Atom structures of relation algebras

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Abstract

We study *atom structures* of relation algebras. We prove that the class of atom structures that arise from representable atomic relation algebras is elementary, but is not definable by a sentence of the infinitary logic $L^{\omega}_{\infty\omega}$ (the fragment of $L_{\infty\omega}$ consisting of formulas using finitely many variables). Hence it is not finitely axiomatisable in first-order logic. We also show that representability of an atomic relation algebra is not determined by its atom structure, by exhibiting two (countable) relation algebras with the same atom structure, one representable, the other, not.

It is therefore of interest to study the class of atom structures $\mathcal{C}$ such that *any* atomic relation algebra with atom structure $\mathcal{C}$ is representable. This is the class of all atom structures whose complex algebra is representable. We show that this class includes any finite representable atom structure, any atom structure satisfying the Lyndon conditions, and more. We also prove that the class is not definable by any sentence of $L^{\omega}_{\infty\omega}$, even modulo the atom structures arising from representable atomic relation algebras.

1 Introduction

This paper is concerned with representability of atom structures arising from relation algebras in algebraic logic. The relation algebra structure of an atomic relation algebra is determined by its boolean algebra structure and by the way the product operation behaves on its atoms. Hence, we can extract the atom structure of an atomic relation algebra, and endow it with natural first-order operations from which the relation algebra structure of the given algebra may be recovered. This is well known: see, for example, [L1], and definition 3.2 and theorem 3.13(1) of [M2]. Now the difficulties in finding representations for relation algebras mostly arise from the product structure — it is easy to find representations of a boolean algebra, and even complete representations. So the question naturally arises as to whether these problems can be pinned down to the atom structure, in the case of atomic relation algebras. That is, does representability of an atomic relation algebra depend only on its atom structure?

It turns out that the answer is 'no': there is more to the issue than that. We will give an example of two countable relation algebras with the same atom structure, one being representable, the other, not.

This prompts us to distinguish some cases. An atom structure is said to be weakly representable if it is the atom structure of a representable relation algebra; and strongly representable if every relation algebra based on it is representable. The two notions coincide for finite atom structures, because (up to isomorphism) there is a unique relation algebra with a given finite atom structure. But the example just mentioned shows that they do not coincide in general. We are able to give explicit axioms for the weakly representable atom structures, by the game technique used in [HH1]. As for the strongly representable ones, we do not know if there are first-order axioms characterising them. We can at least show that an atom structure is strongly representable if it satisfies the Lyndon conditions of [L1]. These conditions are not necessary, however.

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We also give non-finite axiomatisability results, paralleling those in relation algebra (e.g., [Mo]). One feature of our results in this direction is the use of the infinitary logic $L^{\omega}_{\infty\omega}$. This is the fragment of $L_{\infty\omega}$ consisting of the formulas using only finitely many variables. It includes first-order logic but is considerably stronger. However, it has a very well-behaved game-theoretic characterisation, and is quite easy to use. We will show that neither the weakly nor the strongly representable atom structures are definable in this logic. (In fact, we show rather more.) Thus, in the language of atom structures, no first-order sentence, and no first-order theory using finitely many variables, can define these properties.

We expect that similar results can be proved for finite dimensional cylindric algebras, along the lines of [HH2]. We will not do so here.

We should mention that the proofs of most of our negative results rely on a construction introduced in [Hi,HH2], which we will generalise here, as theorem 7.2. Starting with an arbitrary countable, integral, weakly associative relation algebra, $\mathcal{R}$, we enmesh it in a relation algebra, which will be representable if and only if $\mathcal{R}$ has a (relativised) representation of a special kind: namely, it contains an infinite subset of points, any pair of which are 'live' (related by the top element of $\mathcal{R}$). This may be of independent interest.

Contents of the paper

Section 2 Here, we give the main definitions to do with relation algebras and atom structures, and list some basic results. We state our main theorem (theorem 2.10), and discuss the relevance of the Lyndon conditions to atom structures. Then we prove the first two parts of the main theorem.

Section 3 Here we axiomatise the class of weakly representable atom structures.

Section 4 We recall some facts about weakly associative relation algebras, which are used later. We outline a convenient way of specifying a weakly associative relation algebra, by its class of 'consistent triangles'.

Section 5 This section contains the construction of a relation algebra from a weakly associative relation algebra, mentioned above.

Section 6 We discuss a game-theoretic method of proving that a weakly associative relation algebra is associative, and that a relation algebra is representable.

Secton 7 We use the games described in section 6 to establish the main properties of the construction of section 5.

Section 8 We use the construction of section 5 and results of section 7 to read off counterexamples that prove the rest of our main theorem. In doing so, we also say more about $L^{\omega}_{\infty\omega}$.

Appendix Here, we show that the Lyndon conditions are not necessary for an atom structure to be strongly representable, using an example of Maddux [M1].

Notation If $\mathcal{A}$ is a structure, we will sometimes write A for its domain (or universe, as some say). But mostly, we will use the same notation for a structure as for its domain. This is standard algebraic/model-theoretic practice. Thus, for example, if $\mathcal{A}$ is a relation algebra, with domain $\text{dom}(\mathcal{A})$, we will write $S \subseteq \mathcal{A}$ to mean that S is a subset of the domain of $\mathcal{A}$. If X is a set, $\wp X$ will denote the power set (the set of all subsets) of X.

We will write $\bar{x}, \bar{a}$ etc., for *tuples* — finite sequences of variables, elements of a structure, etc. $\bar{x}\bar{y}$ is the tuple formed by concatenating $\bar{x}$ and $\bar{y}$. If M is a structure, we will write $\bar{a} \in M$ to mean that $\bar{a}$ is a tuple of elements of the domain of M. If $t(\bar{x})$ is a term in the signature L of M, and $\bar{a} \in M$, we write $t^M(\bar{a})$ for the value of t in M when the variables $\bar{x}$ are assigned to $\bar{a}$. Similarly, if R is a relation symbol of L, we denote the interpretatin of R in M by R^M.

2 Definitions

Here we set up our system of notation, and recall some basic facts about the objects we will be working with.

2.1 Relation algebras

Definition 2.1 A relation algebra is a structure $\mathcal{A} = (A, 0, 1, \neg, \wedge, Id, \smile, ;)$, where $A \neq \emptyset$, $\wedge$ and ; are binary functions, $\smile$ and $\neg$ are unary functions, and $0, 1$, and Id are constants. We require that the boolean reduct $(A, 0, 1, \neg, \wedge)$ of $\mathcal{A}$ is a boolean algebra (so we can use $\vee$ and $\leq$ as abbreviations), and that for all $x, y, z \in A$, the following hold:

1. $x; Id = Id; x = x$
2. The function ';' is associative
3. $x^{\smile\smile} = x$
4. $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$
5. $(\neg x)^{\smile} = \neg(x^{\smile})$
6. $(x; y)^{\smile} = y^{\smile}; x^{\smile}$
7. $(x; y) \wedge z^{\smile} = 0 \rightarrow (y; z) \wedge x^{\smile} = 0$ ('triangle axiom').

As is shown in [CT,JT2], axioms 3–7 can be replaced by the 'left and right Peircean laws'

$$x; y \wedge z = 0 \Rightarrow x^{\smile}; z \wedge y = z; y^{\smile} \wedge x = 0.$$

(In this expression, ';' binds more strongly than $\wedge$.) But we have found it convenient here to keep the definition above. Other useful properties of relation algebras that follow from axioms 1 and 3–7 include:

8. ';' is monotonic: if $x \leq x'$ and $y \leq y'$ then $x; y \leq x'; y'$.
9. ';' distributes over arbitrary joins: if S is a set of elements with least upper bound $\bigvee S$, then $x; \bigvee S = \bigvee\{x; s : s \in S\}$, and $\bigvee S; x = \bigvee\{s; x : s \in S\}$ for all x. In particular, these least upper bounds exist.
10. $x \leq Id \Rightarrow x^{\smile} = x$. In particular, $Id^{\smile} = Id$.

See [CT, JT2, part II, theorem 4.3], and [M2, theorem 1.13].

Recall that an *atom* of a boolean algebra is a minimal non-zero element of it; a boolean algebra is said to be *atomic* if each of its non-zero elements contains ($\geq$) some atom. By an 'atom' of a relation algebra, we mean an atom of its boolean reduct. A relation algebra is said to be *integral* if Id is an atom, and *atomic* if its boolean reduct is an atomic boolean algebra.

We write **RA** for the class of relation algebras. **RA** is a variety (it can be equationally axiomatised).

2.2 Representations of relation algebras

A relation algebra $\mathcal{A}$ is intended to be a collection of binary relations. So let us regard $\mathcal{A}$ as a binary relational signature (similarity type). That is, we regard each element $r \in \mathcal{A}$ as a binary relation symbol. This will not lead to ambiguity: for $r \in \mathcal{A}$, if we write $r(x, y)$, we are thinking of r as a relation symbol, but if we write simply r, we are thinking of r as an element of $\mathcal{A}$. The following definition is now natural to make.

Definition 2.2 1. A *representation* of a relation algebra $\mathcal{A}$ is a model of the theory $T_{\mathcal{A}}$ consisting of:

$$\begin{array}{lcl}
\forall x, y[Id(x,y) \leftrightarrow (x=y)] & : & \text{for each } r,s,t \in \mathcal{A} \text{ with:} \\
\exists x, y\, r(x,y) & : & \mathcal{A} \models r \neq 0 \\
\forall x, y[r(x,y) \leftrightarrow s(x,y) \vee t(x,y)] & : & \mathcal{A} \models r = s \vee t \\
\forall x, y[1(x,y) \rightarrow (r(x,y) \leftrightarrow \neg s(x,y))] & : & \mathcal{A} \models r = \neg s \\
\forall x, y[r(x,y) \leftrightarrow s(y,x)] & : & \mathcal{A} \models r = s^{\smile} \\
\forall x, y[r(x,y) \leftrightarrow \exists z(s(x,z) \wedge t(z,y))] & : & \mathcal{A} \models r = s;t
\end{array}$$

2. A relation algebra $\mathcal{A}$ is said to be *representable* if it has a representation — i.e., if $T_{\mathcal{A}}$ is consistent.

3. We write **RRA** for the class of all representable relation algebras.

4. A representation M of a relation algebra $\mathcal{A}$ is said to be *complete* if for all $x, y \in M$ such that $M \models 1(x,y)$, there is an atom $a \in \mathcal{A}$ such that $M \models a(x,y)$. [1]

The axioms in definition 2.2 above are not enough to guarantee that a relation algebra is representable, so that we have a 'real' collection of binary relations:

Theorem 2.3 (Jónsson & Tarski [JT], Monk [Mo]) *The class* **RRA** *is a variety (equationally axiomatised). However, it is not finitely axiomatisable.*

This is the chief negative result in the area. (Compare with the boolean algebra case, where *every* boolean algebra is representable as a set algebra.)

2.3 Atom structures

These are our main topic in this paper. We begin by defining them, and then we discuss their links with relation algebras.

Definition 2.4 An *atom structure* (cf. [M2, theorem 2.2]) is a structure $\mathcal{C} = (C, ^{\smile}, R, Id)$, where $^{\smile}$ is a unary function (for converse), R a ternary relation (with meaning $R(a,b,c)$ iff $a;b \geq c^{\smile}$), and Id a unary relation (with meaning $Id(a)$ iff $a \leq Id$), satisfying the following first-order axioms:

1. $\forall x(x^{\smile\smile} = x)$.

2. $\forall xy(x = y \leftrightarrow \exists z(Id(z) \wedge R(x,z,y^{\smile})))$.

3. R is invariant under 'Peircean transforms':

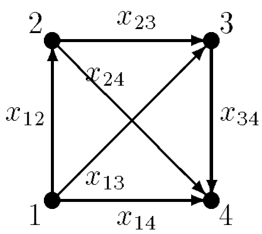
$$\forall xyz(R(x,y,z) \rightarrow R(y,z,x) \wedge R(z^{\smile}, y^{\smile}, x^{\smile})).$$

4. (Associativity)

$$\begin{array}{lll}
\forall x_{12}x_{23}x_{34}x_{14} & \Big(\exists x_{13}(R(x_{12}, x_{23}, x_{13}^{\smile}) \wedge R(x_{13}, x_{34}, x_{14}^{\smile})) \\
& \rightarrow \quad \exists x_{24}(R(x_{23}, x_{34}, x_{24}^{\smile}) \wedge R(x_{12}, x_{24}, x_{14}^{\smile}))\Big).
\end{array}$$

Here, the x_{ij} are arbitrary distinct variables. See the diagram.

[1] These are sometimes called *atomic* representations, the word 'complete' being used for representations in which arbitrary disjunctions are respected. The equivalence of the two notions was proved in [HH2]. Only atomic relation algebras can have complete representations. However, not every atomic representable relation algebra does have a complete representation: in fact, the class of all relation algebras having a complete representation is not an elementary class. For more information, see [HH2].



An atom structure is intended to abstract the properties induced by a relation algebra on its atoms. So let us check that atom structures in the above sense do arise from relation algebras in the natural way.

Definition 2.5 Let $\mathcal{A}$ be an atomic relation algebra. The *atom structure of* $\mathcal{A}$ is the structure $\mathcal{C}$ whose domain is $\{\text{atoms of } \mathcal{A}\}$, with R, Id, and $\smile$ interpreted as suggested above. That is, for all atoms x, y, z of $\mathcal{A}$,

1. $\mathcal{C} \models Id(x)$ iff $\mathcal{A} \models x \leq Id$
2. $\mathcal{C} \models x^\smile = y$ iff $\mathcal{A} \models x^\smile = y$
3. $\mathcal{C} \models R(x, y, z)$ iff $\mathcal{A} \models z^\smile \leq x; y$.

Any atomic relation algebra has an atom structure. Moreover:

Proposition 2.6 *Let $\mathcal{A}$ be any atomic relation algebra. Let $\mathcal{C}$ be the atom structure of $\mathcal{A}$, as defined in definition 2.5. Then $\mathcal{C}$ is an atom structure in the sense of definition 2.4.*

Proof. Exercise — compare [M2, theorem 2.2]. ■

Conversely, if $\mathcal{C}$ is an atom structure, with domain C, then there exist atomic relation algebras $\mathcal{A}$ with atom structure (isomorphic to) $\mathcal{C}$. The two most important examples are:

- the *full power set algebra* over $\mathcal{C}$:

  $$(\wp C, \neg, \cup, \varnothing, C, {}^\smile, ;, I),$$

  where, for $s, t \subseteq C$, $s^\smile = \{a^\smile : a \in s\}$, $s; t = \{a : \mathcal{C} \models R(b, c, a^\smile)$ for some $b \in s, c \in t\}$, and $I = \{a \in C : \mathcal{C} \models Id(a)\}$. If we are willing to be a little lax, we could also call it the *complex algebra of* $\mathcal{C}$. One may check [M2, theorem 2.2(5)] that it is a relation algebra whose atom structure is isomorphic to $\mathcal{C}$; the isomorphism is $\{a\} \mapsto a$.

  Any atomic relation algebra $\mathcal{A}$ with atom structure $\mathcal{C}$ embeds into the full power set algebra over $\mathcal{C}$, via $r \mapsto \{c \in C : \mathcal{A} \models c \leq r\}$. It follows that if $\mathcal{A}$ and $\mathcal{B}$ are atomic relation algebras with the same boolean reduct and the same atom structure, then $\mathcal{A} = \mathcal{B}$.

- the *term algebra* over $\mathcal{C}$. This is the subalgebra of the power set algebra $\mathcal{P}$ over $\mathcal{C}$ generated by the atoms of $\mathcal{P}$. Hence, its domain is $\{t^{\mathcal{P}}(\bar{a}) : t(\bar{x})$ a relation algebra term, $\bar{a}$ a tuple of atoms of $\mathcal{P}\}$.

  For reference, we list the terms $t(\bar{x})$ (with free variables among $\bar{x} = x_1, \ldots, x_n$) of the signature of relation algebras. They are defined by induction, of course, as follows.

  - $0, 1, Id$ are terms.
  - Any variable is a term.
  - If $t(\bar{x}), u(\bar{y})$ are terms, then the following are also terms: $\neg t(\bar{x})$, $(t \wedge u)(\bar{x}\bar{y})$, $t^\smile(\bar{x})$,[2] and $(t; u)(\bar{x}\bar{y})$.

[2] Notice that any term of the signature of atom structures is of the form $x^{\smile\smile\cdots\smile}$, where x is a variable. So it is also a relation algebra term, with the same 'meaning'. This will be convenient later.

The term algebra over $\mathcal{C}$ embeds naturally into any atomic relation algebra $\mathcal{A}$ with atom structure $\mathcal{C}$, via the embedding $t^{\mathcal{P}}(\{a_1\}, \ldots, \{a_n\}) \mapsto t^{\mathcal{A}}(a_1, \ldots, a_n)$. It can be checked that this is a well-defined relation algebra embedding. Hence, up to isomorphism, the term algebra over $\mathcal{C}$ is the unique minimal atomic relation algebra with atom structure $\mathcal{C}$. The term algebra and the full power set algebra over $\mathcal{C}$ are equal iff $\mathcal{C}$ is finite.

An intrinsic definition of the term algebra over an atom structure $\mathcal{C}$ can also be given, along the lines of lemma 3.6 below.

Thus, we have seen:

Proposition 2.7 1. *An atom structure can be obtained canonically from the atoms of any atomic relation algebra.*

2. *Given any atom structure $\mathcal{C}$, there exist atomic relation algebras whose atom structure is isomorphic to $\mathcal{C}$.*

3. *Any atomic relation algebra $\mathcal{A}$ with atom structure $\mathcal{C}$ embeds into the full power set algebra over $\mathcal{C}$, via $r \mapsto \{c \in C : \mathcal{A} \models c \leq r\}$.*

4. *The term algebra of an atom structure $\mathcal{C}$ embeds into any atomic relation algebra with atom structure $\mathcal{C}$, via $t^{\mathcal{P}}(\{a_1\}, \ldots, \{a_n\}) \mapsto t^{\mathcal{A}}(a_1, \ldots, a_n)$.*

5. *If $\mathcal{A}$ and $\mathcal{B}$ are atomic relation algebras with the same boolean reduct and the same atom structure, then $\mathcal{A} = \mathcal{B}$.*

Note that part 5 formalises our remark in the introduction to the effect that a relation algebra is determined by its boolean reduct and its atom structure.

2.4 Representability of atom structures; the main theorem

We now come to the crucial issue (for this paper) of representability of atom structures.

Definition 2.8 Let $\mathcal{C}$ be an atom structure.

1. We say that $\mathcal{C}$ is *weakly representable* if there exists an atomic $\mathcal{A} \in \mathbf{RRA}$ with atom structure $\mathcal{C}$.

2. We say that $\mathcal{C}$ is *strongly representable* if every atomic $\mathcal{A} \in \mathbf{RA}$ with atom structure $\mathcal{C}$ is representable.

Recall (cf. theorem 2.3) that any subalgebra of a representable relation algebra is also representable. So, given proposition 2.7, the following should be evident.

Lemma 2.9 *Let $\mathcal{C}$ be an atom structure.*

1. *If $\mathcal{C}$ is strongly representable then it is weakly representable.*

2. *$\mathcal{C}$ is weakly representable iff its term algebra is representable.*

3. *$\mathcal{C}$ is strongly representable iff the full power set algebra over $\mathcal{C}$ is representable.*

In this paper we will show:

Theorem 2.10 1. If $\mathcal{C}$ is a finite atom structure, weak and strong representability coincide for $\mathcal{C}$.

2. If the atom structure $\mathcal{C}$ satisfies the Lyndon conditions (see below), then it is strongly representable. The converse fails.

3. The class of weakly representable atom structures is elementary.[3]

4. The class of weakly representable atom structures is not definable by any sentence of $L^{\omega}_{\infty\omega}$.[4] (Hence it is not finitely axiomatisable in first-order logic.)

5. There exist weakly representable atom structures that are not strongly representable.

6. The class of strongly representable atom structures is also not definable by any sentence of $L^{\omega}_{\infty\omega}$, even over the class of weakly representable atom structures.

The proofs of parts 1 and 2 of this theorem can be found at the end of this section and in the appendix. Part 3 is proved in section 3. Parts 4, 5, and 6 are proved in the final section, in corollaries 8.8, 8.10, and 8.12, respectively.

The following was asked in [M2, problem 2.3], and, as far as we know, it remains open.

Question 2.11 Do the strongly representable atom structures form an elementary class?

We can also ask:

Question 2.12 Is the class of all strongly representable atom structures closed under elementary equivalence? Is the notion of 'strongly representable' set-theoretically absolute?

2.5 Lyndon conditions and strong representability

The *Lyndon conditions* are certain first-order conditions λ_i $(i < \omega)$ on a relation algebra, given by R. Lyndon [L1, L2] in the 1950s. Importantly, all quantifiers in the Lyndon conditions are relativised to atoms, so we can equally regard them as first-order conditions on an atom structure. They are rather complicated to write out, so we will not do so; see proposition 6.7, and [HH1,HH2], for an equivalent definition. Another equivalent characterisation, established in [HH2], is as follows.

Fact 2.13 *An atomic relation algebra $\mathcal{A}$ satisfies the Lyndon conditions iff $\mathcal{A}$ is elementarily equivalent to a relation algebra that has a complete representation.*

Since any representation of a finite relation algebra is necessarily complete, and since, on finite structures, elementary equivalence is the same as isomorphism, a finite relation algebra is representable if and only if it satisfies the Lyndon conditions. It follows that a finite atom structure is strongly representable iff it satisfies the Lyndon conditions.

In the light of this and theorem 2.10(2), one might hope to answer question 2.11 by proving that the Lyndon conditions themselves axiomatise the strongly representable atom structures. However, the Lyndon conditions are not in general necessary for an atom structure to be strongly representable. An example is obtainable from a construction of Maddux [M1]: see the appendix.

2.6 Proof of the first two parts of the main theorem

We can prove the first two parts of theorem 2.10 now. (1) is immediate, because the term algebra and full power set algebra of a *finite* atom structure are one and the same. There is, up to isomorphism, a unique relation algebra with a given finite atom structure.

For (2), let $\mathcal{A}$ be any atomic relation algebra whose atom structure satisfies the Lyndon conditions. We show that $\mathcal{A}$ is representable. By fact 2.13, there must exist a representable relation algebra $\mathcal{B}$ elementarily equivalent to $\mathcal{A}$. As **RRA** is elementary, and $\mathcal{B} \in$ **RRA**, we see that $\mathcal{A} \in$ **RRA**, as required. ■

[3] We will explicitly axiomatise it.

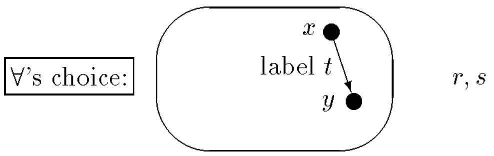
[4] Here, the signature L is that of atom structures: $\{R, \smile, Id\}$, plus equality. $L_{\infty\omega}$ is the closure of the atomic formulas under $\neg, \bigvee, \bigwedge, \forall, \exists$, where $\bigwedge, \bigvee$ may be over arbitrary sets of formulas. $L^{\omega}_{\infty\omega}$ is the class of all $L_{\infty\omega}$-formulas written with only finitely many variables. See §8 for more details.

3 Axiomatising the weakly representable atom structures

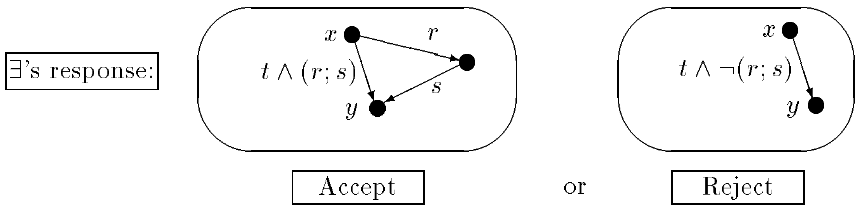
We now find axioms for the class of weakly representable atom structures. As the proofs may appear somewhat technical, it may be worth describing roughly what we will do.

In [HH1], the class of all representable relation algebras was axiomatised using a game-theoretic argument. Given a countable relation algebra, $\mathcal{A}$, two players, $\forall$ (male) and $\exists$ (female), played a game to build and refine an approximation to a representation of $\mathcal{A}$. This 'approximation' was called a network, and consisted simply of a finite set of nodes, each pair of nodes (or 'edge') being labelled by an element of $\mathcal{A}$, and the whole satisfying certain consistency conditions corresponding to the 'universal' part of the theory $T_{\mathcal{A}}$ of definition 2.2. In particular, every 'triangle' had to be consistently labelled: if edges $(x, y), (y, z), (x, z)$ were labelled by elements $r, s, t \in \mathcal{A}$, respectively, then we required that $r; s \wedge t \neq 0$ in $\mathcal{A}$.

In each round of the game, $\forall$ selected, at whim, an edge (x, y) in the existing network, and two elements $r, s \in \mathcal{A}$.



This represented a challenge to $\exists$ to add on a 'triangle' to the edge chosen by $\forall$, with one edge labelled r, the other, s. $\exists$ had to decide whether to accept or reject this challenge, bearing in mind the consequences for network consistency both at the time and in later rounds. If she accepted, she had to define a new network including such a triangle. Otherwise, she had to claim that it was impossible to add the triangle, by altering the label — t, say — on the chosen edge to be disjoint from $r; s$.



If her choice resulted in an inconsistent network, she lost; otherwise, the players went on to the next round, using the new network set down by $\exists$. $\exists$ won the game if she never lost any round; otherwise, $\forall$ won.

It was shown that $\mathcal{A}$ was representable iff $\exists$ could win any game like this of any fixed finite length. The chief idea of the proof was that the addition of triangles forced the $\forall\exists$ conditions in the theory $T_{\mathcal{A}}$ to be met, so that a representation was formed. Moreover, a first-order sentence ψ_n on $\mathcal{A}$ was given, expressing directly that $\exists$ could win the network game of length n. Using these two results, it was shown that **RRA** is axiomatised by $\{\psi_n : n < \omega\}$, plus the relation algebra axioms. The sentences ψ_n were universal ($\forall_1$), their universal quantifiers corresponding to the choices of elements r, s made by $\forall$ in his moves.

This 'game' approach works for atom structures, too, but some small modifications are needed.

We know (lemma 2.9) that an atom structure $\mathcal{C}$ is weakly representable iff its term algebra is representable. So we will use the game on the term algebra, $\mathcal{A}$, over $\mathcal{C}$. However, the ψ_n are axioms on the algebra, $\mathcal{A}$, whereas we are looking for axioms on the atom structure, $\mathcal{C}$. So we will label our networks not by elements of $\mathcal{A}$, but by sets of atoms from $\mathcal{C}$: recall that $\mathcal{A}$ is a subalgebra of the power set algebra over $\mathcal{C}$.

The axioms ψ_n involved universally quantifying over all elements of $\mathcal{A}$, to capture the choices that $\forall$ might make in the game. But now, $\forall$ chooses *sets* of atoms of $\mathcal{C}$, not elements of $\mathcal{A}$. To represent his choices syntactically in the axioms, we will show that each set of atoms corresponding to a term algebra element is actually first-order-definable in $\mathcal{C}$. By restricting $\forall$'s choice in each round to the sets definable by formulas of limited complexity, we will be able to represent all his possible choices by a first-order sentence. The permitted complexity will increase according to how may moves of the game remain. This weakens the constraint on $\forall$: he can choose any set he likes, if the game is long enough. It weakens it sufficiently for us to prove that $\mathcal{A}$ is representable iff $\exists$ can win any finite-length game against $\forall$ when he is restricted in this way. This is the content of proposition 3.4, below.

However, this all requires that the atom structure $\mathcal{C}$ is countable. We have to allow for uncountable atom structures, too. This occupies the first part of the full proof, below.

3.1 Reducing to the countable case

Lemma 3.1 *It suffices to find axioms characterising the countable weakly representable atom structures.*

Proof. Let $\mathcal{C}$ be an arbitrary atom structure, and $\mathcal{A}$ the term algebra over $\mathcal{C}$. Take a countable elementary substructure $\mathcal{A}_0 \preceq \mathcal{A}$. (This can be done using the downward Löwenheim–Skolem theorem: see [Hodg2, corollary 3.1.5].) Then $\mathcal{A}_0$ is atomic. Let its atom structure be $\mathcal{C}_0$.

Claim 1. As atom structures, $\mathcal{C}_0 \preceq \mathcal{C}$.
Proof of claim. It is easily seen that for any formula $\varphi(\bar{x})$ of atom structures there is a formula $\varphi\!\uparrow\!(\bar{x})$ of relation algebras such that for *any* relation algebra $\mathcal{A}'$ with atom structure $\mathcal{C}'$, and any tuple $\bar{a} \in \mathcal{C}'$ of atoms, $\mathcal{C}' \models \varphi(\bar{a}) \iff \mathcal{A}' \models \varphi\!\uparrow\!(\bar{a})$. (To obtain $\varphi\!\uparrow$, just relativise φ to the set of atoms — this set is definable in a relation algebra by the formula $\forall y(y < x \leftrightarrow y = 0)$ — and replace all atomic subformulas of the form $Id(t)$ by $t \leq Id$, and $R(t, u, v)$ by $t; u \geq v^{\smile}$, where t, u, v are terms.) Hence, for any atom structure formula $\varphi(\bar{x})$, and any tuple $\bar{a} \in \mathcal{C}_0$, we have $\mathcal{C} \models \varphi(\bar{a})$ iff $\mathcal{A} \models \varphi\!\uparrow\!(\bar{a})$ iff $\mathcal{A}_0 \models \varphi\!\uparrow\!(\bar{a})$ (because $\mathcal{A}_0 \preceq \mathcal{A}$), iff $\mathcal{C}_0 \models \varphi(\bar{a})$. This proves the claim.

Claim 2. $\mathcal{A}_0$ is the term algebra over $\mathcal{C}_0$.
Proof of claim. If $r \in \mathcal{A}_0$, then because $\mathcal{A}$ is the term algebra over $\mathcal{C}$, there is a relation algebra term $t(\bar{x})$ and atoms $\bar{c} \in \mathcal{C}$ such that $\mathcal{A} \models r = t(\bar{c})$. (Of course, t depends on r.) So $\mathcal{A} \models \exists$ atoms $\bar{x}(r = t(\bar{x}))$. Because $\mathcal{A}_0 \preceq \mathcal{A}$, we have $\mathcal{A}_0 \models \exists$ atoms $\bar{x}(r = t(\bar{x}))$, also. So there are atoms $\bar{d} \in \mathcal{C}_0$ such that $\mathcal{A}_0 \models r = t(\bar{d})$. Thus, any element of $\mathcal{A}_0$ is generated by atoms, so it is the term algebra over $\mathcal{C}_0$.

Now suppose that we have found a first-order theory Σ such that any *countable* atom structure is a model of Σ iff it is weakly representable. Then, using lemma 2.9, $\mathcal{C}$ is weakly representable iff $\mathcal{A}$ is representable, iff $\mathcal{A}_0$ is representable (as $\mathcal{A}_0 \preceq \mathcal{A}$ and **RRA** is elementary), iff $\mathcal{C}_0$ is weakly representable (by claim 2), iff $\mathcal{C}_0 \models \Sigma$ (as $\mathcal{C}_0$ is countable), iff $\mathcal{C} \models \Sigma$ (by claim 1). This proves the lemma. ∎

So we fix, until the end of section 3, a countable atom structure $\mathcal{C}$, with term algebra $\mathcal{A}$. We identify $a \in \mathcal{C}$ with $\{a\} \in \mathcal{A}$; the domain of $\mathcal{C}$ is now contained in that of $\mathcal{A}$. We will find axioms that are true in $\mathcal{C}$ iff $\mathcal{A}$ is representable.

3.2 Networks

We will use 'forcing conditions' called *networks.* A network is an approximation to a representation of $\mathcal{A}$.

Definition 3.2 A *pre-network* is a pair $N = (N_1, N_2)$, where N_1 is a non-empty finite set of nodes, and $N_2 : N_1 \times N_1 \to \mathcal{A}$ is a map assigning an element of $\mathcal{A}$ to each pair of nodes.

A *pre-network* $N = (N_1, N_2)$ is said to be a *network* if, for all nodes x, y, z:

1. $N_2(x,x) \leq Id$
2. $(N_2(x,y); N_2(y,z)) \wedge N_2(x,z) \neq 0$.

To free up some suffixes, from now on we will write simply N for any of N, N_1 and N_2, distinguishing them by context. We will sometimes refer to $N(x,y)$ as being the *labelling* of the *edge* (x,y) of N.

For networks N, N', the notation $N \subseteq N'$ will mean that every node of N is a node of N', and $\mathcal{A} \models N'(x,y) \leq N(x,y)$ for all nodes x, y of N.

3.3 The game G_n

We define a game, $G_n(N)$ (where N is a finite pre-network), of length $n \leq \omega$, between players $\forall$ and $\exists$, to build a chain $N = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n$ of networks if n is finite, and an infinite chain $N = N_0 \subseteq N_1 \subseteq \cdots$ if $n = \omega$.

There are n rounds, the network N_i being built from N_{i-1} in round i. In round i, with $1 \leq i \leq n$, $i < \omega$, if N_{i-1} has already been constructed,

- $\forall$ chooses $x, y \in N_{i-1}$, and elements $r, s \in \mathcal{A}$ definable from atoms by relation algebra terms each with at most $n-i$ function symbols — that is, at most $n-i$ occurrences of symbols from $\{\wedge, \neg, 0, 1, \smile, ;, Id\}$. For example, if $n = i$ then $\forall$ must choose *atoms* r, s, or else $0, 1, Id$. Here, $\omega - i = \omega$; this means that in the game $G_\omega(N)$ there is no restriction whatever on $\forall$'s choices of elements of $\mathcal{A}$.
- $\exists$ must respond with a pre-network $N_i \supseteq N_{i-1}$ such that one of the following holds:

  (reject) N_i is the same as N_{i-1}, except that $N_i(x,y) = N_{i-1}(x,y) \wedge \neg(r;s)$;

  (accept) – The nodes of N_i are those of N_{i-1} plus a new node, z
    - $N_i(x,z) = r$, $N_i(z,y) = s$, and $N_i(z,z) = Id$
    - $N_i \lceil dom(N_{i-1}) = N_{i-1}$ except that $N_i(x,y) = N_{i-1}(x,y) \wedge (r;s)$
    - all other node pairs in N_i not yet determined are labelled by 1.

$\exists$ wins the play if all the $N_0, N_1, \ldots$ played are networks. Otherwise, $\forall$ wins the play.

Informally, in each round, $\forall$ challenges $\exists$ to add a certain triangle to an edge, and $\exists$ must either do so, or claim instead that the edge is outside one side of the triangle. This is as we described it above.

A *strategy* for $\exists$ is simply a set of rules telling her what to do in each position — reject or accept. A strategy is *winning* if, if $\exists$ follows its rules, she will win regardless of how $\forall$ plays. As a trivial example, if N is a pre-network, $\exists$ has a winning strategy for $G_0(N)$ iff N is a network.

3.4 Games and representations

Here, we relate these games to the weak representability of $\mathcal{C}$.

Definition 3.3 An *identity atom* is an element $a \in \mathcal{C}$ such that $\mathcal{C} \models Id(a)$. If a is an identity atom, we write I_a for the network with a single node, 0, the labelling being given by $I_a(0,0) = a$.

Proposition 3.4 *$\mathcal{A} \in$ **RRA** iff $\exists$ has a winning strategy in $G_n(I_a)$ for all $n < \omega$ and all identity atoms $a \in \mathcal{C}$.*

Proof. $\Rightarrow$: If M is a representation of $\mathcal{A}$, $\exists$ can use M to help her decide whether to accept or reject in each round. In more detail, she preserves the condition that for each i there's a map $' : N_i \to M$ with $M \models [N_i(x,y)](x', y')$ for all nodes x, y of N_i.

Recall that $M \models T_{\mathcal{A}}$ (see definition 2.2). So $M \models \exists xy\, a(x,y)$. As a is an identity atom, it follows that if $M \models a(x,y)$ then $x = y$. $\exists$ can therefore choose an element $0' \in M$ with $M \models a(0', 0')$. This choice sets up the map $' : I_a \to M$ initially.

In round i, if $\forall$ plays $\langle x, y, r, s \rangle$:

1. If $M \models [r;s](x',y')$ then $\exists$ accepts, and extends the map $'$ by mapping the z of the game definition above to any $z' \in M$ with $M \models r(x',z') \wedge s(z',y')$. Such a z' exists, because $M \models T_{\mathcal{A}}$.
2. If $M \not\models [r;s](x',y')$, she rejects. Since $M \models N_{i-1}(x',y')$, we have $M \models 1(x',y')$ and so $M \models [\neg(r;s)](x',y')$. This keeps the required properties of the map $'$ into the next round.

This clearly wins the game for $\exists$.

$\Leftarrow$: First, we show:

Claim 1. $\exists$ has a winning strategy in $G_\omega(I_a)$.
Proof of claim. Let N be a finite network, and assume that $\exists$ has a winning strategy σ_n in $G_n(N)$, for infinitely many $n < \omega$. Let $\forall$ play $x, y \in N$ and arbitrary elements $r, s \in \mathcal{A}$ as his first move in $G_\omega(N)$. Choose n_0 so that r, s are definable in $\mathcal{A}$ from atoms by terms with at most n_0 function symbols. Then $\forall$'s move is a valid move of $G_n(N)$ for all $n \geq n_0$, and so $\exists$ may apply σ_n for infinitely many $n \geq n_0$. These strategies say whether to accept or reject $\forall$'s triangle. If infinitely many of them tell her to accept, then she accepts as her move in $G_\omega(N)$. If not, then infinitely many will advise rejection, and she rejects in $G_\omega(N)$. In either case, she survives the round, and, moreover, arrives in a position N' where infinitely many of the winning strategies σ_n are still running. Thus, she has a winning strategy in $G_n(N')$ for infinitely many n.

$\exists$'s strategy in any $G_\omega(I_a)$ is now as follows.

> 'If the current network is N, assume inductively that I have a winning strategy in $G_n(N)$ for infinitely many n. I play to preserve this condition.'

By assumption, $\exists$ has a winning strategy for $G_n(I_a)$ for all $n < \omega$ and all identity atoms a, so the condition holds initially. We outlined above how she can keep it. Hence she can survive each round, and so win $G_\omega(I_a)$. This is true for any identity atom a. The claim is proved.

Now, in a play of $G_\omega(I_a)$, let $\exists$ use her winning strategy, but also persuade $\forall$ to play, in some round, $\langle x, y, r, s\rangle$, for each pair x, y of nodes that arise during the game, and for every $r, s \in \mathcal{A}$. This is possible, as both $\mathcal{A}$ and the networks N_i ($i < \omega$) in the game are countable. Let N^* be the set of nodes built during play: $N^* = \bigcup_{i<\omega} N_i$. Define a binary relation $\sim$ on N^* by:

$$x \sim y \quad \text{iff} \quad \exists i < \omega (x, y \in N_i \text{ and } N_i(x,y) \leq Id).$$

To reduce clutter, when we write $N_i(x,y)$, below, it will be implicit that $x, y \in N_i$.

Claim 2. $\sim$ is an equivalence relation.
Proof of claim. Reflexivity is clear, since if $x \in N^*$ then $N_i(x,x) \leq Id$ for any i with $x \in N_i$. We check symmetry and transitivity by proving that for all $x, y, z \in N^*$, if $x \sim y$ and $z \sim y$ then $x \sim z$. By taking $x = y$, this gives symmetry of $\sim$, and then transitivity of $\sim$ follows easily.

Suppose, then, that there are $i, i' < \omega$ with $N_i(x,y) \leq Id$ and $N_{i'}(z,y) \leq Id$. By assumption, $\forall$ played $\langle x, z, Id, Id\rangle$ in some round, j, in the game. Choose $k \geq i, i', j$. If $\exists$ rejected $\forall$'s challenge in round j, then $N_j(x,z) = N_{j-1}(x,z) \wedge \neg(Id;Id) \leq \neg Id$. Hence, because $N_k \supseteq N_i, N_{i'}, N_j$, and ';' is monotonic, we would have $N_k(x,y) \wedge (N_k(x,z); N_k(z,y)) \leq Id \wedge (\neg Id; Id) = 0$. So N_k is not a network, contradicting the fact that $\exists$ won $G_\omega(I_a)$. Hence $\exists$ accepted in round j, so $N_j(x,z) \leq Id; Id = Id$ and $x \sim z$, as required. This proves the claim.

For a node $x \in N^*$, write $x/{\sim}$ for the $\sim$-equivalence class of x. Define an $\mathcal{A}$-structure M_a as follows. The domain of M_a is $N^*/{\sim}$, the set of $\sim$-classes in N^*. Relations are defined on M_a by:

$$\text{for any } r \in \mathcal{A}, \quad M_a \models r(x/{\sim}, y/{\sim}) \text{ iff } \exists i < \omega\, (N_i(x,y) \leq r).$$

It can be checked that this is well-defined, the argument being similar to that in claim 2.

Claim 3. M_a satisfies all the axioms of the theory $T_{\mathcal{A}}$ of definition 2.2, except perhaps those of the form $\exists xy\, r(x,y)$ for non-zero $r \in \mathcal{A}$.

Proof of claim. We will check that

$$M_a \models \forall x, y[r(x,y) \leftrightarrow \exists z(s(x,z) \wedge t(z,y))] \quad \text{whenever } \mathcal{A} \models r = s;t,$$

the arguments to establish the other axioms being similar. Let $r = s;t$ in $\mathcal{A}$, let $x, y, z \in N^*$, and suppose first that $M \models s(x/\sim, z/\sim) \wedge t(z/\sim, y/\sim)$. So there are $i, i' < \omega$ with $N_i(x,z) \leq s$ and $N_{i'}(z,y) \leq t$. $\forall$ played $\langle x, y, s; t, Id\rangle$ in some round, j, of $G_\omega(I_a)$; choose $k \geq i, i', j$. Then if $\exists$ rejected in round j, we would have $N_j(x,y) \leq \neg((s;t); Id) = \neg(s;t)$. So $N_k(x,y) \wedge (N_k(x,z); N_k(z,y)) \leq \neg(s;t) \wedge (s;t) = 0$. This means that N_k is not a network, and contradicts the fact that $\exists$ won. Thus, she must have accepted, so that $N_j(x,y) \leq (s;t); Id = s;t$, and thus, by definition, $M \models r(x/\sim, y/\sim)$.

Conversely, assume that $M \models r(x/\sim, y/\sim)$, so that $N_i(x,y) \leq s;t$ for some $i < \omega$. Suppose that $\forall$ played $\langle x, y, s, t\rangle$ in round j. If $\exists$ rejected, then $N_j(x,y) \leq \neg(s;t)$, so $N_k(x,y) \leq N_i(x,y) \wedge N_j(x,y) \leq s;t \wedge \neg(s;t) = 0$ for any $k \geq i, j$. This contradicts the fact that N_k is a network. So she must have accepted. This means that there is $z \in N_j$ with $N_j(x,z) = s$ and $N_j(z,y) = t$. Hence $M \models s(x/\sim, z/\sim)$ and $M \models t(z/\sim, y/\sim)$, as required. This ends our proof of the claim.

However, M may not satisfy the axioms $\exists xy\ r(x,y)$ for all non-zero r. We must fix this final problem. Build M_a as above for each identity atom a of $\mathcal{C}$. Let M be the disjoint union of the M_a, defined as an $\mathcal{A}$-structure in the conventional model-theoretic fashion. So the domain of M is the disjoint union of the domains of the M_a (for all identity atoms a of $\mathcal{A}$), and for all $x, y \in M$ we let $M \models r(x,y)$ iff $x, y \in M_a$ and $M_a \models r(x,y)$ for some a.

Claim 4. $M \models \exists xy\ r(x,y)$ for all non-zero $r \in \mathcal{A}$.
Proof of claim. Given such an r, we have $Id; r \wedge r \neq 0$, so by the triangle axiom on $\mathcal{A}$, $r; r^\smile \wedge Id \neq 0$. This implies that there is an identity atom $e \in \mathcal{C}$ with $\mathcal{A} \models e \leq r; r^\smile$. Consider the game $G_\omega(I_e)$. We know that $N_0(0,0) = e$, so $M_e \models r; r^\smile(0/\sim, 0/\sim)$. By claim 3, there is $z \in M_e$ with $M_e \models r(0/\sim, z)$. It follows that $M \models \exists xy\ r(x,y)$, proving the claim.

The form of the remaining axioms of $T_\mathcal{A}$ guarantees that they also hold in M, given that they hold in every M_a. Hence $M \models T_\mathcal{A}$, so M is a representation of $\mathcal{A}$. ■

3.5 Axioms for the games

Here, we write down axioms expressing that $\exists$ can win the games. Unfortunately, but necessarily, the work is rather syntactic and technical. $\mathcal{A}$ continues to be the term algebra of the countable atom structure $\mathcal{C}$.

Remark 3.5 When we write a relation algebra term as '$t(\bar{x})$', we will mean that the variables occurring in t all occur in the tuple $\bar{x}$ of variables. Not every variable in $\bar{x}$ need actually crop up in t. This is the usual convention. The same goes for first-order formulas (such as $\psi_t(\bar{x}, y)$, below).

We will soon need the following technical lemma.

Lemma 3.6 *For any relation algebra term $t(\bar{x})$, there is a formula $\psi_t(\bar{x}, y)$ of the language of atom structures, such that for all tuples $\bar{a}$ of atoms of $\mathcal{A}$, and all atoms b of $\mathcal{A}$,*

$$\mathcal{A} \models b \leq t(\bar{a}) \iff \mathcal{C} \models \psi_t(\bar{a}, b).$$

The formula ψ_t depends only on t, not on $\mathcal{C}$.

Proof. A simple induction on t. We let $\psi_x = (y = x)$, $\psi_0 = (y \neq y)$, $\psi_1 = (y = y)$, and $\psi_{Id} = Id(y)$. Then, given $\psi_t(\bar{x}, y), \psi_u(\bar{x}, y)$, which we can assume involve the same tuple $\bar{x}$, since not all the variables of $\bar{x}$ need actually be used, we define:

1. $\psi_{t \wedge u}(\bar{x}, y) = \psi_t(\bar{x}, y) \wedge \psi_u(\bar{x}, y)$
2. $\psi_{\neg t}(\bar{x}, y) = \neg \psi_t(\bar{x}, y)$

3. $\psi_{t^\smile}(\bar{x}, y) = \psi_t(\bar{x}, y^\smile)$

4. $\psi_{t;u}(\bar{x}, y) = \exists zv(R(z, v, y^\smile) \wedge \psi_t(\bar{x}, z) \wedge \psi_u(\bar{x}, v))$.

■

The following technical definition will also help.

Definition 3.7 1. A *term network* is a pair $N = (N_1, N_2)$, where N_1 is a non-empty finite set of nodes, and N_2 is a map from $N_1 \times N_1$ into the set of relation algebra terms. As before, we write N for any of N, N_1, N_2.

2. Given a term network N and an assignment ι of the variables of the terms in N to elements of $\mathcal{C}$, we can obtain a pre-network N^ι as follows. The nodes of N^ι are the nodes of N, and the label on the edge (x, y) of N^ι is the value of the term $N(x, y)$ in $\mathcal{A}$ under the assignment ι. N^ι is a pre-network; it may or may not be a network.

3. If N is a term network, $x, y \in N$ are nodes, and t, u are any terms, we write $\mathsf{Acc}(N, x, y, t, u)$ for the term network N' obtained from N by adding a new node, z, say, and defining

   - $N'(x, y) = N(x, y) \wedge t; u$
   - $N'(z, z) = Id$
   - $N'(x, z) = t$
   - $N'(z, y) = u$
   - $N'(x', y') = N(x', y')$ for all pairs $(x', y') \in (N \times N) \setminus \{(x, y)\}$.
   - all labels of node pairs of N' not yet specified are 1.

   We write $\mathsf{Rej}(N, x, y, t, u)$ for the term network N' which is the same as N, except that $N'(x, y) = N(x, y) \wedge \neg(t; u)$.

To axiomatise the weakly representable atom structures, it only remains to prove:

Proposition 3.8 *For any $n < \omega$, there is a formula $\xi_n(x)$ (in the language of atom structures) such that for any identity atom a of $\mathcal{C}$, $\mathcal{C} \models \xi_n(a)$ if and only if $\exists$ has a winning strategy in the game $G_n(I_a)$. The ξ_n do not depend on $\mathcal{C}$.*

Proof. We will actually define formulas $\xi_n[N]$, where N is any term network. The free variables of $\xi_n[N]$ will include the variables occurring in the terms of N. We will build the formulas so that the following holds. If ι is any interpretation of the free variables of $\xi_n[N]$ as elements of $\mathcal{C}$, we will have

$$\mathcal{C}, \iota \models \xi_n[N] \iff \exists \text{ has a winning strategy for } G_n(N^\iota).$$

We will define the formulas $\xi_n[N]$ by induction on n. First, we let $\xi_0[N]$ be the conjunction of the following two formulas, saying that N^ι is a network:

$$\begin{array}{ll} \bigwedge_{x \in N} & \forall u(\psi_{N(x,x)}(\bar{v}_{xx}, u) \to Id(u)) \\ \bigwedge_{x,y,z \in N} & \exists uvw(R(u, v, w) \wedge \psi_{N(x,y)}(\bar{v}_{xy}, u) \wedge \psi_{N(y,z)}(\bar{v}_{yz}, v) \wedge \psi_{N(z,x)}(\bar{v}_{zx}, w)). \end{array}$$

Here, the formulas ψ_t are as in lemma 3.6. We wrote $\bar{v}_{xy}$ for an arbitrary tuple of variables containing those in the term $N(x, y)$, for each pair of nodes x, y of N. It is assumed that u, v, w are variables not occurring in the terms of N.

For the inductive case, let us first set down some terminology. If t, t' are relation algebra terms, we say that they are *equivalent* if we can substitute variables for other variables in t in a (1-1) fashion to obtain t'. It is clear that, up to equivalence, there are only finitely many relation algebra terms containing at most $n + 1$ function symbols (taken from $\wedge, \neg, {}^\smile, ;, Id$).

Now we define $\xi_{n+1}[N]$ to be the conjunction

$$\bigwedge_{t_1,t_2,x,y} \forall \bar{w}\Big(\xi_n[\mathsf{Acc}(N,x,y,t_1,t_2)] \ \vee \ \xi_n[\mathsf{Rej}(N,x,y,t_1,t_2)]\Big),$$

where:

1. The conjunction is taken over all of the finitely many pairs of relation algebra terms t_1, t_2, each with at most $n+1$ function symbols, up to equivalence, and all of the finitely many pairs of nodes $x, y \in N$.

2. The terms t_1, t_2 are representatives of their equivalence classes, chosen so that no variable occurring in either of them also occurs in the other or in any term in N. By definition of term equivalence, we can substitute new variables for old in t_1, t_2. So we can choose such representatives.

3. $\bar{w}$ is a tuple of variables containing precisely those occurring in t_1, t_2. So no variable in $\bar{w}$ occurs in any $\bar{v}_{xy}$ $(x, y \in N)$.

We show by induction on n that these work. For $n = 0$, it is evident that $\mathcal{C}, \iota \models \xi_0[N]$ iff N^ι is a network, which, by definition of G_0, is iff $\exists$ has a winning strategy in $G_0(N^\iota)$. Assume the result for n. Now evidently, $\exists$ has a winning strategy in $G_{n+1}(N^\iota)$ iff (i) N^ι is a network, and (ii) for all possible first moves that $\forall$ could make in $G_{n+1}(N^\iota)$, $\exists$ can respond to leave a pre-network $N^+ \supseteq N^\iota$ such that she has a winning strategy in $G_n(N^+)$. Note that (ii) implies (i). So this is to say that for all relation algebra terms $t_1(\bar{x}), t_2(\bar{y})$ with at most $n+1$ function symbols (where $\bar{x}$ and $\bar{y}$ are disjoint from each other and from the variables occurring in N), all atoms $\bar{a}, \bar{b} \in \mathcal{C}$, and all $x, y \in N$, if $\forall$ plays $\langle x, y, t_1^{\mathcal{A}}(\bar{a}), t_2^{\mathcal{A}}(\bar{b})\rangle$ then $\exists$ can respond with a pre-network $N^+ \supseteq N^\iota$ giving her a winning strategy in $G_n(N^+)$. By the game definition, if she accepts then her response is $\mathsf{Acc}(N,x,y,t_1,t_2)^{\iota^+}$, whereas if she rejects then it is $\mathsf{Rej}(N,x,y,t_1,t_2)^{\iota^+}$, where $\iota^+(\bar{x}) = \bar{a}$, $\iota^+(\bar{y}) = \bar{b}$, and ι^+ agrees with ι on all other variables. So inductively, (ii) is equivalent to: for all terms $t_1(\bar{x}), t_2(\bar{y})$ as above, all $\bar{a}, \bar{b} \in \mathcal{C}$, and all $x, y \in N$, either $\mathcal{C}, \iota^+ \models \xi_n[\mathsf{Acc}(N,x,y,t_1,t_2)]$ or $\mathcal{C}, \iota^+ \models \xi_n[\mathsf{Rej}(N,x,y,t_1,t_2)]$, where ι^+ is defined as above. This is clearly the same as $\mathcal{C}, \iota^+ \models \xi_{n+1}[N]$, completing the induction.

We then obtain the formulas $\xi_n(x)$ (where x is any variable) as $\xi_n[I]$, where I is the term network consisting of a single node, 0, with $I(0,0) = x$. The construction of $\xi_n(x)$ is clearly independent of the choice of $\mathcal{C}$. ■

By lemmas 2.9, 3.1, and propositions 3.4 and 3.8, an atom structure is weakly representable iff it satisfies $\forall x(Id(x) \to \xi_n(x))$ for all $n < \omega$, where ξ_n is as in the preceding proposition. This proves part 3 of theorem 2.10. ■

4 Weakly associative relation algebras

To prove the rest of theorem 2.10, we will need to consider weakly associative relation algebras, as in [M2]. Here we recall the basic definitions and facts about them.

- A *weakly associative relation algebra* obeys all the axioms for relation algebras except, possibly, the associativity axiom, but satisfies instead

$$(Id \wedge r);(1;1) = ((Id \wedge r);1);1$$

  for all r belonging to the algebra.[5]

- As in a relation algebra, ';' in a weakly associative relation algebra is monotonic and distributes over arbitrary joins ($\bigvee$), and we still have $Id^\smile = Id$. The proofs of these properties did not use associativity.

[5] Of course, $1;1 = 1$, but the equation is left in this form to indicate the connection with associativity.

- A weakly associative relation algebra is said to be *integral* if Id is an atom.[6]
- A *relativised representation* of a weakly associative relation algebra $\mathcal{R}$ is a model of the following first-order theory in the signature consisting of a binary relation symbol $r(x, y)$ for every $r \in \mathcal{R}$.

$$\begin{array}{lcl}
\forall x, y[1(x,y) \to (Id(x,y) \leftrightarrow (x = y))] & : & \forall r, s, t \in \mathcal{R} \text{ with:} \\
\exists xy\; r(x,y) & : & r \neq 0 \\
\forall x, y[r(x,y) \leftrightarrow s(x,y) \vee t(x,y)] & : & r = s \vee t \\
\forall x, y[1(x,y) \to (r(x,y) \leftrightarrow \neg s(x,y))] & : & r = \neg s \\
\forall x, y[r(x,y) \leftrightarrow s(y,x)] & : & r = s^{\smile} \\
\forall x, y[1(x,y) \to (r(x,y) \leftrightarrow \exists z(s(x,z) \wedge t(z,y)))] & : & r = s;t
\end{array}$$

- A relativised representation M of such an algebra $\mathcal{R}$ is said to be *complete* if for all $x, y \in M$, if $M \models 1(x, y)$ then $M \models a(x, y)$ for some atom a of $\mathcal{R}$.

Axioms 1 and 6 for relativised representations differ from the corresponding axioms for representations of relation algebras. These axioms enforce the soundness of the interpretations of Id and ';' relativised to the top element, 1. Crucially, if $M \models s(x, z) \wedge t(z, y)$, we only require that $M \models r(x, y)$ if already $M \models 1(x, y)$. Relativising the other axioms to 1 does not change their meaning, so we have kept their original form.

It is much easier to build a relativised representation of a weakly associative relation algebra than to build a representation of a relation algebra. Any atomic weakly associative relation algebra has a complete relativised representation: see [M2, theorems 5.13, 5.19], or [HH1]. Indeed, the following can be shown:

Lemma 4.1 *Let $\mathcal{R}$ be any countable weakly associative relation algebra. Let K be an $\mathcal{R}$-structure satisfying the first five axioms above, plus:*

$$\forall x, y[\exists z(r(x,z) \wedge s(z,y)) \to [r;s](x,y)],$$

for all $r, s \in \mathcal{R}$. Then K embeds into a relativised representation of $\mathcal{R}$.

Assume further that for all $x, y \in K$, $K \models a(x, y)$ for some atom a of $\mathcal{R}$. Then K embeds into a complete relativised representation of $\mathcal{R}$.

Proof. A straightforward game-theoretic argument. See [HH1, theorem 33]. ■

4.1 Consistent triangles

Now we recall a useful way of specifying a weakly associative relation algebra. It is similar to that presented in [L1,§4], but it deals with non-atomic, incomplete, and weakly associative algebras. We will only use it for atomic algebras, but we thought it worthwhile to take the more general approach.

Definition 4.2 Let $\mathcal{B}$ be any boolean algebra (not necessarily atomic). A *notion of consistency* for $\mathcal{B}$ is a triple $\langle Id, {}^{\smile}, \mathcal{T}\rangle$ consisting of an element $Id \in \mathcal{B}$, a boolean algebra automorphism ${}^{\smile} : \mathcal{B} \to \mathcal{B}$ satisfying (i) $Id^{\smile} = Id$ and (ii) $r^{\smile\smile} = r$ for all $r \in \mathcal{B}$, and a set $\mathcal{T} \subseteq \mathcal{B}^3$ of 'inconsistent triangles' satisfying:

1. axiom for Id: for all $r, s \in \mathcal{B}$, $(r, s, Id) \in \mathcal{T}$ iff $r \wedge s^{\smile} = 0$.
2. $\mathcal{T}$ is closed under Peircean transforms:

$$(r, s, t) \in \mathcal{T} \quad \Rightarrow \quad (s, t, r) \in \mathcal{T} \text{ and } (t^{\smile}, s^{\smile}, r^{\smile}) \in \mathcal{T}.$$

[6] This does not imply $r; s = 0 \Rightarrow r = 0 \vee s = 0$.

3. distribution over $\bigvee$: for all $S \subseteq \mathcal{B}$ and $r, t \in \mathcal{B}$, if $\bigvee S$ exists, then
$$(r, \bigvee S, t) \in \mathcal{T} \iff \bigwedge_{s \in S} ((r, s, t) \in \mathcal{T}).$$

Remarks 4.3 1. By the distributivity axiom, $\mathcal{T}$ as above is 'closed downwards': that is, if $(r, s, t) \in \mathcal{T}$, and $r \geq r'$, $s \geq s'$, and $t \geq t'$, then $(r', s', t') \in \mathcal{T}$.

2. The infinitary distributivity law for $\mathcal{T}$ can be replaced by the stronger first-order axiom $\forall xy \exists z \forall t((x, y, t) \in \mathcal{T} \leftrightarrow t \leq z)$.

Example 4.4 Let $\mathcal{R}$ be a weakly associative relation algebra, with identity Id and converse $^{\smile}$. Define $\mathcal{T} = \{(r, s, t) \in \mathcal{R}^3 : r; s \wedge t^{\smile} = 0\}$. Then $\langle Id, ^{\smile}, \mathcal{T} \rangle$ is a notion of consistency for the underlying boolean algebra of $\mathcal{R}$.

We now try to reverse this example, to obtain a weakly associative relation algebra from a notion of consistency. Suppose that $\mathcal{B}$ is a boolean algebra, and $\langle Id, ^{\smile}, \mathcal{T} \rangle$ is a notion of consistency for $\mathcal{B}$. For $r, s \in \mathcal{B}$, define
$$l_{r,s} = \{t \in \mathcal{B} : (r, s, t^{\smile}) \in \mathcal{T}\}.$$
For any $r, s \in \mathcal{B}$, the distributivity axiom (with $S = \varnothing$) gives $(r, 0, s^{\smile}) \in \mathcal{T}$, so $(r, s, 0) \in \mathcal{T}$ and $0 \in l_{r,s}$. It is now easily seen that $l_{r,s}$ is an ideal in $\mathcal{B}$. Moreover, if the least upper bound $t = \bigvee l_{r,s}$ exists in $\mathcal{B}$, then by the distributivity law, $t \in l_{r,s}$, so that $l_{r,s}$ is a principal ideal in this case.

Proposition 4.5 *Let $\langle Id, ^{\smile}, \mathcal{T} \rangle$ be a notion of consistency for the boolean algebra $\mathcal{B}$. Define a product operation ';' on $\mathcal{B}$ by $r; s = \neg \bigvee l_{r,s}$, if the right-hand side exists. Suppose that it does exist, for all $r, s \in \mathcal{B}$. Then:*

1. *For all $r, s, t \in \mathcal{B}$, we have $r; s \wedge t^{\smile} = 0$ iff $(r, s, t) \in \mathcal{T}$.*
2. *$(\mathcal{B}, Id, ^{\smile}, ;)$ satisfies all axioms for a weakly associative relation algebra, except possibly the equation $(Id \wedge x); (1; 1) = ((Id \wedge x); 1); 1$.*
3. *If Id is an atom of $\mathcal{B}$, then $(\mathcal{B}, Id, ^{\smile}, ;)$ is a weakly associative relation algebra.*

Proof.

1. For '$\Leftarrow$', if $(r, s, t) \in \mathcal{T}$ then $t^{\smile} \in l_{r,s}$, so $r; s = \neg \bigvee l_{r,s} \leq \neg t^{\smile}$. Hence, $r; s \wedge t^{\smile} = 0$.

   For '$\Rightarrow$', if $r; s \wedge t^{\smile} = 0$ then $\neg(r; s) = \bigvee l_{r,s} \geq t^{\smile}$. Now as we said, $l_{r,s}$ is actually a principal ideal. So $\neg(r; s) \in l_{r,s}$. Hence, $t^{\smile} \in l_{r,s}$, which gives $(r, s, t) \in \mathcal{T}$, as required.

2. Axioms 3–5 of definition 2.1 are given, since $^{\smile}$ is a boolean algebra automorphism. Axiom 2, associativity, is not involved. We check axioms 1, 6, and 7. To check that for any $r \in \mathcal{B}$ we have $Id; r = r$, observe that $Id; r = \neg \bigvee l_{Id,r} = \neg \bigvee \{t : (Id, r, t^{\smile}) \in \mathcal{T}\} = \neg \bigvee \{t : r \wedge t = 0\} = r$. To show that $r; Id = r$ is similar. This proves axiom (1).

   We check that $(r; s)^{\smile} = s^{\smile}; r^{\smile}$ for all $r, s \in \mathcal{B}$. As $\mathcal{T}$ is closed under Peircean transforms, and $x \mapsto x^{\smile}$ is a boolean algebra automorphism, we have $(r; s)^{\smile} = \neg \bigvee \{t : (r, s, t) \in \mathcal{T}\}$ $= \neg \bigvee \{t : (s^{\smile}, r^{\smile}, t^{\smile}) \in \mathcal{T}\} = s^{\smile}; r^{\smile}$, as required for axiom (6).

   As $\mathcal{T}$ is closed under the Peircean transforms, the validity of the triangle axiom (7) follows from part 1.

3. If Id is an atom, then the weak associativity axiom reduces to the two equations $Id; (1; 1) = (Id; 1); 1$ and $0; (1; 1) = (0; 1); 1$. We know that $Id; r = r$ for all r; so $(Id; 1); 1 = 1; 1 = Id; (1; 1)$. Now obviously $1; r^{\smile} \wedge 0 = 0$ for any r, so by the triangle law (see part 2 above), $0; r \wedge 1 = 0; r = 0$. The second equation follows easily from this.

■

Notions of consistency (or inconsistency?) can be studied in their own right. In the next section, however, we will merely use them to specify weakly associative relation algebras in a convenient fashion.

5 The construction

Here, we provide a method for proving most of the remaining parts of the theorem. It is a modified form of the 'rainbow' algebra constructed in [HH2]. Let $\mathcal{R}$ be any integral weakly associative relation algebra. We will define a new weakly associative relation algebra, denoted $\mathcal{A}(\mathcal{R})$, which contains $\mathcal{R}$ as a subset, and will turn out to be a relation algebra. We do so by first defining a weakly associative relation algebra $\mathcal{P}$, and then letting $\mathcal{A}(\mathcal{R})$ be a certain subalgebra in $\mathcal{P}$.

5.1 Definition of $\mathcal{P}$

Definition 5.1 We obtain $\mathcal{P}$ as follows. For convenience, we will view any element of $\mathcal{R}$ except Id as being coloured *red.* Id is colourless. Starting with $\mathcal{R}$, we add the following set $\mathcal{N}$ of new atoms:

- a black atom, B
- a yellow atom, Y
- green atoms g_i $(i < \omega)$
- white atoms w_s $(s \subseteq \omega, |s| \leq 2$ or $s = \omega)$.

These atoms are disjoint from all elements of $\mathcal{R}$. We define the domain and the boolean algebra structure of $\mathcal{P}$ by stating that it is the closure of $\mathcal{R}$ and the power set of $\mathcal{N}$ under the boolean operations.

Formally, as a boolean algebra, $\mathcal{P}$ is the product algebra $\mathcal{R} \times \wp(\mathcal{N})$, so that for $r, r' \in \mathcal{R}$, $n, n' \subseteq \mathcal{N}$, we have $\neg(r, n) = (\neg r, \mathcal{N} \setminus n)$ and $(r, n) \vee (r', n') = (r \vee r', n \cup n')$ in $\mathcal{P}$. But we take it that the idea is clear, so to keep the notation as simple as we can, we will not use this formal representation very much. Rather, we will identify each $n \in \mathcal{N}$ with $(0, \{n\}) \in \mathcal{P}$, and each $r \in \mathcal{R}$ with $(r, \varnothing) \in \mathcal{P}$. Thus, using this identification, $\mathcal{N} \cup \mathrm{dom}(\mathcal{R}) \subseteq \mathrm{dom}(\mathcal{P})$.

Bear in mind that although $\mathcal{R}$ is a subset of $\mathcal{P}$, it is not a subalgebra — in particular, the terms 1 and $\neg x$ (and, as we will see, ';') have different meanings in $\mathcal{P}$. For clarity, we will write $\boldsymbol{R}$ for the top element $1^{\mathcal{R}}$ when regarded as an element of $\mathcal{P}$.

We will now expand $\mathcal{P}$ into a weakly associative relation algebra, by defining a notion of consistency $\langle Id, \smile, \mathcal{T} \rangle$ on it. The identity element $Id^{\mathcal{P}}$ of $\mathcal{P}$ will be the identity $Id^{\mathcal{R}}$ of $\mathcal{R}$. Converses can be given as follows. Since $\smile$ distributes over $\bigvee$, it is enough to state the following.

1. All atoms in $\mathcal{N}$ are self-converse.
2. $\smile$ on $\mathcal{R}$ is unchanged: $(dom(\mathcal{R}), \wedge, \vee, \smile)$ is a substructure of $(dom(\mathcal{P}), \wedge, \vee, \smile)$.

So, using the formal presentation of $\mathcal{P}$, we have $(r, n)^\smile = (r^\smile, n)$ for all $r \in \mathcal{R}, n \subseteq \mathcal{N}$. It remains to define the set $\mathcal{T}$ of inconsistent triangles of $\mathcal{P}$, as in definition 4.2.

Notation 5.2 Let $\mathcal{Q} = \mathcal{N} \cup \{Id\} \cup \{r \in \mathcal{R} : r \neq 0, r \wedge Id = 0\}$. For $r, s, t \in \mathcal{P}$, we write $(x, y, z) \sqsubseteq (r, s, t)$ to mean that $x, y, z \in \mathcal{Q}$ and $x \leq r$, $y \leq s$, $z \leq t$.

Definition 5.3 A triangle $(r, s, t) \in \mathcal{P}^3$ is in $\mathcal{T}$ ('is inconsistent') iff for every $(x, y, z) \sqsubseteq (r, s, t)$, (x, y, z) or some Peircean transform of it has the form of one of the seven clauses in table 1.

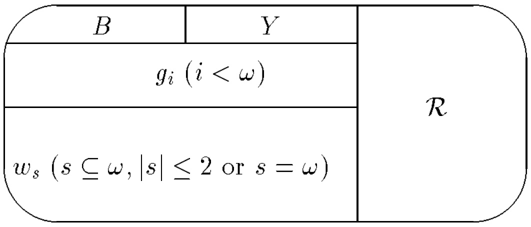
1.	$x, y, z \leq \boldsymbol{R}$ and $\mathcal{R} \models x; y \wedge z^{\smile} = 0$
2.	$(x \in \mathcal{N}$ or $y \in \mathcal{N})$, $z = Id$, $x \wedge y^{\smile} = 0$
3.	(g_i, g_j, g_k), for any $i, j, k < \omega$
4.	(g_i, g_j, w_s), for any $i, j < \omega$ and any white atom w_s
5.	(Y, Y, Y)
6.	(Y, Y, B)
7.	(g_i, Y, w_s) where $i \notin s$

Table 1.

We write $\mathcal{T}_1$ for the set of all $(x, y, z) \in \mathcal{Q}^3$ with a Peircean transform having the form given in table 1. Thus, $(r, s, t) \in \mathcal{P}^3$ is inconsistent iff every $(x, y, z) \sqsubseteq (r, s, t)$ is in $\mathcal{T}_1$.

By now, the reader may have blanched, balked, or bolted. For those who have not taken the latter course, let us try to offer some guidance. The idea behind this definition comes from [HH2].[7] What we are doing is attaching to $\mathcal{R}$ a number of additional atoms.

$\mathcal{P}$:



These have some rather interesting properties, which we will not be able to discuss fully until section 7. They will have the effect of 'stretching' a relativised representation of $\mathcal{R}$.

For now, our concerns are simpler. As we will prove here, we have defined a notion of consistency, which gives rise to a product operation on $\mathcal{P}$ making it into a weakly associative relation algebra. Broadly, this product agrees with the one on the original algebra $\mathcal{R}$ when used on elements of $\mathcal{R}$ — the whole idea is to transfer properties of relativised representations of $\mathcal{R}$ to representations of $\mathcal{A}(\mathcal{R}) \subseteq \mathcal{P}$. This can be seen in the definition: a triangle (r, s, t) of elements of $\mathcal{R}$ is in $\mathcal{T}$ iff it is inconsistent in $\mathcal{R}$. See example 5.8(1) below.

5.2 $\mathcal{P}$ is a weakly associative relation algebra

We need two lemmas to prove this.

Lemma 5.4 *$\langle Id, ^{\smile}, \mathcal{T}\rangle$ is a notion of consistency for $\mathcal{P}$ (see definition 4.2).*

Proof. Recall that $\mathcal{T}_1$ is the set of all $(x, y, z) \in \mathcal{Q}^3$ with a Peircean transform arising from table 1 in definition 5.3; and $\mathcal{T} = \{(r, s, t) \in \mathcal{P}^3 : (x, y, z) \sqsubseteq (r, s, t) \Rightarrow (x, y, z) \in \mathcal{T}_1\}$.

[7] For readers who have seen [HH2], the 'red atoms' of that paper have simply been replaced by the whole algebra $\mathcal{R}$.

1. We check the identity law. Let $r, s \in \mathcal{P}$. Then $(x, y, z) \sqsubseteq (r, s, Id)$ implies $z = Id$. By inspection of table 1 in definition 5.3, only (1) or (2) can apply to (x, y, Id). So $(r, s, Id) \in \mathcal{T}$ iff any $(x, y, z) \sqsubseteq (r, s, Id)$ satisfies $x, y \leq \boldsymbol{R}$ and $\mathcal{R} \models x; y \wedge Id^{\smile} = 0$ (by (1)), or else $\neg(x, y \leq \boldsymbol{R})$ and $x \wedge y^{\smile} = 0$ (by (2)). As $\mathcal{R}$ satisfies the triangle axiom of definition 2.1, the second conjuncts here are both equivalent to $x \wedge y^{\smile} = 0$. So $(r, s, Id) \in \mathcal{T}$ iff $x \wedge y^{\smile} = 0$ for all $(x, y, z) \sqsubseteq (r, s, Id)$. Since clearly, $r = \bigvee\{x : (x, y, z) \sqsubseteq (r, s, Id)\}$, and similarly for s, we see that for all $r, s \in \mathcal{P}$, $(r, s, Id) \in \mathcal{T}$ iff $r \wedge s^{\smile} = 0$, as required.

2. Evidently, $\mathcal{T}$ is closed under Peircean transforms.

3. Let $r, t \in \mathcal{P}$, $S \subseteq \mathcal{P}$, and suppose that $\bigvee S$ exists in $\mathcal{P}$. We must show that $(r, \bigvee S, t)$ is inconsistent iff (r, s, t) is inconsistent for every $s \in S$.

   '$\Rightarrow$' is trivial, since $(x, y, z) \sqsubseteq (r, s, t) \Rightarrow (x, y, z) \sqsubseteq (r, \bigvee S, t)$. We pass to '$\Leftarrow$'. Assume that $(r, s, t) \in \mathcal{T}$ for all $s \in S$. Let $(x, y, z) \sqsubseteq (r, \bigvee S, t)$. We must show that $(x, y, z) \in \mathcal{T}_1$.

   If $y \in \mathcal{N} \cup \{Id\}$, then y is an atom of $\mathcal{P}$, so $(x, y, z) \sqsubseteq (r, s, t)$ for some $s \in S$. But $(r, s, t) \in \mathcal{T}$, so $(x, y, z) \in \mathcal{T}_1$.

   Otherwise, $0 < y \leq \boldsymbol{R} \wedge \neg Id$. As $y \leq \bigvee S$, there exist $s \in S$ for which $y \wedge s \neq 0$. For any such s, $y \wedge s \in \mathcal{Q}$ and $(x, y \wedge s, z) \sqsubseteq (r, s, t)$. But $(r, s, t) \in \mathcal{T}$, so $(x, y \wedge s, z) \in \mathcal{T}_1$. Inspecting table 1, we see that $(x, y \wedge s, z)$ must be obtainable via (1) or (2).

   If there is $s \in S$ such that (2) applies to $(x, y \wedge s, z)$, then either $z = Id$ and $x \leq \neg\boldsymbol{R}$, or the same swapping x, z. But then (2) also applies to (x, y, z), since $y \leq \boldsymbol{R}$. We are done.

   So assume that every $(x, y \wedge s, z)$ with $y \wedge s \neq 0$ arises from (1) of table 1. As (1) is invariant under Peircean transforms, this means that $\mathcal{R} \models y \wedge s \wedge (x^{\smile}; z^{\smile}) = 0$ for all $s \in S$. By distributivity, $\mathcal{R} \models y \wedge \bigvee S \wedge (x^{\smile}; z^{\smile}) = 0$, i.e., $\mathcal{R} \models y \wedge (x^{\smile}; z^{\smile}) = 0$ so that (1) applies to (x, y, z), too. This completes the proof.

■

Lemma 5.5 *Define ';' on $\mathcal{P}$ as in proposition 4.5. Then $r; s$ exists, for all $r, s \in \mathcal{P}$.*

Proof. We must show that if $r, s \in \mathcal{P}$ and $l_{r,s} = \{t \in \mathcal{P} : (r, s, t^{\smile}) \in \mathcal{T}\}$, then $\neg \bigvee l_{r,s}$ exists in $\mathcal{P}$. Intuitively, it exists because it can be split by $\boldsymbol{R}$ and Id into three parts. The part in $\neg\boldsymbol{R}$ will certainly exist, because all bounds of set of elements beneath $\neg\boldsymbol{R}$ exist, by definition of $\mathcal{P}$. The part in Id will obviously exist. The final third, the part in $\boldsymbol{R} \wedge \neg Id$, will exist, because it can be obtained from a product of elements of $\mathcal{R}$.

Formally, consider

$$\begin{aligned}
\alpha &= \bigvee\{u \wedge \boldsymbol{R} \wedge \neg Id : u \in \mathcal{P}, (r, s, u) \in \mathcal{T}\} \\
\beta &= \bigvee\{v \wedge \neg\boldsymbol{R} : v \in \mathcal{P}, (r, s, v) \in \mathcal{T}\} \\
\gamma &= \bigvee\{Id : (r, s, Id) \in \mathcal{T}\}.
\end{aligned}$$

If these three elements exist, then, using the de Morgan laws and the fact that $l_{r,s}$ is an ideal and $^{\smile}$ a boolean algebra automorphism, we have $\neg \bigvee l_{r,s} = \neg(\alpha \vee \beta \vee \gamma)^{\smile}$. So it suffices to prove that α, β and γ exist.

γ trivially exists — it is 0 or Id. β certainly exists — any set of elements beneath $\neg\boldsymbol{R}$ has a least upper bound in $\mathcal{P}$. The problem is α. Let

$$S = \{w : w \leq \boldsymbol{R} \wedge \neg Id,\ (r, s, w) \in \mathcal{T}\}.$$

Then $\alpha = \bigvee S$, so it suffices to prove that $\bigvee S$ exists. Now $w \in S$ iff $w \leq \boldsymbol{R} \wedge \neg Id$ and every $(x, y, z) \sqsubseteq (r, s, w)$ is in $\mathcal{T}_1$. There are three cases.

Case 1: $r, s \leq \boldsymbol{R}$. Then only rule (1) of table 1 can apply to (x, y, z): that is, $\mathcal{R} \models x; y \wedge z^{\smile} = 0$. So $w \in S$ iff $w \leq \boldsymbol{R} \wedge \neg Id$ and $\mathcal{R} \models x; y \wedge z^{\smile} = 0$ for all $(x, y, z) \sqsubseteq (r, s, w)$. By distributivity of ';' over $\vee$ in $\mathcal{R}$, this is iff $w \leq \boldsymbol{R} \wedge \neg Id$ and $\mathcal{R} \models r; s \wedge w^{\smile} = 0$ — that is, iff $w \leq \neg Id \wedge q$, where

$\mathcal{R} \models q = \neg(r;s)^\smile$. Clearly, then, $\bigvee S = \neg Id \wedge q$. Notice that we obtained $\bigvee S$ using the existence of the product $r;s$ in $\mathcal{R}$.

Case 2: $r \not\leq \boldsymbol{R}$, $s = Id$, or vice versa. In this case, $x \in \mathcal{N}$ and $z \leq \boldsymbol{R}$, so (x,y,z) can only be obtained, if at all, by clause (2) of table 1. So $w \in S$ iff $w \leq \boldsymbol{R} \wedge \neg Id$ and every $(x,y,z) \sqsubseteq (r,s,w)$ satisfies $x \wedge z^\smile = 0$. By distributivity, this holds iff $w \leq \boldsymbol{R} \wedge \neg Id$ and $r \wedge w^\smile = 0$ — i.e., iff $w \leq \boldsymbol{R} \wedge \neg Id \wedge \neg r^\smile$. So $\bigvee S = \boldsymbol{R} \wedge \neg Id \wedge \neg r^\smile$. The case where $s \not\leq \boldsymbol{R}$, $r = Id$ is similar.

Case 3: $r \not\leq \boldsymbol{R}$, $s \neq Id$, or vice versa. If $w \leq \boldsymbol{R} \wedge \neg Id$ and $w \neq 0$, there is $(x,y,z) \sqsubseteq (r,s,w)$ with $x \in \mathcal{N}$ and $y \leq \neg Id$. As $z \leq \boldsymbol{R} \wedge \neg Id$, no clause from table 1 applies to (x,y,z). Thus, $(x,y,z) \notin \mathcal{T}_1$, so $w \notin S$. In this case, $\bigvee S = \bigvee\{0\} = 0$. The case $s \not\leq \boldsymbol{R}$, $r \neq Id$ is similar. ■

Taken with proposition 4.5, this tedious lemma shows that the notion of consistency $\langle Id, ^\smile, \mathcal{T}\rangle$ on $\mathcal{P}$ yields an integral weakly associative relation algebra, also written $\mathcal{P}$.

5.3 Definition of $\mathcal{A}(\mathcal{R})$

Definition 5.6 *We now let the weakly associative relation algebra $\mathcal{A}(\mathcal{R})$ be the subalgebra of $\mathcal{P}$ generated by $\mathcal{R}$ and $\mathcal{N}$.*

Importantly, if $\mathcal{R}$ is countable, then so is $\mathcal{A}(\mathcal{R})$. We will show below (corollary 7.4) that ';' is associative, so that $\mathcal{A}(\mathcal{R})$ is always a relation algebra.

5.4 Simple properties of $\mathcal{A}(\mathcal{R})$

Lemma 5.7 *Let $r, s \in \mathcal{Q}$. Then $r;s = \neg\bigvee\{z \in \mathcal{Q} : (r,s,z^\smile) \in \mathcal{T}_1\}$.*

Proof. Well, $r;s = \neg\bigvee\{t \in \mathcal{P} : (x,y,z) \sqsubseteq (r,s,t^\smile) \Rightarrow (x,y,z) \in \mathcal{T}_1\}$. Note that for any $t \in \mathcal{P}$, we have $t = \bigvee\{z : z \in \mathcal{Q}, z \leq t\}$. So $r;s = \neg\bigvee\{z : x,y,z \in \mathcal{Q}, x \leq r, y \leq s \Rightarrow (x,y,z^\smile) \in \mathcal{T}_1\}$. Now for any $z \in \mathcal{Q}$, we have $[\forall xy \in \mathcal{Q}(x \leq r \wedge y \leq s \Rightarrow (x,y,z^\smile) \in \mathcal{T}_1)]$ iff $(r,s,z^\smile) \in \mathcal{T}_1$. ('$\Rightarrow$' is trivial, as $r,s \in \mathcal{Q}$; for '$\Leftarrow$', observe that $\mathcal{T}_1$ is closed downwards.) Hence $r;s = \neg\bigvee\{z : (r,s,z^\smile) \in \mathcal{T}_1\}$, as required. ■

We can use this lemma to obtain some sample products in $\mathcal{P}$.

Example 5.8 1. Let $r,s,t \in \mathcal{R}$ and suppose that $\mathcal{R} \models r = s;t$. We evaluate $s;t$ in $\mathcal{A}(\mathcal{R})$. We will see that $\mathcal{A}(\mathcal{R}) \models s;t \wedge \boldsymbol{R} = r$. Thus, the new atoms do not interfere with the product operation on $\mathcal{R}$.

It is clear that if $s \leq Id$ or $t \leq Id$ then $\mathcal{A}(\mathcal{R}) \models s;t \wedge \boldsymbol{R} = r$. So assume not. We write '$;^{\mathcal{R}}$' for the product in $\mathcal{R}$, reserving ';' for the product in $\mathcal{A}(\mathcal{R})$; '$\neg$' will denote negation in $\mathcal{A}(\mathcal{R})$. Write s^0 for $s \wedge Id$ and s^1 for $s \wedge \neg Id$, and similarly for t. So $s^1, t^1 \in \mathcal{Q}$. By the lemma, $s^1;t^1 = \neg\bigvee\{u \in \mathcal{Q} : (s^1,t^1,u^\smile) \in \mathcal{T}_1\} = \neg\bigvee\{u \leq \boldsymbol{R} : s^1;^{\mathcal{R}} t^1 \wedge u = 0\} = \neg\boldsymbol{R} \vee s^1;^{\mathcal{R}} t^1$. Then by distribution in $\mathcal{A}(\mathcal{R})$ and the fact that Id is the identity of both $\mathcal{R}$ and $\mathcal{A}(\mathcal{R})$,

$$\begin{aligned} s;t &= s^0;t^0 \vee s^1;t^0 \vee s^0;t^1 \vee s^1;t^1 \\ &= s^0;^{\mathcal{R}} t^0 \vee s^1;^{\mathcal{R}} t^0 \vee s^0;^{\mathcal{R}} t^1 \vee s^1;t^1 \\ &= s^0;^{\mathcal{R}} t^0 \vee s^1;^{\mathcal{R}} t^0 \vee s^0;^{\mathcal{R}} t^1 \vee (\neg\boldsymbol{R} \vee s^1;^{\mathcal{R}} t^1). \end{aligned}$$

Hence, by distribution in $\mathcal{R}$,

$$\begin{aligned} s;t \wedge \boldsymbol{R} &= [s^0;^{\mathcal{R}} t^0 \vee s^1;^{\mathcal{R}} t^0 \vee s^0;^{\mathcal{R}} t^1 \vee (\neg\boldsymbol{R} \vee s^1;^{\mathcal{R}} t^1)] \wedge \boldsymbol{R} \\ &= s^0;^{\mathcal{R}} t^0 \vee s^1;^{\mathcal{R}} t^0 \vee s^0;^{\mathcal{R}} t^1 \vee s^1;^{\mathcal{R}} t^1 \\ &= s;^{\mathcal{R}} t. \end{aligned}$$

So $\mathcal{A}(\mathcal{R}) \models s;t \wedge \boldsymbol{R} = r$ in all cases, as claimed.

2. If $a \in \mathcal{N}$ and $r \leq \boldsymbol{R}$, then $a;r = \begin{cases} 0, & \text{if } r = 0 \\ a, & \text{if } r = id \\ \neg Id, & \text{otherwise.} \end{cases}$

   For assume first that $0 < r \leq \boldsymbol{R} \wedge \neg Id$. Then $a, r \in \mathcal{Q}$, so by the lemma, $a;r = \neg \bigvee \{z \in \mathcal{Q} : (a, r, z^{\smile}) \in \mathcal{T}_1\} = \neg \bigvee \{Id\} = \neg Id$. In general,

$$a;r = a;(r \wedge Id) \vee a;(r \wedge \neg Id) = \begin{cases} 0, & \text{if } r = 0 \\ a, & \text{if } r = Id \\ \neg Id, & \text{otherwise} \end{cases}$$

   as required.

3. If $a, b \in \mathcal{N}$, then $a;b = \neg \bigvee \{z \in \mathcal{Q} : (a, b, z^{\smile}) \in \mathcal{T}_1\} = \neg \bigvee (\{Id : a \neq b\} \cup \{c \in \mathcal{N} : (a, b, c) \in \mathcal{T}_1\})$. We see that

$$a;b = \begin{cases} \bigvee \{c \in \mathcal{N} : (a, b, c) \notin \mathcal{T}_1\} \vee \boldsymbol{R}, & \text{if } a = b \\ \bigvee \{c \in \mathcal{N} : (a, b, c) \notin \mathcal{T}_1\} \vee (\boldsymbol{R} \wedge \neg Id), & \text{if } a \neq b. \end{cases}$$

   We can immediately apply this in some concrete cases, reading off products of new atoms simply by inspection of table 1.

4. $Y;Y = \neg(Y \vee B)$.

5. If $i \neq j$, then $g_i;g_j = (\boldsymbol{R} \wedge \neg Id) \vee Y \vee B$.

6. Hence, if $i \neq j$, then $Y;Y \wedge g_i;g_j = \neg(Y \vee B) \wedge ((\boldsymbol{R} \wedge \neg Id) \vee Y \vee B) = \boldsymbol{R} \wedge \neg Id$.

7. $g_i;w_{s_0} = \begin{cases} (\boldsymbol{R} \wedge \neg Id) \vee B \vee \bigvee_s w_s \vee Y, & \text{if } i \in s_0 \\ (\boldsymbol{R} \wedge \neg Id) \vee B \vee \bigvee_s w_s, & \text{otherwise.} \end{cases}$

8. $g_i;Y = \neg(Id \vee \bigvee_{i \notin s} w_s) = (\boldsymbol{R} \wedge \neg Id) \vee Y \vee B \vee \bigvee_i g_i \vee \bigvee_{i \in s} w_s$.

   Note that the infinite disjunctions in this expression exist in $\mathcal{A}(\mathcal{R})$. For by (7), $\bigvee_i g_i = \neg(g_0;w_{\{0\}} \vee Id)$, and $\bigvee_{i \in s} w_s = g_0;w_\varnothing \wedge \neg(\boldsymbol{R} \vee B)$.

6 Ultrafilter networks and games

Here, we introduce a rather different kind of game than the one seen earlier. It is played on networks whose 'labels' are not *elements* of an algebra, but *ultrafilters.* A similar game was used in [HH1], and related ideas appear in work of Lyndon, Maddux, and others — see, e.g., [L1, M2]. We can use the game to characterise associativity and representability of an algebra.

Let $\mathcal{R}$ be an arbitrary weakly associative relation algebra.

6.1 Ultrafilter networks

Notation 6.1 For subsets f, g of $\mathcal{R}$, we write

- $f^{\smile}$ for $\{r^{\smile} : r \in f\}$,
- $f;g$ for $\{r;s : r \in f, s \in g\}$.

Definition 6.2 An *ultrafilter network* over $\mathcal{R}$ is a pair $N = (N_1, N_2)$, where N_1 is a finite non-empty set of nodes, and $N_2 : N_1 \times N_1 \to \{\text{ultrafilters of } \mathcal{R}\}$ is a map. We require, for all nodes $x, y, z \in N_1$,

1. $Id \in N_2(x, y)$ iff $x = y$;
2. $N_2(x, y); N_2(y, z) \subseteq N_2(x, z)$.

If $N = (N_1, N_2)$, and $N' = (N'_1, N'_2)$ are ultrafilter networks, we write $N \subseteq N'$ if $N_1 \subseteq N'_1$ and $N'_2 \lceil (N_1 \times N_1) = N_2$.

As before, from now on we will write simply N for any of N, N_1 and N_2, distinguishing them by context.

Lemma 6.3 *Let N be any ultrafilter network. Then $N(y,x) = N(x,y)^{\smile}$, for any nodes $x, y \in N$.*

Proof. If not, there is $r \in N(x,y)$ with $\neg r^{\smile} \in N(y,x)$. Then by the triangle axiom in $\mathcal{R}$ and the two defining properties of ultrafilter networks, we have $0 = r;\neg r^{\smile} \wedge Id \in N(x,x)$, which is impossible. ∎

6.2 Games on ultrafilter networks

Definition 6.4 We describe a game $F_\omega(\mathcal{R})$, with players $\forall$, $\exists$ as before, but to build a chain $N_1 \subseteq N_2 \subseteq \cdots$ of ultrafilter networks, of length ω.

In round 1, $\forall$ chooses an element $r \in \mathcal{R}$ with $0 < r \leq Id$, and $\exists$ responds with an ultrafilter network N_1 with a single node, say x, satisfying $r \in N_1(x,x)$. (She can always do this, as $\{r\}$ extends to an ultrafilter of $\mathcal{R}$ containing Id.)

In a later round, i, assuming that $N_1 \subseteq \cdots \subseteq N_{i-1}$ have been built,

- $\forall$ chooses nodes $x, y \in N_{i-1}$ and $r, s \in \mathcal{R}$ with $r;s \in N_{i-1}(x,y)$.
- $\exists$ must respond, if possible, with an ultrafilter network $N_i \supseteq N_{i-1}$ containing a node z with $r \in N_i(x,z)$ and $s \in N_i(z,y)$.

Possibly, such a node z already exists in N_{i-1}, in which case $\exists$ may play $N_i = N_{i-1}$. Otherwise, N_i will contain at least one more node, z.

Notation 6.5 In this case, as a kind of notational convention, we will say that $\exists$ chose the labels on the (unordered) edges $(z,t), (t,z)$ for every $t \in N_{i-1} \setminus \{x,y\}$; and that $\forall$ chose the labels on the (unordered) edges $(x,z), (z,y)$. (We appreciate that $\forall$ only chose r, s in the ultrafilters labelling (x,z) and (z,y), leaving the other elements to $\exists$: as we said, this is merely notation.)

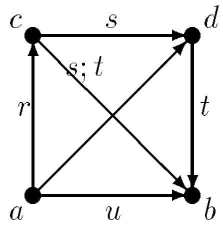
If $\exists$ can continue like this for ω rounds, she wins the play.

If $n < \omega$, we define the game $F_n(\mathcal{R})$ as above, but there are only n rounds. $\exists$ wins if she survives all of them. (In $F_0(\mathcal{R})$, she wins vacuously.)

Proposition 6.6 *Suppose that $\mathcal{A}$ is a weakly associative relation algebra, and that $\exists$ has a winning strategy for $F_4(\mathcal{A})$. Then $\mathcal{A}$ is a relation algebra. Cf. [L1, M1].*

Proof. We show that $\mathcal{A} \models \forall xyz(x;(y;z) = (x;y);z)$. Let $r, s, t \in \mathcal{A}$, and assume for a contradiction that $\exists$ has a winning strategy for $F_4(\mathcal{A})$, and that $r;(s;t) \neq (r;s);t$.

Suppose that $u = r;(s;t) \wedge \neg((r;s);t) \neq 0$; the other case is similar. By the triangle axiom in $\mathcal{A}$, we have $v = (u;u^{\smile}) \wedge Id \neq 0$. $\forall$ begins the game $F_4(\mathcal{A})$ by playing v. $\exists$ responds with an ultrafilter network N_1, with a single node, a, say, and $v \in N_1(a,a)$. $\forall$ continues by playing $a, a, u, u^{\smile}$. Note that $u;u^{\smile} \geq v \in N_1(a,a)$, so, as $N_1(a,a)$ is an ultrafilter, $u;u^{\smile} \in N_1(a,a)$; so this is a valid move for him. $\exists$ uses her winning strategy to respond with an ultrafilter network $N_2 \supseteq N_1$, with nodes {a,b} (not necessarily distinct) and $u \in N_2(a,b)$. $\forall$ then plays $a, b, r, (s;t)$ — this is legal, as $r;(s;t) \geq u \in N_2(a,b)$. Let $\exists$'s response be $N_3 \supseteq N_2$, with nodes a, b, c, and $r \in N_3(a,c)$, $s;t \in N_3(c,b)$. Finally, $\forall$ plays c, b, s, t. $\exists$'s response is an ultrafilter network $N_4 \supseteq N_3$ containing a node d such that $s \in N_4(c,d)$, $t \in N_4(d,b)$.



Now we see that $r; s \in N_4(a, d)$, and so $(r; s); t \in N_4(a, b)$. Hence, $(r; s); t \wedge u \neq 0$. By definition of u, this is a contradiction, proving the proposition. ■

Now a theorem which is essentially in [HH1].

Proposition 6.7 *Let $\mathcal{A}$ be a relation algebra. Then:*

1. *$\mathcal{A}$ is representable if and only if $\exists$ has a winning strategy in $F_\omega(\mathcal{A})$.*
2. *Assume that $\mathcal{A}$ is countable (or merely that it has only countably many atoms). Then $\mathcal{A}$ has a complete representation if and only if $\exists$ has a winning strategy in $F_\omega(\mathcal{A})$ that always chooses principal ultrafilters.*
3. *$\mathcal{A}$ satisfies the Lyndon conditions if and only if for all $n < \omega$, $\exists$ has a winning strategy in $F_n(\mathcal{A})$ that always chooses principal ultrafilters.*

Proof. (sketch)

1. '$\Rightarrow$' is proved much as in '$\Rightarrow$' of proposition 3.4, by using a representation of $\mathcal{A}$ to tell $\exists$ how to win.

   If $\mathcal{A}$ is countable, the proof of '$\Leftarrow$' is also similar to the corresponding argument in '$\Leftarrow$' of proposition 3.4. Note that the first step of that proof, showing that $\exists$ can win the game of length ω, is not needed here: we are given it. See [HH1] for more details.

   If $\mathcal{A}$ is uncountable, take a countable elementary subalgebra $\mathcal{B} \preceq \mathcal{A}$. Assume that $\exists$ can win $F_\omega(\mathcal{A})$. Then she can win $F_\omega(\mathcal{B})$, too. For, in a play $N_0 \subseteq N_1 \subseteq \cdots$ of $F_\omega(\mathcal{B})$, $\exists$ keeps a private play $N_0^+ \subseteq N_1^+ \subseteq \cdots$ of $F_\omega(\mathcal{A})$ running simultaneously, such that for all $i < \omega$, N_i and N_i^+ have the same nodes, and $N_i(x, y) = N_i^+(x, y) \cap \mathcal{B}$ for all nodes x, y of N_i. It is easily checked that $N_i^+(x, y) \cap \mathcal{B}$ is an ultrafilter of $\mathcal{B}$, and that if N_i^+ is an ultrafilter $\mathcal{A}$-network, and N_i is related to N_i^+ as above, then N_i will be an ultrafilter $\mathcal{B}$-network. If $\forall$ plays $\langle x, y, r, s\rangle$ in round i of $F_\omega(\mathcal{B})$, then $r, s \in \mathcal{B}$ and $r; s \in N_{i-1}(x, y)$. So $r, s \in \mathcal{A}$ and $r; s \in N_{i-1}^+(x, y)$, too. Hence $\langle x, y, r, s\rangle$ is a valid move for $\forall$ in $F_\omega(\mathcal{A})$, and $\exists$ can respond with her winning strategy in this game. This provides her with an ultrafilter $\mathcal{A}$-network $N_i^+ \supseteq N_{i-1}^+$, with $r \in N_i^+(x, z)$, $s \in N_i^+(z, y)$ for some z. Defining N_i as above, $\exists$ responds with it to $\forall$'s original move in $F_\omega(\mathcal{B})$. This approach will win for her.

   So by the result for countable algebras, $\mathcal{B}$ is representable. As **RRA** is elementary, so is $\mathcal{A}$, proving (1).

2. This is similar to (1), the point to '$\Leftarrow$' being that if the networks $N_1 \subseteq N_2 \subseteq \cdots$ in the game are all labelled by principal ultrafilters then the representation built from them will be complete. See [HH2] for the details.

3. We will take this as a definition of an algebra satisfying the Lyndon conditions, so we will not give any proof here! For the original conditions, see [L1].

■

Of course, both (2) and (3) imply that $\mathcal{A}$ is atomic. In fact, the Lyndon conditions are in effect conditions on the atom structure of $\mathcal{A}$, as can be seen from (3).

7 Representations of $\mathcal{A}(\mathcal{R})$

Here we use the games just described to relate the representations of the weakly associative relation algebra $\mathcal{A}(\mathcal{R})$ defined in section 5 to the relativised representations of the integral weakly associative relation algebra $\mathcal{R}$. First, a simple definition.

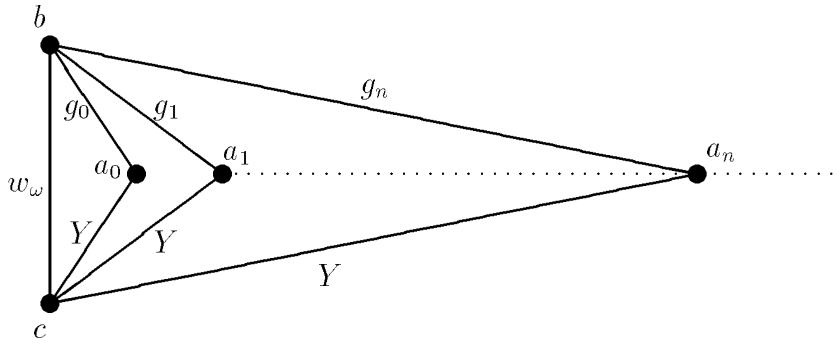
Definition 7.1 Let M be a relativised representation of a weakly associative relation algebra $\mathcal{R}$ with top element 1. A *clique* in M is a set $K \subseteq M$ such that $M \models 1(x, y)$ for all $x, y \in K$.

7.1 Discussion

We can no longer postpone our discussion of the architecture of $\mathcal{A}(\mathcal{R})$. There are two main points, which are linked.

1. The new atoms — those in $\mathcal{N}$ — are designed to gently 'stretch' parts of a representation of $\mathcal{A}(\mathcal{R})$ into an infinite $\mathcal{R}$-clique.

For the following pattern must occur in any representation M of $\mathcal{A}(\mathcal{R})$.



To see this, note first that there must be nodes b, c with $M \models w_\omega(b, c)$. Referring to example 5.8, we notice that $w_\omega \leq g_i; Y$ for all i. So for any such b, c, there must also be nodes a_i $(i < \omega)$ with $M \models g_i(b, a_i) \wedge Y(a_i, c)$. Clearly, the a_i are all distinct. Also, $M \models (g_i; g_j \wedge Y; Y)(a_i, a_j)$, so that $M \models \boldsymbol{R}(a_i, a_j)$ for all $i < j < \omega$. So the a_i form an infinite $\mathcal{R}$-clique.

So any edge labelled w_ω has the effect of 'stretching' a part of the representation into an infinite set whose edges are all red. It follows that if $\mathcal{A}(\mathcal{R})$ is representable then $\mathcal{R}$ has a relativised representation containing an infinite clique. Is the converse true?

What are the possible obstacles to making a representation of $\mathcal{A}(\mathcal{R})$, given that $\mathcal{R}$ has a relativised representation with an infinite clique? Could it be the wrong kind of clique? No! We repeat: the form of the product operation in $\mathcal{A}(\mathcal{R})$ will 'stretch' parts of any representation, forcing there to be an infinite set of points pairwise related by red elements (from $\mathcal{R}$). But *which* red elements are involved is left undetermined by the structure of $\mathcal{A}(\mathcal{R})$. Any elements permitted by $\mathcal{R}$ are allowed. So, if $\mathcal{R}$ has *some* relativised representation with an infinite clique, that clique can be used as a template to 'fill in' the stretched part of a representation of $\mathcal{A}(\mathcal{R})$.

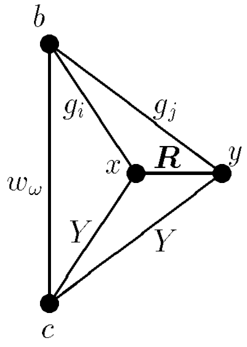
The real danger is that $\mathcal{R}$ might also have elements that cannot occur in an infinite clique. These elements are in $\mathcal{A}(\mathcal{R})$, too, and must have non-empty interpretations in any representation of $\mathcal{A}(\mathcal{R})$. What if they, too, get 'stretched' by the new atoms? The result will be an infinite clique involving them — impossible.

The second idea behind the definition of $\mathcal{A}(\mathcal{R})$ deals with this:

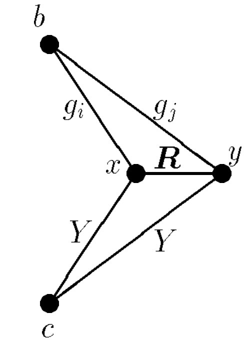
2. Roughly, no 'stretching' need occur on any particular x, y in a representation of $\mathcal{A}(\mathcal{R})$.

It is hard to express this precisely at the moment: the real meaning will come through in theorem 7.2 below. But as a taster, suppose that x, y are points of a representation M of $\mathcal{A}(\mathcal{R})$

that are related by a red element. It may be that x, y have the form a_i, a_j as above. Must they *necessarily* have this form? That is, must they necessarily be in a part of the representation that gets stretched? This will be so if the following pattern occurs in M:



Now $g_i;g_j \geq \boldsymbol{R} \wedge \neg Id$ and $Y;Y \geq \boldsymbol{R}$ in $\mathcal{A}(\mathcal{R})$. Since M is a representation of $\mathcal{A}(\mathcal{R})$, if $M \models \boldsymbol{R}(x, y)$ then there will certainly be nodes $b, c \in M$ as follows:



But the relation between b and c need not be w_ω. For example, it might be $w_{\{i,j\}}$. This white atom does not generate infinite cliques. So x, y are not necessarily stretched.

To ensure that in such cases the relation between b, c actually *is* $w_{\{i,j\}}$, rather than w_ω or some other dangerous atom, we will use the game-theoretic forcing techniques outlined in the preceding section. The argument has the character of an omitting-types theorem, as in [Hodg1, Hodg2], for example, in that every possible place where a w_ω could arise has to be looked after. The extra atom B is needed to get the construction to work.

7.2 Constructing a representation of $\mathcal{A}(\mathcal{R})$

We now proceed to the formal result and proof.

Theorem 7.2 *Let $\mathcal{R}$ be a countable integral weakly associative relation algebra. Then:*

1. *$\mathcal{A}(\mathcal{R})$ has a representation iff $\mathcal{R}$ has a relativised representation with an infinite clique.*
2. *$\mathcal{A}(\mathcal{R})$ has a complete representation iff $\mathcal{R}$ has a complete relativised representation with an infinite clique.*
3. *$\mathcal{A}(\mathcal{R})$ satisfies the Lyndon conditions iff for all finite n, $\mathcal{R}$ has a complete relativised representation containing an n-element clique.*

Remark 7.3 (2) and (3) imply that $\mathcal{R}$ is atomic. Clearly, this is iff $\mathcal{A}(\mathcal{R})$ is atomic.

Proof. This is similar to the proof in [HH2]. We will prove parts 1 and 2 simultaneously.

For '$\Rightarrow$', let M be a representation of $\mathcal{A}(\mathcal{R})$. Consider the reduct $M\lceil\mathcal{R}$ to the signature $\mathcal{R}$.

Claim 1. $M\lceil\mathcal{R}$ is a relativised representation of $\mathcal{R}$.

Proof of claim. As the checks are really purely routine, we will only look at the last axiom: namely, if $r, s, t \in \mathcal{R}$ and $\mathcal{R} \models r = s;t$, then a relativised representation should satisfy $\forall x, y[1(x, y) \to (r(x, y) \leftrightarrow \exists z(s(x, z) \wedge t(z, y)))]$. So let $x, y \in M$, and suppose that $M \models \boldsymbol{R}(x, y)$ (remember that we write $\boldsymbol{R}$ for the '1' of $\mathcal{R}$). We require that $M \models r(x, y) \leftrightarrow \exists z(s(x, z) \wedge t(z, y))$. Because M is a representation of $\mathcal{A}(\mathcal{R})$, this is to say that $M \models r(x, y) \leftrightarrow [s;t](x, y)$, the product $s;t$ being evaluated in $\mathcal{A}(\mathcal{R})$.

Recall from example 5.8 that $\mathcal{A}(\mathcal{R}) \models s;t \wedge \boldsymbol{R} = r$. As M is a representation of $\mathcal{A}(\mathcal{R})$, and $M \models \boldsymbol{R}(x, y)$, we see that $M \models r(x, y) \leftrightarrow [s;t](x, y)$, as required. This proves the claim.

We show that $M\lceil\mathcal{R}$ contains an infinite clique. Choose $x, y \in M$ with $M \models w_\omega(x, y)$. By example 5.8, $w_\omega \leq g_i;Y$. So there is $z_i \in M$ with $M \models g_i(x, z_i) \wedge Y(z_i, y)$.

Claim 2. $M \models \boldsymbol{R}(z_i, z_j)$ for all $i, j < \omega$.

Proof of claim. This is clear if $i = j$, since $Id \leq \boldsymbol{R}$. If $i \neq j$, then $M \models [Y;Y \wedge g_i;g_j](z_i, z_j)$. So by example 5.8, $M \models \boldsymbol{R}(z_i, z_j)$. This proves the claim.

If $i \neq j$ then $M \models [g_i;g_j](z_i, z_j)$, so $z_i \neq z_j$. Hence, $\{z_i : i < \omega\}$ is an infinite clique. Evidently, if M is a complete representation of $\mathcal{A}(\mathcal{R})$ then $M\lceil\mathcal{R}$ is a complete relativised representation of $\mathcal{R}$. This proves '$\Rightarrow$' of parts 1 and 2.

Conversely, assume that $\mathcal{R}$ has a relativised representation V containing an infinite clique, K. If $x, y \in K$, we will write $K(x, y)$ for the ultrafilter $\{r \in \mathcal{R} : V \models r(x, y)\}$ of $\mathcal{R}$. We show that $\exists$ has a winning strategy in the game $F_\omega(\mathcal{A}(\mathcal{R}))$. By proposition 6.7(1), this will yield the result.

The game proceeds by building ultrafilter networks $N_1, N_2, \ldots$ To help her win, $\exists$ will keep the following properties, for each $i < \omega$.

- α For all $x, y \in N_i$, if $\neg\boldsymbol{R} \in N_i(x, y)$ then $N_i(x, y)$ is principal.
- β There is an injective (1–1) map $' : N_i \to K$.
- γ For all distinct $x, y \in N_i$, if $\boldsymbol{R} \in N_i(x, y)$ and $\exists$ chose the label on the edge (x, y) (recall notation 6.5) then $N_i(x, y) = K(x', y')\!\uparrow$. (If f is an ultrafilter of $\mathcal{R}$, we write $f\!\uparrow$ for the unique ultrafilter of $\mathcal{A}(\mathcal{R})$ generated by f.)

In round 1, $\forall$ has to play Id, and $\exists$ responds with the ultrafilter network N_1 with a single node, x, say, with $N_1(x, x)$ the principal ultrafilter generated by Id. She also defines x' to be an arbitrary point of K.

Assume inductively that in some round, i, of the game, the ultrafilter network $N = N_{i-1}$ and the map $' : N \to K$ have been built, so that properties α–γ hold. Let $\forall$ begin the next round of the game by choosing points $x, y \in N$ and elements $r, s \in \mathcal{A}(\mathcal{R})$ with $r;s \in N(x, y)$. $\exists$ must respond with a network $N^+ = N_i \supseteq N$ containing a node z with $r \in N^+(x, z)$ and $s \in N^+(z, y)$. If necessary, she must also extend the map $'$ to N^+ so that α–γ hold for N^+.

Write r^0, r^1, r^2 for $r \wedge Id$, $r \wedge \boldsymbol{R} \wedge \neg Id$, $r \wedge \neg\boldsymbol{R}$, respectively, and make a similar definition for s. By distributivity of ';' over $\vee$ in $\mathcal{A}(\mathcal{R})$, and the fact that $N(x, y)$ is an ultrafilter, there are $e, e' < 3$ with $r^e; s^{e'} \in N(x, y)$. Suppose that $\forall$ had played $\langle x, y, r^e, s^{e'}\rangle$ instead of $\langle x, y, r, s\rangle$. If $\exists$ can find a suitable N^+, then the ultrafilter $N^+(x, y)$ will contain r^e, and hence also r, as required; and similarly for s. So we can assume without loss of generality that r and s intersect exactly one of $Id, \boldsymbol{R} \wedge \neg Id, \neg\boldsymbol{R}$.

If there is already a node $z \in N$ with $r \in N(x, z)$ and $s \in N(z, y)$, then $\exists$ need do nothing. So assume otherwise. In particular, this means that r, s are disjoint from Id. $\exists$ proceeds as follows.

1. First, she picks some arbitrary $z' \in K \setminus \{t' : t \in N\}$. This is possible, since K is infinite.

2. Next, she begins to define N^+. She lets $N^+(z,z)$ be the principal ultrafilter generated by Id, of course.

3. Now she must define $N^+(x,z)$ and $N^+(z,y)$ and their converses. To keep property α, we require:

   Claim 3. There are ultrafilters f, h of $\mathcal{A}(\mathcal{R})$ with $r \in f$, $s \in h$ and $f; h \subseteq N(x,y)$ such that if $\neg \boldsymbol{R} \in f$ then f is principal, and similarly for h.

   Proof of claim. Using Zorn's lemma and the triangle law, it is easy to find f, h if we do not require them to be principal. The technique is well known and we will not go into it here.

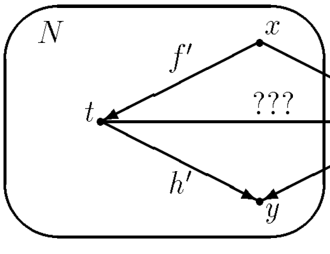
   We can obtain the full claim as follows. Consider only r; the situation for s is similar. If $r \leq \boldsymbol{R}$, there is nothing further to prove. Assume then that $r \leq \neg\boldsymbol{R}$. We wish to replace r by some atom $a \leq r$. So long as we still have $a; s \in N(x,y)$, we may continue as before, using Zorn's lemma.

   There are two cases. First, assume that $N(x,y)$ is principal, generated by the atom b, say. In this case, $r; s \wedge b \neq 0$, so by the triangle axiom for $\mathcal{A}(\mathcal{R})$, $r \wedge b; s^{\smile} \neq 0$. So, as $r \leq \neg\boldsymbol{R}$, there is an atom $a \leq r$ with $a \wedge b; s^{\smile} \neq 0$. Because b is an atom, the triangle axiom yields $a; s \geq b$, so that $a; s \in N(x,y)$.

   Now assume that $N(x,y)$ is non-principal. So the atom Id is not in $N(x,y)$, and, by property α, $\boldsymbol{R} \in N(x,y)$. Let $t \in N(x,y)$ with $t \leq \boldsymbol{R} \wedge \neg Id$. As $r; s \in N(x,y)$, the triangle $(r, s, t^{\smile})$ is consistent ($\notin \mathcal{T}$). So there must be some $(a,b,c) \sqsubseteq (r,s,t^{\smile})$ that is not in $\mathcal{T}_1$, so is *not* obtainable from table 1, definition 5.3. As $a \leq \neg\boldsymbol{R}$ and $c \leq \boldsymbol{R}$, only item 2 is relevant; applying it, we see that $b \neq Id$. It follows that for all $u \in N(x,y)$, $(a, b, (t \wedge u)^{\smile}) \sqsubseteq (a, s, u^{\smile})$ and $(a, b, (t \wedge u)^{\smile})$ is not obtainable from table 1. So $(a, s, u^{\smile})$ is consistent, for all $u \in N(x,y)$. But $N(x,y)$ is an ultrafilter, so $a; s \in N(x,y)$. As a is obviously an atom of $\mathcal{A}(\mathcal{R})$, this proves the claim.

   $\exists$ then defines $N^+(x,z) = f$, $N^+(z,x) = f^{\smile}$, and $N^+(z,y) = h$, $N^+(y,z) = h^{\smile}$ (from claim 3). It can be checked that if $x = y$ then $f^{\smile} = h$, so this is well-defined. Notice that because r, s are disjoint from Id, we have $Id \notin f, h, f^{\smile}, h^{\smile}$. Also note that $\forall$ chose the labels on the edges $\{x,z)\}, \{z,y)\}$. So the condition on the map $v \mapsto v'$ is kept, so far.

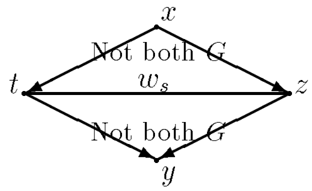
4. If the only nodes of N are x, y, then $\exists$ has finished. Assume otherwise. Pick $t \in N$, $t \neq x, y$. $\exists$ must define $N^+(z,t)$. Let $N(x,t) = f'$ and $N(t,y) = h'$ (see the diagram below, but remember that x and y might be equal).

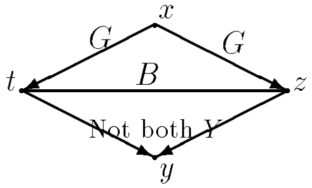
   Write G for $\neg(Id \vee g_0; w_{\{0\}})$. Note that $G = \bigvee\{g_i : i < \omega\}$ (cf. example 5.8).

   There are three cases.

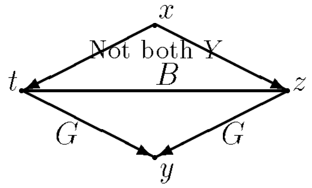
   Case 1: Not both of f, f' contain G, and, moreover, not both of h, h' contain G. Then $\exists$ defines both $N^+(z,t)$ and $N^+(t,z)$ to be the (self-converse) principal ultrafilter of $\mathcal{A}(\mathcal{R})$ generated by w_s, where $s = \{k < \omega : g_k \in f \cup f' \cup h \cup h'\}$. By the case assumption, $|s| \leq 2$.



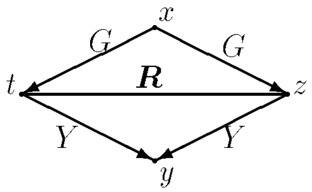
Case 2: $G \in f \cap f'$ and not both of h, h' contain Y; or the mirror situation: $G \in h \cap h'$ and not both of f, f' contain Y. In this case, $\exists$ defines $N^+(z,t)$ and $N^+(t,z)$ to be the principal ultrafilter generated by B.



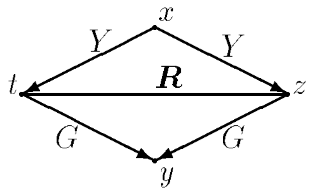
or



Case 3: $G \in f \cap f'$ and $Y \in h \cap h'$; or the mirror situation, $Y \in f \cap f'$ and $G \in h \cap h'$. In this case, $\exists$ has no choice but to include $\boldsymbol{R}$ in $N^+(z,t)$. She defines $N^+(z,t) = K(z',t')\!\uparrow$ and $N^+(t,z) = K(t',z')\!\uparrow$.



or



This completes the definition of N^+. Notice that the conditions α–γ have been preserved.

We check that N^+ is an ultrafilter network (see definition 6.2). First, it is clear from claim 3 and the three cases above that Id is not in $N^+(z,t)$ or $N^+(t,z)$ for any $t \in N$. So the first defining property of ultrafilter networks holds. The second property is 'for all $x', y', z' \in N^+$, we have $N^+(x',y'); N^+(y',z') \subseteq N^+(x',z')$'. After a little thought on the case where two or more of x', y', z' are equal, we see that it is sufficient to show

($*$) If $t, u \in N$ are distinct, $q_1, q_2, q_3 \in \mathcal{Q}$, and $q_1 \in N^+(z,t)$, $q_2 \in N^+(t,u)$, $q_3 \in N(u,z)$, then (q_1, q_2, q_3) is not obtainable from table 1, definition 5.3.

We refer to this condition as 'the triangle (z,t,u) is consistent', and we call such a (q_1,q_2,q_3) a *basic triple for z,t,u.*

For assume ($*$), let t, u be distinct nodes of N, and suppose that $r_1 \in N^+(z,t)$, $r_2 \in N^+(t,u)$, and $r_3 \in N(u,z)$. Now by property α, there is $q_1 \in \mathcal{Q}$ with $q_1 \in N^+(z,t)$ and $q_1 \leq r_1$; and similarly for r_2, r_3. By ($*$), $(r_1,r_2,r_3) \notin \mathcal{T}$. It follows easily that $N^+(z,t); N^+(t,u) \subseteq N^+(z,u)$, and similarly for all permutations of z,t,u.

Case A: $\{t,u\} = \{x,y\}$. The consistency of the triangle (z,x,y) follows immediately from claim 3.

Case B: $|\{t,u\} \cap \{x,y\}| = 1$. This case covers triangles of the form (z,x,t) and (z,y,t) for $t \neq x,y$. Consider (z,x,t); the other side is similar. Suppose that $\exists$ used case 1 above. Let (w_s, q_2, q_3) be a basic triple for z,x,t. Then q_2, q_3 are both disjoint from Id, and either q_2 or q_3 is disjoint from G. If by chance we have $\{q_2,q_3\} = \{g_j, Y\}$ for some j, then $j \in s$. So in any case, (w_s, q_2, q_3) is not obtainable from table 1.

If $\exists$ used case 2, a basic triple for z,x,t has the form (B, g_i, g_j) or (B, q_2, q_3) with q_2, q_3 not both Y. This is not obtainable from table 1.

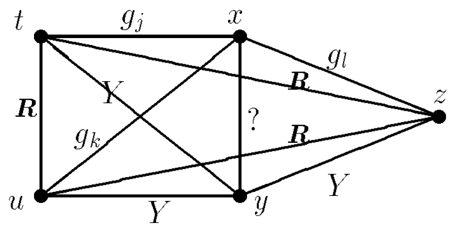
If case 3 was used, a basic triple for z,x,t has the form (q, g_i, g_j) or (q, Y, Y) with $q \leq \boldsymbol{R} \wedge \neg Id$. Again, this is not obtainable from table 1.

Case C: $\{t,u\}\cap\{x,y\}=\varnothing$, and at least one of $N^+(z,t), N^+(z,u)$ contains $\neg\boldsymbol{R}$. So $\exists$ chose the labels on (z,t) and (z,u), and at least one of them contains B or a white atom. The other cannot contain G or Y, since $\exists$ never chooses these colours. So a basic triple of z,t,u has the form (B,q_1,q_2) or (w_s,q_1,q_2), where q_1 is not green, yellow, or Id, and $q_2 \neq Id$ — and consistency is assured.

Case D: $\{t,u\}\cap\{x,y\}=\varnothing$, and both of $N^+(z,t), N^+(z,u)$ contain $\boldsymbol{R}$. So $\exists$ used K to define $N^+(z,t), N^+(z,u)$. If $\neg\boldsymbol{R}\in N(t,u)$, consistency is not in doubt, since no basic triple not involving Id and with exactly one element disjoint from $\boldsymbol{R}$ can arise from table 1. So assume $\boldsymbol{R}\in N(t,u)$. Then if $\exists$ chose the label on (t,u), we have $N^+(z,t)=K(z',t')\uparrow$, $N^+(z,u)=K(z',u')\uparrow$, and $N(t,u)=K(t',u')\uparrow$. So if (q_1,q_2,q_3) is a basic triple of z,t,u, then $q_1,q_2,q_3\leq\boldsymbol{R}$; and as K is a clique in a representation of $\mathcal{R}$, we have $\mathcal{R}\models q_1;q_2\wedge q_3^{\smile}\neq 0$. So (q_1,q_2,q_3) is not obtainable from table 1 in definition 5.3. Hence, z,t,u is consistent.

But what if $\forall$ chose the label on (t,u)? This would have happened in some earlier move, of course. In fact, this is impossible.

Assume for contradiction that he did. Also assume — inductively — that $\exists$ used the strategy we have described in all previous moves of the game. Then, as case 3 above was used to define $N^+(z,t)$ and $N^+(z,u)$, we must have the situation shown in the diagram below (or its mirror image, swapping x and y). In particular, there are indices $j,k<\omega$, not necessarily distinct, with $g_j\in N(x,t)$ and $g_k\in N(x,u)$ — we know this because these ultrafilters are principal and contain G.



Suppose that v was the last point of $\{x,y,t,u\}$ to be built, as the game proceeded. Now in each round, $\forall$ prescribes elements of exactly two ultrafilters. Hence, in the round when v was built, $\exists$ must have chosen the label on at least one of the edges in $\{x,y,t,u\}$. But she never chooses an ultrafilter containing G or Y, and $\forall$ chose elements of $N(t,u)$. It follows that $\exists$ must have chosen the label on (x,y). This also shows that $v\in\{x,y\}$, and that in the round when v was built, $\forall$ chose t,u as his two nodes. Now, examining $\exists$'s strategy, which we assumed was used, we see that she must have chosen $N(x,y)$ containing the white atom $w_{\{j,k\}}$.

Returning to the current round, we note that the triangle (x,y,z) is certainly consistent ($\forall$ chose it). Hence, $l\in\{j,k\}$. *But this makes the new node z into a duplicate of either t or u.* That is, since $g_l\leq r$ and $Y\leq s$, we have $r\in N(x,v)$ and $s\in N(v,y)$ for some $v\in\{t,u\}$. So $\exists$ could have defined $N^+=N$, since there is already an appropriate node, t or u, in N. Since we assumed at the outset that she could not do this, we have a contradiction.

We are done: we have shown that N^+ is an ultrafilter network. $\exists$ now plays $N_i=N^+$ in response to $\forall$ in round i. She continues to use this strategy in subsequent rounds. She will survive for ω rounds without getting stuck. Because $\mathcal{A}(\mathcal{R})$ is countable, proposition 6.7 implies that it is representable, as we said.

Notice that if all ultrafilters $K(x,y)$ are principal, then by property α, $\exists$ can always choose principal ultrafilters throughout the game. Hence, by proposition 6.7(2), if $\mathcal{R}$ has a complete relativised representation with an infinite clique, $\mathcal{A}(\mathcal{R})$ will have a complete representation. This ends the proof of parts 1 and 2.

The proof of part 3 is similar, and we only summarise it. By proposition 6.7(3), $\mathcal{A}(\mathcal{R})$ satisfies the Lyndon conditions if and only if for all finite n, $\exists$ has a winning strategy for $F_n(\mathcal{A}(\mathcal{R}))$ that plays only principal ultrafilters. Now, for '$\Leftarrow$', just note that if $\mathcal{R}$ has a complete relativised representation containing a clique of size n, then the strategy described above allows $\exists$ to win $F_n(\mathcal{A}(\mathcal{R}))$ while playing only principal ultrafilters. Conversely, if she has such a strategy in $F_n(\mathcal{A}(\mathcal{R}))$, $\forall$ first forces her to produce an ultrafilter network N with nodes 1, 2 such that $w_\omega \in N(1,2)$; in subsequent moves, he plays $\langle 1, 2, g_i, Y\rangle$ for $i = 3, 4, \ldots, n$. Let the outcome be N', with nodes $1, 2, \ldots, n$. As before, $\boldsymbol{R} \in N'(i,j)$ for all $3 \leq i < j \leq n$. So $N' \lceil \{3, 4, \ldots, n\}$ is essentially an ultrafilter network over $\mathcal{R}$, of size $n-2$, all of whose ultrafilters are principal. By lemma 4.1, it may be extended to a complete relativised representation of $\mathcal{R}$, in which it is a clique of size $n-2$. ■

The following is a corollary of the proof of theorem 7.2, rather than its statement.

Corollary 7.4 *If $\mathcal{R}$ is an integral weakly associative relation algebra, then $\mathcal{A}(\mathcal{R})$ is a relation algebra.*

Proof. We know (remark 5.4) that $\mathcal{A}(\mathcal{R})$ is a weakly associative relation algebra. It is easily seen that the strategy we described allows $\exists$ to win the ultrafilter network game $F_4(\mathcal{A}(\mathcal{R}))$ of length 4, without any assumptions on $\mathcal{R}$. ($\exists$ will only have to choose a red edge not containing Id at most once; and if we examine case 3 above, which she will use in such a situation, we see that she can use any ultrafilter of $\mathcal{R}$ containing $\neg Id$.) Associativity of $\mathcal{A}(\mathcal{R})$ follows by proposition 6.6. ■

8 Counterexamples

We can now reap the fruit of our hard work by reading off some counterexamples to complete the proof of theorem 2.10.

We will apply the term 'atom structure' to the structure induced on the atoms of an atomic *weakly associative* relation algebra in the same way as a conventional atomic relation algebra induces structure on its atoms. Of course, such an atom structure need not satisfy the associativity axiom. We extend the meaning of the terms 'full power set algebra' and 'term algebra' to these 'weakly associative atom structures' in a similar way.

Definition 8.1

1. For each $n \leq \omega$, $n \geq 3$, let the countable integral weakly associative relation algebra $\mathcal{R}_n$ be the term algebra whose atoms are Id, r^i_j $(3 \leq i < \omega, j < i, j < n)$. All elements are self-converse, and the only consistent triangles are those extending:

   (a) Peircean transforms of (r^i_j, r^i_j, Id), for all i, j

   (b) (r^i_j, r^i_k, r^i_l), if j, k, l are distinct (i is arbitrary).

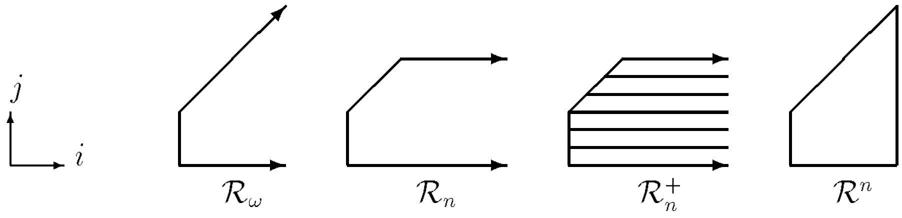
   It is easily checked that the requirements of a class of consistent triangles are met. Hence, by proposition 4.5, $\mathcal{R}_n$ is a weakly associative relation algebra.

2. Observe that every element of $\mathcal{R}_n$ is a finite or cofinite subset of the set of atoms of $\mathcal{R}_n$.

3. If $n < \omega$, let $\mathcal{R}^+_n$ be the closure of $\mathcal{R}_n \cup \{R_j : j < n\}$ under the algebra operations in the full power set algebra, where $R_j = \{r^i_j : 3 \leq i < \omega, i > j\}$. Notice that $\mathcal{R}^+_n$ is countable, atomic, and has the same atom structure as $\mathcal{R}_n$.

4. We also define $\mathcal{R}^n$ to be the finite integral algebra whose atoms are precisely $\{r^i_j : 3 \leq i < n, j < i\}$, consistent triangles and converse being as above.

The following diagram may help.



8.1 Complete representations

In [HH2], what was essentially $\mathcal{A}(\mathcal{R}_\omega)$ was used to show that the class of completely representable relation algebras is not elementary. We can prove this here, too.

Proposition 8.2 *For all $m < \omega$, $\mathcal{R}_\omega$ has a complete relativised representation containing a clique of size at least m; but it has no complete relativised representation containing an infinite clique.*

Proof. The first part follows easily from lemma 4.1. Let K be the complete graph on m vertices $(3 \leq m < \omega)$. To each graph edge in K, allocate a unique atom, r_j^i, where $i = m(m-1)/2$, and $j < i$ is unique to that edge. This essentially defines a consistent $\mathcal{R}_\omega$-network, which, by lemma 4.1, extends to a complete relativised representation of $\mathcal{R}_\omega$. In this relativised representation, K is a clique of size m.

For the second part, suppose for a contradiction that some complete relativised representation M of $\mathcal{R}_\omega$ contains an infinite clique, K. Fix $x \in K$. Then by definition of consistency in $\mathcal{R}_\omega$, for all $y \in K, y \neq x$, we must have $M \models r^i_{j(y)}(x,y)$ for some fixed i and some $j(y) < i$. Since all the $j(y)$ must be distinct, this is impossible. ∎

Corollary 8.3 *$\mathcal{A}(\mathcal{R}_\omega)$ satisfies the Lyndon conditions, but has no complete representation.*

Proof. This was proved in [HH2], and here it follows immediately from the preceding proposition and theorem 7.2(2,3). ∎

By fact 2.13, $\mathcal{A}(\mathcal{R}_\omega)$ has an elementary extension that has a complete representation. Hence, the class of completely representable relation algebras is not elementary.

8.2 The logic $L^\omega_{\infty\omega}$

Theorem 2.10 is couched in terms of the logic $L^\omega_{\infty\omega}$. Let us define this more formally than before.

Let L be any first-order signature — here, it will be the signature of atom structures, $\{Id, {}^\smile, R\}$. Let $x_1, x_2, \ldots$ be an ω-sequence of distinct variables. Let $0 < k < \omega$. The formulas of $L^k_{\infty\omega}$ are defined as follows.

1. Any atomic formula, written with variables $x_1, \ldots, x_k$ only, is an $L^k_{\infty\omega}$-formula. Thus, the atomic formulas are just $Id(x_i^{\smile\smile\cdots\smile})$, $R(x_i^{\smile\smile\cdots\smile}, x_j^{\smile\smile\cdots\smile}, x_l^{\smile\smile\cdots\smile})$, and $x_i^{\smile\smile\cdots\smile} = x_j^{\smile\smile\cdots\smile}$, where $1 \leq i,j,l \leq k$, and, of course, $x_i^{\smile\smile\cdots\smile}$ could be just x_i.
2. If Φ is any set of $L^k_{\infty\omega}$-formulas, then $\bigvee \Phi$ and $\bigwedge \Phi$ are $L^k_{\infty\omega}$-formulas.
3. If φ is an $L^k_{\infty\omega}$-formula, then $\neg\varphi$ and $\exists x_i \varphi$ are also $L^k_{\infty\omega}$-formulas, for each $i = 1, \ldots, k$.

Then φ is an $L^\omega_{\infty\omega}$-formula if and only if it is an $L^k_{\infty\omega}$-formula for some $k < \omega$. In symbols, this is usually written as $L^\omega_{\infty\omega} = \bigcup_{k<\omega} L^k_{\infty\omega}$. Clearly, any first-order formula is an $L^\omega_{\infty\omega}$-formula.

The semantics of these logics are defined in the obvious way. If M, N are L-structures, and $k \leq \omega$, we say that M and N are $L^k_{\infty\omega}$-equivalent, and write $M \equiv^k_{\infty\omega} N$, if for all sentences σ of $L^k_{\infty\omega}$, we have $M \models \sigma \iff N \models \sigma$.

An elegant game characterisation of $\equiv^k_{\infty\omega}$ for finite k was given essentially by Barwise in [B]; also see [I,P], and the earlier, related paper [K]. Players $\forall$ and $\exists$ play a back-and-forth Ehrenfeucht–Fraïssé-style k-pebble game between two L-structures, M and N. There are k pairs of pebbles. Initially, there are no pebbles in play. There are ω rounds. In each round, $\forall$ picks up one pebble of any pair, and places it on an element of one of the structures. $\exists$ has to place the other pebble of the pair on an element of the other structure. (If all pebbles are already in play, this involves *moving* pebbles.) $\exists$ loses at this stage if the relation $\subseteq M \times N$ defined by the pebble pairs is not a partial isomorphism from M to N (that is, it does not preserve all atomic and negated atomic L-formulas). Otherwise, the game proceeds for another round. As usual, if $\exists$ can survive each round, she wins.

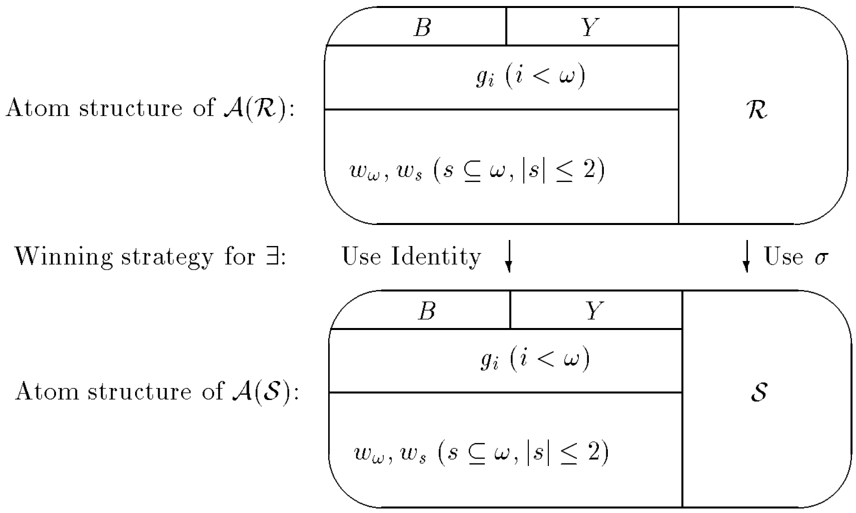
Proposition 8.4 (Barwise, [B]) *Let M, N be L-structures, and let $k < \omega$. Then $M \equiv^k_{\infty\omega} N$ if and only if $\exists$ has a winning strategy in the k-pebble game played on M, N.* ■

As an example, consider the following proposition.

Proposition 8.5 *Let $\mathcal{R}, \mathcal{S}$ be atomic weakly associative relation algebras, and suppose that their atom structures are $L^k_{\infty\omega}$-equivalent, for some $k < \omega$. Then the atom structures of $\mathcal{A}(\mathcal{R})$ and $\mathcal{A}(\mathcal{S})$ are also $L^k_{\infty\omega}$-equivalent.*

Proof. This is proved by a Feferman–Vaught-style argument. Assume that $\exists$ has a winning strategy σ in the k-pebble game on the atom structures of $\mathcal{R}, \mathcal{S}$. She can adopt the following strategy in the k-pebble game on the atom structures of $\mathcal{A}(\mathcal{R})$ and $\mathcal{A}(\mathcal{S})$.

- If $\forall$ moves a pebble onto an atom in $\mathcal{R}$ or $\mathcal{S}$, then $\exists$ removes all pebbles *not* on atoms in $\mathcal{R}$ or $\mathcal{S}$, responds using σ, and then puts these pebbles back in their original places.
- If $\forall$ moves a pebble onto a yellow, green, white, or black atom, then $\exists$ moves the other pebble of the pair onto the same atom in the other structure.



It is easily seen that this will win for her. The result now follows from proposition 8.4. ■

8.3 Non-definability of weak representability in $L^\omega_{\infty\omega}$

Now we can show that the class of weakly representable atom structures is not definable by any sentence of $L^\omega_{\infty\omega}$. This will establish part 4 of theorem 2.10.

Lemma 8.6 *For any $n < \omega$, $n \geq 3$, the atom structure of $\mathcal{A}(\mathcal{R}^n)$ is not weakly representable.*

Proof. $\mathcal{R}^n$ is finite, so any relativised representation of it is complete. Now, as in proposition 8.2, it can be seen that $\mathcal{R}^n$ has no relativised representation containing an infinite clique. So by theorem 7.2, $\mathcal{A}(\mathcal{R}^n)$ is not representable. Recalling that $\mathcal{A}(\mathcal{R}^n)$ is by definition the term algebra over its atom structure, it follows that this atom structure is not weakly representable. ■

Lemma 8.7 *Let $k < \omega$. Then, for all large enough $n < \omega$, the atom structures of $\mathcal{A}(\mathcal{R}^n)$ and $\mathcal{A}(\mathcal{R}_\omega)$ are $L^k_{\infty\omega}$-equivalent.*

Proof. If $n \geq 2k$, then there are insufficient pebbles for $\forall$ to distinguish the atom structures of $\mathcal{R}^n$ and $\mathcal{R}_\omega$ in the k-pebble game, even in ω moves. Hence, by proposition 8.4, $\mathcal{R}^n \equiv^k_{\infty\omega} \mathcal{R}_\omega$ The result now follows from proposition 8.5. ■

Corollary 8.8 *The class of weakly representable atom structures is not definable by any sentence of $L^\omega_{\infty\omega}$.*

Proof. Suppose for contradiction that there is a sentence σ of $L^k_{\infty\omega}$, for some $k < \omega$, such that a (relation algebra) atom structure satisfies σ if and only if it is weakly representable. By corollary 8.3, $\mathcal{A}(\mathcal{R}_\omega)$ satisfies the Lyndon conditions. Hence, by theorem 2.10(2) and lemma 2.9(1), its atom structure is weakly representable, so it satisfies σ. By proposition 8.7, for all large enough n, the atom structure of $\mathcal{A}(\mathcal{R}^n)$ also satisfies σ. Since this atom structure is not weakly representable, we have a contradiction. ■

8.4 Weak and strong representability differ

Here, we show that representability of an atomic relation algebra does not depend on its atom structure alone. This will establish part 5 of theorem 2.10.

Lemma 8.9 *For all $3 \leq n < \omega$, $\mathcal{R}_n$ has a relativised representation containing an infinite clique, but $\mathcal{R}_n^+$ does not.*

Proof. As we said, any $r \in \mathcal{R}_n$ is either a finite or a cofinite set of atoms of $\mathcal{R}_n$. It follows that there is a unique non-principal ultrafilter $f \subseteq \mathcal{R}_n$, which consists of all the cofinite sets of atoms. Since the product in $\mathcal{R}_n$ of any two cofinite sets of atoms is also cofinite, we have $f;f \subseteq f$. Obviously, $f^\smile = f$ and $Id \notin f$.

Consider a countably infinite set K, made an $\mathcal{R}_n$-structure by: for all $r \in \mathcal{R}_n$ $K \models r(x,x)$ iff $r \geq Id$, and, if $x \neq y$, $K \models r(x,y)$ iff $r \in f$. Then K is essentially an ultrafilter-network over $\mathcal{R}_n$. By lemma 4.1, it embeds into a relativised representation of $\mathcal{R}_n$, in which it is an infinite clique. This proves the first part.

For the second, suppose for a contradiction that M is a relativised representation of $\mathcal{R}_n^+$ with an infinite clique, K. Fix $x \in K$. For each $y \in K$, $y \neq x$, there is $j(y) < n$ with $M \models R_{j(y)}(x,y)$. Choose distinct $y, z \in K \setminus \{x\}$ such that $j(y) = j(z) = j$, say. Notice that $(R_j, R_j, \neg Id)$ is inconsistent. As $M \models 1(y,z) \wedge R_j(y,x) \wedge R_j(x,z)$, we must have $M \models Id(y,z)$. This is impossible, and completes the proof. ■

Corollary 8.10 *For all $3 \leq n < \omega$, $\mathcal{A}(\mathcal{R}_n)$ is representable, but $\mathcal{A}(\mathcal{R}_n^+)$, with the same atom structure, is not. Hence, the atom structure of $\mathcal{A}(\mathcal{R}_n)$ is weakly representable but not strongly representable.*

Proof. Immediate from the preceding lemma and theorem 7.2. ■

8.5 Non-definability of strong representability

Finally, we show that the strongly representable atom structures are not $L^\omega_{\infty\omega}$-definable, even among the weakly representable ones. This will establish part 6 of theorem 2.10.

Lemma 8.11 *For all $k < \omega$, the atom structures of $\mathcal{R}_n$ and $\mathcal{R}_\omega$ are $L^k_{\infty\omega}$-equivalent for all large enough $n < \omega$.*

Proof. Similar to lemma 8.7. Again, $n \geq 2k$ is adequate. ∎

Corollary 8.12 *No sentence of $L^{\omega}_{\infty\omega}$ axiomatises the class of strongly representable atom structures, even over the class of weakly representable atom structures.*

Proof. Assume for contradiction that there is a sentence σ of $L^{k}_{\infty\omega}$, for some finite k, such that any weakly representable atom structure $\mathcal{C}$ satisfies σ iff $\mathcal{C}$ is strongly representable. The atom structure of $\mathcal{A}(\mathcal{R}_\omega)$ satisfies the Lyndon conditions, so, by theorem 2.10(2), is strongly representable, and satisfies σ. But the atom structures of $\mathcal{A}(\mathcal{R}_n)$, for each $n < \omega$, are weakly but not strongly representable, so they do not satisfy σ.

By the preceding lemma, we may choose n so large that the atom structures of $\mathcal{R}_n$ and $\mathcal{R}_\omega$ agree on all sentences of $L^{k}_{\infty\omega}$. By proposition 8.5, the same holds for the atom structures of $\mathcal{A}(\mathcal{R}_n)$ and $\mathcal{A}(\mathcal{R}_\omega)$. This is impossible, as they disagree on σ. ∎

This completes the proof of theorem 2.10.

Appendix: Lyndon conditions and strong representability

Here, we give an example of a strongly representable atom structure that does not satisfy the Lyndon conditions. It is a slightly simpler version of an example given by Maddux in [M1, examples 23, p. 154ff]. We have taken some of the proofs from there, too. Because so much of this is due to Maddux, we relegated it to an appendix. Special thanks go to Robin Hirsch and Yde Venema for discussions and help here.

Fix q with $6 \leq q < \omega$. We write $\mathcal{M}_q$ for the following atom structure. The atoms in $\mathcal{M}_q$ are:

1. e_i $(i < q)$
2. d^{lr}_{ij} $(r < \omega,\ i, j, l < q,\ l \neq i, j)$.

In d^{lr}_{ij}, i, j are called the start and end indices, respectively, l is called the middle index, and r the rank. In e_i, i is called both the start and the end index.

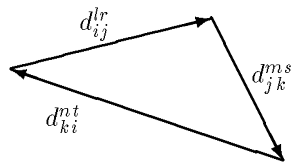
The identity relation $Id^{\mathcal{M}_q}$ is defined to be $\{e_i : i < q\}$. We must also specify the converse function:

3. each e_i is self-converse ($e_i^{\smile} = e_i$)
4. $d^{lr\,\smile}_{ij} = d^{lr}_{ji}$.

Finally, we specify R, the consistency relation. R will hold precisely on all Peircean transforms of the following triples of atoms:

5. $(e_i, e_i, e_i), (d^{lr}_{ij}, e_j, d^{lr}_{ji})$
6. $(d^{lr}_{ij}, d^{ms}_{jk}, d^{nt}_{ki})$, where $r > s, t$ and $l \neq k$.

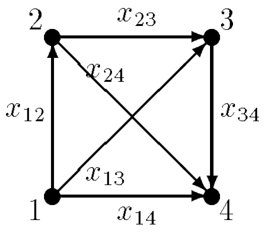
The crucial point of this definition is that a triangle of the form



is consistent iff there is a d-atom among the three that has maximum rank (strictly larger than the others), say d^{lr}_{ij}, and its middle index l is different from the 'induced index', k, of the node opposite.

Lemma 8.13 *$\mathcal{M}_q$ is an atom structure.*

Proof. We refer to definition 2.4. Properties 1–3 are clear. For associativity (4), consider the diagram.



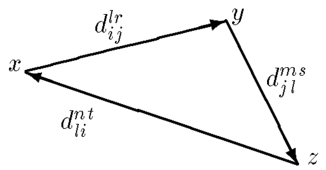
We have to find a value of x_{24} that makes triangles 124 and 234 consistent, for all values of the other variables making triangles 123 and 134 consistent. Fix such values. If one of triangles 123 and 134 involves an identity atom, this immediately forces our choice of x_{24} to be the label on its third side. For example, if $x_{12} = e_k$ for some $k < q$, we must let $x_{24} = x_{14}$. If $x_{14} = e_k$ for some $k < q$, we must let $x_{24} = x_{12}^{\smile}$, and so on. It can be checked that this choice does in fact make triangles 124 and 234 consistent.

So we may assume that no identity atom occurs among the given values. For each node, define its (induced) index to be the start index of (the value of the variable on) any outgoing edge, or the end index of any incoming edge, excluding the edge 24. This is well-defined because triangles 123 and 134 are consistent. At most four indices occur in this way; choose $l < q$ not one of them. Let the induced indices of 2, 4 be i, j, respectively. We may now define x_{24} to be d^{lr}_{ij}, where r is chosen larger than any rank occurring in the diagram. This makes triangles 124 and 234 consistent, as can easily be checked. So associativity holds. ■

Lemma 8.14 (Maddux) *For any q, the atom structure $\mathcal{M}_q$ does not satisfy the qth Lyndon condition.*

Proof. Let $\mathcal{A}$ be the full power set algebra over $\mathcal{M}_q$. We show that $\exists$ has no winning strategy in the game $F_q(\mathcal{A})$ that only ever chooses principal ultrafilters. (This was our definition of the Lyndon conditions, in theorem 6.7(3).)

Assume for contradiction that $\exists$ does have such a strategy. Let her use it in a play of $F_q(\mathcal{A})$, and let $\forall$ use his q moves to ensure that a node is generated of every possible induced index $< q$. So, for example, he could begin by choosing e_0 in round 1. $\exists$ will respond with an ultrafilter network N_1 with sole node x, say, with $N_1(x, x)$ the principal ultrafilter of $\mathcal{A}$ generated by e_0. $\forall$ may now play $\langle x, x, d^{j0}_{0i}, d^{j0}_{i0}\rangle$ in round $i+1$ $(1 \leq i < q)$, where the middle index $j < q$ is arbitrary (with $j \neq 0, i$). The resulting ultrafilter network at the end of play is $N_q = N$, say; and as $\exists$ used her winning strategy, for all distinct $x, y \in N$, the edge (x, y) of N is labelled by a principal ultrafilter — essentially, by an atom, d^{lr}_{ij}. (It cannot be an identity atom, by definition of ultrafilter networks.) Choose (x, y) such that the rank r of the associated atom is maximal. Let $z \in N$ have induced index l; z exists because $\forall$ played as he did. Consider the triangle x, y, z.



If this is consistent, one rank among r, s, t is strictly larger than the rest. It must be r. But as l, the middle index of d_{ij}^{lr}, is *not* different from the induced index of the opposite node, z, the triangle is not consistent.

Thus, N is not a consistent network, and $\exists$ lost the game, a contradiction. ■

It remains to prove:

Lemma 8.15 *$\mathcal{M}_q$ is strongly representable.*

Proof. We wish to show that $\mathcal{A}$, the full power set algebra over $\mathcal{M}_q$, is representable. By theorem 6.7(1), it suffices to show that $\exists$ has a winning strategy in $F_\omega(\mathcal{A})$ — not necessarily choosing principal filters!

An ultrafilter over $\mathcal{A}$ is the same thing as a set ultrafilter over $\mathcal{M}_q$. Consider such an ultrafilter, φ. If it is principal, we can regard it as an atom of $\mathcal{M}_q$. In any event, either it contains some identity atom e_i, or else it contains exactly one of the finitely many sets $D_{ij}^l = \{d_{ij}^{lr} : r < \omega\}$, in which case

$$\varphi^* = \{S \subseteq \omega : \{d_{ij}^{lr} : r \in S\} \in \varphi\}$$

is an ultrafilter on ω. Conversely, any ultrafilter f on ω gives rise to the ultrafilter

$$f_{ij}^l = \{X \subseteq \mathcal{M}_q : \{r < \omega : d_{ij}^{lr} \in X\} \in f\}.$$

Clearly, $(\varphi^*)_{ij}^l = \varphi$ and $(f_{ij}^l)^* = f$, so the correspondence is 1–1.

A 'consistent ultrafilter triangle' is a triple φ, ψ, χ of ultrafilters on $\mathcal{A}$ such that $\varphi; \psi \subseteq \chi^\smile$. Equivalently, for all $r \in \varphi$, $s \in \psi$, $t \in \chi$, there are $a \in r$, $b \in s$, $c \in t$ with $\mathcal{M}_q \models R(a, b, c)$ (cf. definition 6.2). Examining the definition of R, we see that the only consistent ultrafilter triangles are of the following form:

1. e_i, e_i, e_i
2. Peircean transforms of f_{ij}^l, e_j, f_{ji}^l, for any ultrafilter f over ω
3. $f_{ij}^l, g_{jk}^m, h_{ki}^n$, where f, g, h are ultrafilters on ω, and either
   (a) one of the ultrafilters is non-principal (say f), and its middle index (l, here) is not the induced index of the opposite node (k, here); or
   (b) not (a), f, g, h are all principal ultrafilters, and, regarding them as atoms, the triangle is consistent in the sense defined for $\mathcal{M}_q$.

We can now describe a winning strategy for $\exists$ in $F_\omega(\mathcal{A})$. In round 1, $\forall$ plays some e_i, and $\exists$ responds in the obvious (the only possible) way. Suppose that we are in round $n > 1$, so that the current ultrafilter network is N_{n-1}. By definition of ultrafilter networks, $e_i \notin N_{n-1}(x, y)$ for all $i < q$ and all distinct nodes x, y of N_{n-1}. Suppose that $\forall$ elects to play $\langle x, y, r, s\rangle$ in round i, where $x, y \in N$, $r, s \in \mathcal{A}$, and $r; s \in N_{n-1}(x, y)$. $\exists$ must respond with an ultrafilter network $N_n \supseteq N_{n-1}$ with a node z such that $r \in N_n(x, z)$ and $s \in N_n(z, y)$.

If there is such a node $z \in N_{n-1}$ already, $\exists$ sets $N_n = N_{n-1}$. Otherwise, she chooses (using Zorn's lemma) any ultrafilters g, h on $\mathcal{A}$ containing r, s, respectively, and with $g; h \subseteq N_{n-1}(x, y)$. There will be no identity atom in g or in h — otherwise, there would have been a suitable z already in N_{n-1}. $\exists$ now creates a new node $z \notin N_{n-1}$, and defines $N_n(x, z) = g$, $N_n(z, x) = g^\smile$, $N_n(z, y) = h$, and $N_n(y, z) = h^\smile$. It can be checked that this is well-defined if $x = y$. She also defines $N_n(z, z) = e_k$, where $k < q$ is the unique index such that some $D_{ik}^l \in g$, and we identify e_k with the principal ultrafilter of $\mathcal{A}$ generated by it.

She now has the task of defining the remaining values $N_n(z, t)$ and $N_n(t, z)$ for $t \in N_{n-1} \setminus \{x, y\}$. Let the induced indices of x, y in N_{n-1} be i, j, respectively. Choose any permutation $'$ of $s = \{l < q : l \neq i, j, k\}$ without fixed points or 2-cycles. This is possible, as $q \geq 6$, so that $|s| \geq 3$. E.g., if

$s = \{1,2,5\}$, we can let $1' = 2$, $2' = 5$, and $5' = 1$. Also, fix a non-principal ultrafilter f on ω. Then $\exists$ defines
$$N_n(z,t) = f_{im}^{m'} \quad \text{and} \quad N_n(t,z) = f_{mi}^{m'},$$
where t has induced index m in N_{n-1}.

We check that N_n is consistent, so an ultrafilter network.

- The triangle xyz is certainly consistent.
- Let $t \in N_{n-1} \setminus \{x,y\}$. Then there is no identity atom in $N_n(x,z)$ or $N_n(t,x)$, $N_n(z,t) = f_{im}^{m'}$, and m' is not the induced index of x. Since f is non-principal, 3(a) above applies, and the triangle xzt is consistent. The triangle yzt is also consistent, for similar reasons.
- If $t, u \in N_{n-1} \setminus \{x,y\}$ are distinct, the triangle ztu is consistent, for the following reason. Let t have induced index m. By exchanging t, u if necessary, we may assume that m' is not the induced index of u. Since there is no identity atom in $N_n(t,u)$ or $N_n(u,z)$, $N_n(z,t) = f_{im}^{m'}$, and f is non-principal, we see as before that ztu is consistent.
- Other checks, such as $N_n(z,t) = N_n(t,z)^{\smile}$ for all t, are trivial to make.

This completes the round. $\exists$ may use this strategy in every round. Since she never loses, the strategy must win the game for her. The proof is complete. ■

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