there is a quasimodel for } \sigma \text{ over } T.$$

Put the two lemmas together to obtain:

**Theorem.** Assume that $T$ is linear with decidable monadic second-order theory (e.g., $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$). Then it is decidable whether a sentence in $\mathcal{F}$ is satisfiable over $T$.

## Complexity

Seems to be MAX(EXPSPACE, complexity of underlying first-order fragment).

- Monodic one-variable, (two-variable too?), monadic fragments are EXPSPACE-complete over $\mathbb{N}$.
- Monodic guarded fragment is 2EXPTIME-complete over $\mathbb{N}$.

## Reasoning algorithms

- The monodic monadic fragment over $\mathbb{N}$ is axiomatisable (Wolter & Zakharyaschev 2001)
- Resolution-based algorithm (Degtyarev & Fisher 2001, for fragment only)
- Tableau-based algorithms (Kontchakov, Lutz, Wolter & Zakharyaschev 2001)
- Implementation of tableaux (Günsel 2001)

## Extensions

- Can add rigid constants.
- Can't add equality in general: monodic validities not r.e. (Degtyarev & Fisher 2001).
- But can add equality safely to guarded fragments.



## 4. Monodic fragments over branching time

Here, things are not quite so good.

Over any class of trees containing a tree with at least two histories through each time, the *1-variable fragment is undecidable.* (– certainly monodic!)

This can be strengthened in technical ways: bundles, just $G$, modal products, etc.

Proof uses a reduction of tiling problem, via relation algebras.

### What did we do to deserve this?

There are three dimensions again:

- domain point
- time point
- history (fixed in linear case)

Interaction between these is restricted, but still enough for undecidability.

Evaluation at pairs $(t,h)$ breaks quasimodels.



## Positive branching results

We can get quasimodels working again by imposing *extra restrictions* to limit the dimensional interaction and/or dependence on the history $h$.

Two different ways are known.

### Fragment 1: restrict $\exists$

A *state formula* is one made from atomic formulas and formulas of the form E$\psi$ using only the first-order operations.
Semantics of state formulas is history-independent.

By requiring that

- $\exists x$ applies only to state formulas,
- $\mathsf{U}, \mathsf{S}, \bigcirc, \mathsf{E}$ are used monodically
- the 'first-order part' lies in a decidable fragment of first-order logic (guarded fragment, etc),

we obtain a decidable fragment over ω-trees.

Eg: $\forall x \mathsf{A}[(receive(x) \to \bigcirc shred(x)) \,\mathsf{U}\, caught]$ – OK
$\mathsf{E}\forall x[(receive(x) \to \bigcirc shred(x)) \,\mathsf{U}\, caught]$ – not OK



### Fragment 2: restrict U, S, E

By requiring that

- U, S, E apply only to sentences *(nullodic?),*
- $\bigcirc$ is used monodically,
- the 'first-order part' lies in a decidable fragment of first-order logic,

we obtain a decidable fragment over ω-trees.

Eg: $\underbrace{\mathsf{E}\,(\underbrace{[\forall x(receive(x) \to \bigcirc shred(x))]}_{\text{sentence}} \,\mathsf{U}\, \underbrace{caught}_{\text{sentence}})}_{\text{sentence}}$

These results use

1. downward Löwenheim–Skolem theorem, to find a model over a countable ω-tree,
2. quasimodels
3. decidability of monadic second-order theory of countable ω-trees (follows from Rabin's theorem).

The search for the 'best' fragment over ω-trees is still on.

## 5. Applications

Predicate temporal logic is important. First-order structures can represent:

- states of programs (reactive systems): specification and verification, synthesis of programs from (temporal) specifications
- states of databases: e-commerce
- states of knowledge bases: description logics such as $\mathcal{ALC}$
- (some) topological spaces: qualitative spatial reasoning systems like RCC-8
- Kripke frames for epistemic logics: multi-agent systems

So it seems worth studying predicate temporal logics.



As well as being powerful in its own right, predicate temporal logic embeds other useful logics – often monodically – so serves as a unifying framework.

- Agent logics of Halpern *et al.*

  Almost all decidable agent logics can be embedded into decidable monodic fragments.

- Temporal description logics extend standard DLs with temporal operators.

  In $\mathcal{ALC}$, if temporal operators are applied only to subsumptions and concepts (but not to roles), then the language is reducible to the two-variable monodic fragment.

- Spatial logic RCC-8 can be extended with temporal operators.

  The resulting logics can be reduced to decidable monodic fragments if the topology can be 'simulated' in first-order logic.



## 6. Conclusion

Predicate temporal logic is poorly behaved.

Monodic fragments of predicate temporal logic are better behaved.

They give us a new direction of research, new problems, but also new hope of

- understanding temporal logic better,
- using temporal logic in more advanced applications.



## 7. References

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See also `www.dcs.kcl.ac.uk/staff/mz`