Kruskal's theorem and Nash-Williams theory

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This is based on notes I took at Wilfrid's seminar for Ph.D. students at Queen Mary College, around autumn 1985. His seminars drew to some extent on [1, 3, 4, 5].

I started this latex version of the notes in about March 1992, and revised and expanded it beyond the material in the 1985 notes in Jan 2003. Section 6 is still incomplete. Some of the improvements may be bad improvements, and there are surely still some errors due to my bad note-taking and transcribing. Use at your own risk.

I strongly recommend and request that *you do not cite these notes* in your own publications. The notes are made available because they may be helpful, but they are not authoritative or formally published. The original sources are [1, 3, 4] and these should be cited in preference. Any results given here that are not in these papers are essentially just exercises.

Thanks to Sz. Mikulás for helpful comments.

1 Well-quasi-orderings

Definition 1.1 *If S is any set and κ a cardinal, $[S]^\kappa$ is the set of subsets of S of size κ, and $[S]^{<\kappa}$ is the set of subsets of S of size less than κ.*

Definition 1.2 *A* pre-order *is a reflexive and transitive binary relation, usually written $\leq, \sqsubseteq$, etc. Given a pre-order $\leq$, we write $x < y$ to abbreviate $x \leq y \wedge y \not\leq x$.*

Definition 1.3 (wqo) *A* well-quasi-ordering (wqo) *is a pre-order such that (i) it is well-founded (it has no infinite strictly descending ($>$) sequences), and (ii) there is no infinite* antichain *(a set of pairwise incomparable elements).*

Lemma 1.4 *If $\leq$ is a pre-order on a set I, then the following are equivalent:*

1. *$\leq$ is a wqo.*

2. *If $x_0, x_1, \ldots \in I$ then there are $i < j$ with $x_i \leq x_j$.*

3. *If $x_0, x_1, \ldots \in I$ then there is an infinite $X \subseteq \mathbb{N}$ such that $x_i \leq x_j$ for all $i < j$ in X.*

4. *Any $X \subseteq I$ has a finite subset of minimal elements: there is finite $Y \subseteq X$ such that $\forall x \in X \exists y \in Y (y \leq x)$.*

Proof. For the equivalence of 1–3, it is enough to show $1 \Rightarrow 3$. Assume (1). Let $f : [\mathbb{N}]^2 \to \{\leq, >, \perp\}$ be such that $x_i f(i,j) x_j$ for all $i < j < \omega$. Here, $a \perp b$ means a and b are incomparable. By Ramsey's theorem (see corollary 4.4 below), there is infinite $X \subseteq \mathbb{N}$ such that $f \restriction [X]^2$ is constant. Assuming (1), the constant value is not $\perp$ or $>$; hence it is $\leq$, and we are done.

Clearly, $4 \Rightarrow 2$. To show $2 \Rightarrow 4$, assume $X \subseteq I$ fails (4). Define $x_0, x_1, \ldots \in X$ by induction: given $n < \omega$, since $\{x_0, \ldots, x_{n-1}\}$ is finite and X fails (4), there is $x_n \in X$ with $x_i \not\leq x_n$ for all $i < n$. Then the sequence $x_0, x_1, \ldots$ fails (2). □

This is reminiscent of König's tree lemma: well-founded replaces the tree condition, and the lack of infinite antichains replaces the finitely-branching condition. The lemma's conclusion, that any infinite sequence of distinct elements contains an infinite increasing subsequence, corresponds to the infinite branch that König's lemma provides.

Now we see a little Nash-Williams theory.

Definition 1.5 *Given a pre-order $\leq$ on I, a* bad *sequence is an infinite sequence $x_0, x_1, \ldots$ in I such that $x_i \not\leq x_j$ whenever $i < j$. We say such a sequence is* minimal bad *if it is bad, and for each n there is no bad sequence $x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots$ with $y_n < x_n$.*

By the lemma, a wqo is one without bad sequences. Any infinite subsequence of a bad sequence is bad.

Lemma 1.6 *If $\leq$ is well-founded but is not a wqo then there is a minimal bad sequence.*

Proof. By induction. There are bad sequences. Choose $x_0 \in I$ minimal such that it is the first element of a bad sequence — this is possible as $\leq$ is well-founded. Inductively, if x_i $(i < n)$ are chosen, choose x_n such that it is minimal such that $x_0, \ldots, x_n$ extends to a bad sequence. Clearly the result $x_0, x_1, \ldots$ is a minimal bad sequence. □

Lemma 1.7 *Let $x_0, x_1, \ldots$ be minimal bad. Put $Y = \{x \in I : x < x_i$ for some $i\}$. Then $\leq$ is a wqo on Y.*

Proof. If not, then there is a bad sequence $y_0, y_1, \ldots$ in Y. Each y_i is $<$ some $x_{i'}$. Choose $i < \omega$ to make i' as small as possible. By throwing away $y_0, \ldots, y_{i-1}$ we can assume $i = 0$.

Claim: $x_0, \ldots, x_{i'-1}, y_0, y_1, \ldots$ is bad.

Proof of Claim: If not, then as the x-part and y-part are from bad sequences, we must have $x_m \leq y_n$ for some $m < i'$ and some n. Now by definition of Y, we have $y_n < x_{n'}$ for some n'. By choice of i' as small as possible, $n' \geq i'$. So $x_m \leq y_n < x_{n'}$ and $m < n'$, contradicting $x_0, x_1, \ldots$ being bad. This proves the claim.

The claim contradicts the choice of $x_0, x_1, \ldots$ as minimal bad. Thus the lemma is proved. □

Definition 1.8 *Given a pre-order $(I, \leq)$, define $I^{<\omega}$ to be the set of all finite sequences of elements of I, ordered by: $(x_0, \ldots, x_{m-1}) \leq (y_0, \ldots, y_{n-1})$ iff there is a one-one order-preserving map $f : m \to n$ such that $x_i \leq y_{f(i)}$ for all $i < m$.*

Lemma 1.9 *The relation $\leq$ on $I^{<\omega}$ is a pre-order, and if $<$ is well-founded on I then $<$ is well-founded on $I^{<\omega}$.*

Proof. Reflexivity and transitivity of $\leq$ on $I^{<\omega}$ are clear. If there is an infinite strictly decreasing chain $\bar{x}_i$ $(i < \omega)$ in $I^{<\omega}$, then as the lengths of the $\bar{x}_i$ must form a non-increasing sequence of natural numbers, we can assume the $\bar{x}_i$ all have the same length, m say. For each i there is $j < m$ such that $x_{i+1,j} < x_{ij}$. Hence there is j such that the above holds for infinitely many i. Thus $(x_{ij} : i < \omega)$ contains a strictly decreasing subsequence, contradicting the well-foundedness of $<$ on I. □

Theorem 1.10 (Higman) *If $\leq$ is a wqo on I, then $\leq$ is a wqo on $I^{<\omega}$.*

Proof. By lemma 1.9, $\leq$ is well-founded. If the theorem fails, then by lemma 1.9 there is a minimal bad sequence $\bar{x}_0, \bar{x}_1, \ldots$ of sequences $\bar{x}_i = (x_{i,0}, \ldots, x_{i,j_i-1})$. Now the null sequence is vacuously $\leq$ any sequence, so each $\bar{x}_i$ has length ≥ 1. So we can write it as $(x_{i,0}, \mathrm{tail}(\bar{x}_i))$. As $\mathrm{tail}(\bar{x}_i)$ has length $< \mathrm{len}(\bar{x}_i)$, we have $\mathrm{tail}(\bar{x}_i) < \bar{x}_i$ in $I^{<\omega}$ (note the strict $<$). So by lemma 1.7, $Y = \{\mathrm{tail}(\bar{x}_i) : i < \omega\}$ is a subset of a well-quasi-ordered set, and so itself well-quasi-ordered. So by lemma 1.4 we can assume that $\mathrm{tail}(\bar{x}_0) \leq \mathrm{tail}(\bar{x}_1) \leq \ldots$. (This might not preserve *minimal* badness, but minimality is no longer needed.) Also, I is a wqo, so there are $i < j$ with $x_{i,0} \leq x_{j,0}$. Hence by "piecing together", $\bar{x}_i \leq \bar{x}_j$, a contradiction. □

Exercise 1.11 Let $(I, \leq)$ be a wqo. Define a relation $\leq$ on the set $\wp_{<\omega}(I)$ of finite subsets of I, by $s \leq t$ iff there is a map $f : s \to t$ with $i \leq f(i)$ for all $i \in s$. Show that $(\wp_{<\omega}(I), \leq)$ is a wqo (cf. [1, p. 32]). Repeat for the variant of $\leq$ where f is required to be one-one.

Remark 1.12 (Rado) *Theorem 1.10 fails for the set $I^{<\omega_1}$, the set of countable sequences of elements of I.*

Proof. Let $I = \{(i, j) : i < j < \omega\}$, ordered by $(i, j) \leq (k, l)$ iff either $i = k$ and $j \leq l$, or else $i, j < k$. One can check that this is a wqo on I. Now for $i < \omega$ let α_i be the sequence $((i, i + 1), (i, i + 2), \ldots)$. Then for all $i < j < \omega$, $\alpha_i \not\leq \alpha_j$. So the sequence $(\alpha_i : i < \omega)$ is bad. □

Exercise 1.13 Show that exercise 1.11 fails for the full power set $\wp(I)$.

Exercise 1.14 (cf. exercise 1.11) Let $(I, \leq)$ be a wqo. Define a relation $\preccurlyeq$ on $\wp_{<\omega}(I)$ by $s \preccurlyeq t$ iff there is a map $g : t \to s$ with $g(i) \leq i$ for all $i \in t$. Find $(I, \leq)$ such that $(\wp_{<\omega}(I), \preccurlyeq)$ is not a wqo. [I think this non-preservation is well-known but thanks to Sz. Mikulás for pointing this example out to me.]

2 Kruskal's theorem

Definition 2.1 *A* tree *is a finite connected graph without cycles. Let T, S be trees. We say that T* immediately yields *S if a tree isomorphic to S can be got from T by either removing one "leaf" vertex and its only attaching edge, or turning (a) into (b) below:*

(a) *(b)*

We write $S > T$ if there are $n > 1$ and trees $T_1, \ldots, T_n$ such that $S \cong T_1, T_n \cong T$ and T_i immediately yields T_{i+1}. We write $S \leq T$ if $S < T$ or $S \cong T$.

Note that $|S| < |T|$ if $S < T$. Hence $<$ is well-founded on finite trees. Question: is there an infinite sequence of trees $T_0, T_1, \ldots$ such that if $i < j$ then $T_i \not\leq T_j$ (i.e., a bad sequence)? Kruskal answered "no". We'll prove this.

Definition 2.2 *A* pointed tree *is a tree in which some vertex is distinguished as the "root". A tree is then partially ordered by: $x \leq y$ if the path from the root to y passes through x — x is nearer to the root than y. For any vertices*

x, y there's a greatest lower bound $x \wedge y$, the furthest node from the root on both paths to x and to y.

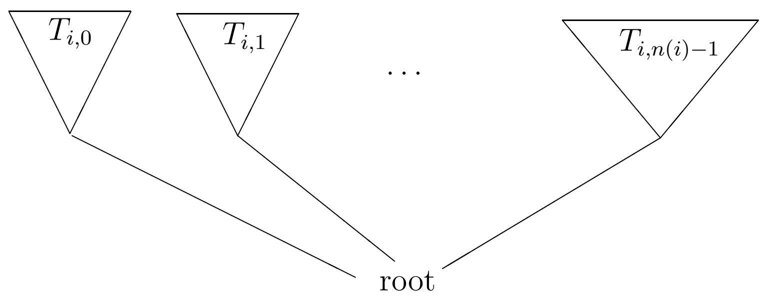
A decent embedding *of pointed trees T, S is a 1-1 map $f : T \to S$ satisfying:*

1. *$u \leq v \Rightarrow f(u) \leq f(v)$*
2. *$f(u) \wedge f(v) = f(u \wedge v)$.*

One can show that if there's a decent embedding $f : T \to S$ then $T \leq S$ as non-pointed trees. So it is enough to show that if we pre-order the class of pointed trees by $T \leq S$ iff there's a decent embedding from T to S, then $\leq$ is a wqo.

Theorem 2.3 *The pointed trees are a wqo under this ordering.*

Proof. As trees are finite, it's clearly well-founded. If it's not a wqo, take a minimal bad sequence of trees, $T_0, T_1, \ldots$. Since the sequence is bad, each T_i has at least two vertices (as the 1-point tree embeds decently into any pointed tree). So T_i can be written as in the diagram below.



Each T_{ij} is $< T_i$ (use the inclusion map; "$\leq$" fails because all trees are finite), and so $Y = \{T_{ij} : i < \omega, j < n(i)\}$ is a wqo by lemma 1.7. Hence by Higman's theorem, $Y^{<\omega}$ is also a wqo. Now replace T_i by the finite sequence $(T_{i,0}, \ldots, T_{i,n(i)-1})$. We deduce that there's $i < j$ such that $(T_{i,0}, \ldots, T_{i,n(i)-1}) \leq (T_{j,0}, \ldots, T_{j,n(j)-1})$. We can now construct a decent embedding : $T_i \to T_j$, contradicting badness. □

Kruskal's theorem is now proved. The argument seems to me to be essentially to be proving that a sequence $T_0, T_1, \ldots$ is not bad by induction on the "slope" of a sequence $T_0, T_1, \ldots$ of trees, where $T_0, T_1, \ldots$ has slope $< U_0, U_1, \ldots$ iff $T_i < U_i$ for all i. This ordering is well-founded in the sense that non-empty "closed" sets of sequences (e.g., the bad sequences) have minimal elements. A tree is shown to be decomposable into a finite sequence of smaller pieces, in such a way that a decent embedding of sequences can be patched together to give a decent embedding of the original trees. Higman's theorem is used to extend the inductive hypothesis to *sequences* of smaller objects.

3 Prikry pairs and Ramsey sets

This section is central to what follows.

Definition 3.1

1. *We say a set $\mathcal{S} \subseteq [\omega]^\omega$ is* Ramsey *if there is some infinite set $M \subseteq \omega$ such that either $[M]^\omega \subseteq \mathcal{S}$ or $[M]^\omega \cap \mathcal{S} = \emptyset$.*
2. *We say $\mathcal{S} \subseteq [\omega]^\omega$ is* open *if for every $X \in \mathcal{S}$ there is $n < \omega$ such that whenever $Y \subseteq \omega$ is infinite and $Y \cap n = X \cap n$ then $Y \in \mathcal{S}$.*

Remark 3.2 $[\omega]^\omega$ is Ramsey. We can use the axiom of choice and diagonalisation to make a non-Ramsey set. $\mathcal{S} \subseteq [\omega]^\omega$ is open if whenever $X \in \mathcal{S}$ then some finite part of X puts it in $\mathcal{S}$.

Theorem 3.3 (Galvin, Prikry) *Every open set is Ramsey. (In fact, Borel sets are Ramsey.)*

Fix $\mathcal{S} \subseteq [\omega]^\omega$. First a definition.

Definition 3.4

1. *If $X, Y \subseteq \omega$ we write $X < Y$ if for all $x \in X$ and $y \in Y$ we have $x < y$.*
2. *We say that an infinite set $M \subseteq \omega$* accepts *a finite set $X \subseteq \omega$ if $X \cup N \in \mathcal{S}$ for every infinite $N \subseteq M$ with $X < N$.*
3. *We say that M* rejects *X if no infinite $N \subseteq M$ accepts X.*
4. *We say that M* determines *X if it either accepts or rejects X.*

Remark 3.5

1. *If M accepts X, and $N \subseteq M$ is infinite, then N accepts X.*
2. *The same holds for "rejects".*
3. *By definition of "rejects", for any finite X and infinite M, there is infinite $N \subseteq M$ that determines X.*

Definition 3.6 *A* Prikry pair *is a pair* (X, N) *where* $X, N \subseteq \omega$*,* X *is finite,* N *is infinite and* $X < N$*. We partially order Prikry pairs by:* $(X, N) < (X', N')$ *iff* X' *is an end extension of* X*,* $N' \subseteq N$*,* $X' \setminus X \subseteq N$*.*

Lemma 3.7 *There is an infinite* $M \subseteq \omega$ *that determines every finite subset of itself.*

Proof. Inductively choose Prikry pairs (X_i, M_i) $(i < \omega)$ so that M_i determines every subset of X_i. We let $X_0 = \emptyset$ and take M_0 to be any infinite set determining $\emptyset$. Inductively, let $X_{i+1} = X_i \cup \min(M_i)$ and choose $M_{i+1} \subseteq M_i \setminus \min(M_i)$ that determines every subset of X_{i+1} (use (3) of the remark repeatedly). Let $M = \bigcup_{i<\omega} X_i$. Observe that $M \subseteq X_i \cup M_i$ for all i. We claim that M works. If $X \subseteq M$ is finite, then for all $i < \omega$, $X \subseteq X_i$ iff $M_i > X$. Let i be least such that this condition holds. Then if $N \subseteq M$ and $N > X$, we have $N \subseteq M_i$. But M_i determines X, and hence so does N. $\square$

Lemma 3.8 *There is infinite* $N \subseteq \omega$ *that either accepts each finite subset of itself, or rejects each finite subset of itself.*

Proof. Take M as in lemma 3.7. Select elements $a_0 < a_1 < \ldots$ of M by induction, so that for each i, if $X \subseteq \{a_0, \ldots, a_{i-1}\}$ and M rejects X then M rejects $X \cup \{a_i\}$. For this, suppose that $a_0, \ldots, a_{i-1}$ are all chosen. Let $X_1, \ldots, X_k$ list all subsets of $\{a_0, \ldots, a_{i-1}\}$ that are rejected by M. We want a_i so that M rejects all of $X_1 \cup \{a_i\}, \ldots, X_k \cup \{a_i\}$.

Suppose there's no such a_i. Then for all $a_{i-1} < a \in M$ there's $j_a \leq k$ such that M doesn't reject $X_{j_a} \cup \{a\}$. By lemma 3.7, M accepts $X_{j_a} \cup \{a\}$. Clearly there is some $j \leq k$ such that $j = j_a$ for infinitely many a. Put $N = \{a : j_a = j\}$. Then $N \subseteq M$ is infinite, and accepts X_j. For let $P \subseteq N$ be infinite with $X_j < P$. If $p = \min(P)$, then by choice of N, M accepts $X_j \cup \{p\}$. So as $P \setminus \{p\} \subseteq M$, we have $X_j \cup \{p\} \cup (P \setminus \{p\}) \in \mathcal{S}$. That is, $X_j \cup P \in \mathcal{S}$ for all such P; this contradicts the fact that M rejects X_j.

So we can choose $a_0, a_1, \ldots$ as stated. Put $N = \{a_i : i < \omega\}$. There are two cases. If N accepts $\emptyset$ then by definition, $[N]^\omega \subseteq \mathcal{S}$. So N accepts any finite subset of itself. If not, then it is easily seen by induction on $|X|$ that N rejects any $X \subseteq N$. $\square$

Proof of theorem 3.3: By lemma 3.8, we can take an infinite N that uniformly decides all its finite subsets. If N accepts them, then as above, it accepts $\emptyset$ so $[N]^\omega \subseteq \mathcal{S}$. If it rejects them, we claim $[N]^\omega \cap \mathcal{S} = \emptyset$. For if not, there's infinite $X \subseteq N$ such that $X \in \mathcal{S}$. Now as $\mathcal{S}$ is open, we can take $n < \omega$ such that if $Y \cap n = X \cap n$ and Y is infinite then $Y \in \mathcal{S}$. We can increase n as we like; so as X is infinite, we can assume that $n-1 \in X$. But now, if $P \subseteq N$

is infinite and $P > X \cap n$ then $\min(P) \geq n$. So $((X \cap n) \cup P) \cap n = X \cap n$, and $(X \cap n) \cup P \in \mathcal{S}$. Hence, N accepts $X \cap n$, a contradiction. □

It's a good exercise to prove Ramsey's theorem now. We will wait a little (see corollary 4.4).

4 Barriers

Definition 4.1 *Let X be an infinite subset of ω.*

1. *We say that $\mathcal{B} \subseteq [X]^{<\omega}$ is a* barrier on X *if:*
   - *for every infinite $Y \subseteq X$, there is an initial segment of Y in $\mathcal{B}$,*
   - *$\mathcal{B}$ is an antichain with respect to $\subseteq$.*
2. *A* barrier *is a barrier on some infinite $X \subseteq \omega$.*
3. *Clearly, for a barrier $\mathcal{B}$ on X and infinite $Y \subseteq X$, there is a* unique *initial segment of Y in $\mathcal{B}$. We write this initial segment as $Y{\restriction}\mathcal{B}$.*
4. *The* base *of a barrier $\mathcal{B}$ is defined to be $\bigcup \mathcal{B}$.*

Remark 4.2 It can be checked that if $\mathcal{B}$ is a barrier on X then $X = \bigcup \mathcal{B}$. Bearing in mind that X is order-isomorphic to ω, we see that any barrier is 'isomorphic' to a barrier on ω. We seem more interested in barrier-ness than in what the base of a barrier is.

For any $n < \omega$, $[\omega]^n$ is a barrier on ω. Thinking of $n = 1$, where $\{\{0\}, \{1\}, \ldots\}$ is a barrier, we will perhaps see (especially in definition 5.1) that a barrier is a kind of generalised sequence, or rather, the index set of such a sequence.

4.1 The Nash-Williams Ramsey theorem

Theorem 4.3 (Nash-Williams Ramsey theorem) *Let $\mathcal{B}$ be a barrier on $X \in [\omega]^\omega$. Suppose that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then there is an infinite subset Y of X such that one of $\mathcal{B}_i \cap [Y]^{<\omega}$ $(i = 1, 2)$ is a barrier on Y.*

Proof. [Rewritten Jan 2003]. X is in order-preserving bijection with ω, so we may assume without loss of generality that $X = \omega$. Let

$$\mathcal{S} = \{Z \in [\omega]^\omega : Z \text{ has an initial segment in } \mathcal{B}_1\}.$$

Clearly, $\mathcal{S}$ is open, and hence (by theorem 3.3) Ramsey. Take $Y \in [\omega]^\omega$ such that $[Y]^\omega \subseteq \mathcal{S}$ or $[Y]^\omega \cap \mathcal{S} = \emptyset$. If $[Y]^\omega \subseteq \mathcal{S}$, then clearly $\mathcal{B}_1 \cap [Y]^{<\omega}$ is a barrier on Y. If on the other hand $[Y]^\omega \cap \mathcal{S} = \emptyset$, then take any $Z \in [Y]^\omega$. Since $\mathcal{B}$ is a barrier on ω, Z has an initial segment in $\mathcal{B}$. But $Z \notin \mathcal{S}$. So that initial segment must be in $\mathcal{B}_2$. Hence, $\mathcal{B}_2 \cap [Y]^{<\omega}$ is a barrier on Y. $\square$

Corollary 4.4 (Ramsey's theorem) *If $f : [\omega]^n \to k$ where $k, n < \omega$, then there is infinite $N \subseteq \omega$ such that $f{\restriction}[N]^n$ is constant.*

Proof. For some fixed n, let $\mathcal{B} = [\omega]^n$. Obviously, $\mathcal{B}$ is a barrier on ω. If $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_k$, iterated application of the theorem gives an infinite $Y \subseteq \omega$ such that some $\mathcal{B}_i \cap [Y]^{<\omega}$ $(i \leq k)$ is a barrier on Y. If e.g., $\mathcal{B}_1 \cap [Y]^{<\omega}$ is a barrier on Y, then every n-element subset s of Y is in $\mathcal{B}_1$. For if we end-extend s to an infinite subset S of Y, then S has an initial segment in $\mathcal{B}_1$. As this must have size n, it must be s. The theorem now follows by letting $\mathcal{B}_i = f^{-1}(i-1)$ for each $1 \leq i \leq k$, since then $[Y]^n \subseteq f^{-1}(j)$ for some $j < k$. $\square$

4.2 The barrier $\mathcal{B}(2)$

Definition 4.5

1. *If s, t are finite subsets of ω, we write $s \lhd t$ to mean that there are $i_1 < \ldots < i_k$ and j $(1 \leq j < k)$ such that $s = \{i_1, \ldots, i_j\}$ and $t = \{i_2, \ldots, i_k\}$.*

2. *Given a barrier $\mathcal{B}$ and a pre-ordered set Q, we say that a map $f : \mathcal{B} \to Q$ is* bad *if there are no $s, t \in \mathcal{B}$ with $s \lhd t$ and such that $f(s) \leq f(t)$.*

3. *A map $f : \mathcal{B} \to Q$ is* perfect *if $f(s) \leq f(t)$ for all $s \lhd t$ in $\mathcal{B}$.*

Note that $j = 1$ is allowed; $k > j$ by barrier-ness. $\lhd$ is not transitive in general.

Example 4.6 Suppose $\mathcal{B}$ is a barrier on ω. For any infinite $X \subseteq \omega$, temporarily write $X_0 = X$ and $X_{n+1} = X_n \setminus \{\min(X_n)\}$. Each X_n has an initial segment $s_n \in \mathcal{B}$. Then $s_0 \lhd s_1 \lhd \cdots$. Moreover, if $s_0 \lhd s_1 \lhd \cdots \lhd s_n$ in $\mathcal{B}$, choose infinite $X \subseteq \omega$ having $s_0 \cup \cdots \cup s_n$ as an initial segment. Then $s_i = X_i{\restriction}\mathcal{B}$ for all $i \leq n$. This shows that $s_0, \ldots, s_n$ are recoverable from $s_0 \cup \cdots \cup s_n$.

If a barrier $\mathcal{B}$ 'is' the index set of a generalised sequence, and $s, t \in \mathcal{B}$, then $s \lhd t$ means that t is a 'higher' index than s.

Definition 4.7 *Let $\mathcal{B}$ be a barrier on X. Write $\mathcal{B}(2) = \{s \cup t : s \lhd t \text{ in } \mathcal{B}\}$.*

Theorem 4.8 *$\mathcal{B}(2)$ is a barrier on X.*

Proof. We show that if $s, t, s', t' \in \mathcal{B}$, $s \lhd t$, $s' \lhd t'$, and $s \cup t \subseteq s' \cup t'$, then $s = s'$ and $t = t'$. For let $s \cup t$ be $\{i_1, \ldots, i_k\}$ and $s' \cup t'$ be $\{j_1, \ldots, j_m\}$, in increasing order. Then $t = \{i_2, \ldots, i_k\}$ and $t' = \{j_2, \ldots, j_m\}$. Clearly $t \subseteq t'$; so as $\mathcal{B}$ is a barrier, we have $t = t'$. Hence $i_2 = j_2$ (as $i_1 = j_1$). Hence s, s' are initial segments of $s \cup t$, and so they are equal as $\mathcal{B}$ is a barrier. So there are no proper inclusions between elements of $\mathcal{B}(2)$.

Let Y be any infinite subset $\{y_0, y_1, \ldots\}$ of X, listed in increasing order. If $\mathcal{B}$ is a barrier then Y has an initial segment $s \in \mathcal{B}$. Similarly, $Y \setminus \{y_0\}$ has an initial segment $t \in \mathcal{B}$. Then $s \lhd t$ by barrier-ness. So $s \cup t$ is an initial segment of Y in $\mathcal{B}(2)$. $\square$

Remark 4.9 The proof (and example 4.6) shows that each set $s \cup t \in \mathcal{B}(2)$ allows s, t to be recovered uniquely. So we will always write elements of $\mathcal{B}(2)$ in the form $s \cup t$, where (implicitly) $s, t \in \mathcal{B}$.

If $s \cup t, s' \cup t' \in \mathcal{B}(2)$ and $s \cup t \lhd s' \cup t'$, then clearly, t is an initial segment of $s' \cup t'$. Since s' is as well, we have $s' \subseteq t$ or $t \subseteq s'$. Since $s', t \in \mathcal{B}$, we have $t = s'$. So $s \lhd s'$ and $t \lhd t'$.

Exercise 4.10 *Define $\mathcal{B}(n)$ for $2 \leq n < \omega$, and generalise the above to it.*

Corollary 4.11 *Given a barrier $\mathcal{B}$ on a set $X \in [\omega]^\omega$, and a map $f : \mathcal{B} \to Q$, there exists a barrier $\mathcal{D}$ on some $Y \in [X]^\omega$ with $\mathcal{D} \subseteq \mathcal{B}$, and such that the restriction $f \restriction \mathcal{D}$ of f to $\mathcal{D}$ is either bad or perfect.*

Proof. Put

$$\begin{aligned} \mathcal{A}_1 &= \{s \cup t \in \mathcal{B}(2) : f(s) \leq f(t)\}, \\ \mathcal{A}_2 &= \{s \cup t \in \mathcal{B}(2) : f(s) \not\leq f(t)\}. \end{aligned}$$

Since $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{B}(2)$, by the Nash-Williams Ramsey theorem, there is infinite $Y \subseteq X$ such that some $\mathcal{C} \subseteq \mathcal{A}_1$ or $\mathcal{C} \subseteq \mathcal{A}_2$ is a barrier on Y.

Suppose $\mathcal{C} \subseteq \mathcal{A}_1$. Put

$$\mathcal{D} = \{s \in \mathcal{B} : s \cup t \in \mathcal{C} \text{ for some } t \text{ with } s \lhd t\}.$$

Then $\mathcal{D}$ is a barrier on Y — certainly $\mathcal{D}$ is an antichain since $\mathcal{D} \subseteq \mathcal{B}$, and further, if $Z \subseteq Y$ is infinite, there is an initial segment $s \cup t$ of Z with $s \cup t \in \mathcal{C}$, so $s \in \mathcal{D}$ is an initial segment of Z.

Let $g = f \restriction \mathcal{D}$; then g is perfect. For suppose $s \lhd t$ in $\mathcal{D}$. End-extend $s \cup t$ to an infinite set in Y. This has an initial segment $s' \cup t' \in \mathcal{C}$. As in the proof of theorem 4.8, $s = s'$ and $t = t'$. So $s \cup t \in \mathcal{C} \subseteq \mathcal{A}_1$.

The case $\mathcal{C} \subseteq \mathcal{A}_2$ is similar; we get a bad g. $\square$

5 Better-quasi-orderings

This is a long and interesting section.

Definition 5.1 (bqo) *Let* $(Q, \leq)$ *be a pre-order. We say that* $\leq$ *is a* better-quasi-ordering (bqo) *if for every barrier* $\mathcal{B}$ *on* ω*, there is no bad map* $f : \mathcal{B} \to Q$*.*

Because any barrier is isomorphic to a barrier on ω, if $\leq$ is a bqo then for every barrier $\mathcal{B}$, there is no bad map : $\mathcal{B} \to Q$.

Example 5.2 $(\omega, \leq)$ is a bqo. For let $\mathcal{B}$ be a barrier on ω, and $f : \mathcal{B} \to \omega$. Let s_n be an initial segment of $\omega \setminus n$ in $\mathcal{B}$. Then $s_0 \lhd s_1 \lhd \cdots$. If f is bad, then $f(s_0) \not\leq f(s_1) \not\leq \cdots$, which is impossible.

Considering the barrier $[\omega]^1 = \{\{n\} : n < \omega\}$, we see that a bqo is a wqo.

5.1 Combining bqos

Lemma 5.3 *Let* $\leq, \sqsubseteq$ *be bqos on the same set* Q*. Then* $\preccurlyeq =_{def.} \leq \cap \sqsubseteq$ *is also a bqo on* Q*.*

Proof. Certainly, $\preccurlyeq$ is a pre-order. Let $\mathcal{B}$ be a barrier on ω, and $f : \mathcal{B} \to Q$. By corollary 4.11, there exists a barrier $\mathcal{D} \subseteq \mathcal{B}$ such that $f{\restriction}\mathcal{D}$ is either bad or perfect with respect to $(Q, \leq)$. Since $\leq$ is a bqo, $f{\restriction}\mathcal{D}$ cannot be bad, so is perfect. Since $\sqsubseteq$ is a bqo, $f{\restriction}\mathcal{D}$ cannot be bad with respect to $(Q, \sqsubseteq)$, so there are $s \lhd t$ in $\mathcal{D}$ with $f(s) \sqsubseteq f(t)$. By perfection, $f(s) \leq f(t)$. So $f(s) \preccurlyeq f(t)$, showing that f is not bad. □

5.2 Power sets of bqos

Definition 5.4 Let $(Q, \leq)$ be a pre-order. Define a pre-order $\leq$ on $\wp(Q)$ by $\Gamma \leq \Delta$ iff for all $\delta \in \Delta$ there is $\gamma \in \Gamma$ with $\gamma \leq \delta$.

The following fails for wqos; cf. exercise 1.14.

Proposition 5.5 *If* $(Q, \leq)$ *is a bqo then so is* $(\wp(Q), \leq)$*.*

Proof. Let $\mathcal{B}$ be a barrier on ω, and $f : \mathcal{B} \to \wp(Q)$. Assume for contradiction that f is bad. So for each $s \cup t \in \mathcal{B}(2)$, $f(s) \not\leq f(t)$, so there is an element $h(s \cup t) \in f(t)$ such that for every $q \in f(s)$ we have $h(s \cup t) \not\geq q$. (Recall from remark 4.9 that s, t are uniquely recoverable from $s \cup t \in \mathcal{B}(2)$.) We have therefore defined a map $h : \mathcal{B}(2) \to Q$.

By theorem 4.8, $\mathcal{B}(2)$ is a barrier on ω. As Q is a bqo, h is not bad, and there are $s \cup t \lhd s' \cup t'$ in $\mathcal{B}(2)$ with $h(s \cup t) \leq h(s' \cup t')$. By remark 4.9, $t = s'$. But now, $h(s' \cup t') \in f(t')$ is $\geq h(s \cup t) \in f(t) = f(s')$. This contradicts the definition of h. □

5.3 Minimal bad maps

The presentation of the results of this and the following two sections is similar to that of [1].

Let Q be pre-ordered by $\leq$. We assume that Q also carries a transitive well-founded relation $\prec$ such that $x \prec y \Rightarrow x < y$ for all $x, y \in Q$. We write $x \preccurlyeq y$ to mean $x \prec y$ or $x = y$.

Definition 5.6

1. *For sets $s, t \subseteq \omega$, we write $t \trianglelefteq s$ to mean that t is an initial segment of s, and $t \vartriangleleft s$ to mean that t is a proper initial segment of s.*

2. *For barriers $\mathcal{B}, \mathcal{C}$ on X, Y, we write $\mathcal{B} \sqsubseteq \mathcal{C}$ ($\mathcal{B}$* foreruns *$\mathcal{C}$) if $Y \subseteq X$ and every element of $\mathcal{C}$ has an initial segment in $\mathcal{B}$.*

3. *We say $\mathcal{B} \sqsubset \mathcal{C}$* (strictly foreruns) *if $\mathcal{B} \sqsubseteq \mathcal{C}$ and some element of $\mathcal{C}$ has a* proper *initial segment in $\mathcal{B}$. Example: $\mathcal{B} \sqsubset \mathcal{B}(2)$.*

4. *Given $f : \mathcal{B} \to Q$ and $g : \mathcal{C} \to Q$, we write $f \sqsubseteq g$ if*

   (a) $\mathcal{B} \sqsubseteq \mathcal{C}$,

   (b) $f(s) = g(s)$ for all $s \in \mathcal{B} \cap \mathcal{C}$,

   (c) for all $b \in \mathcal{B}$, $c \in \mathcal{C}$ with $b \vartriangleleft c$, we have $g(c) \prec f(b)$.

   We write $f \sqsubset g$ if $f \sqsubseteq g$ and $\mathcal{B} \sqsubset \mathcal{C}$.

5. *We say that a bad map $f : \mathcal{B} \to Q$ is* minimal *if there is no bad $g \sqsupset f$. (Minimality is with respect to $\preccurlyeq$, not $\sqsubseteq$!)*

6. *If $f : \mathcal{B} \to Q$ is bad but not minimal bad, then there are $\mathcal{C} \sqsupset \mathcal{B}$ and a bad $g : \mathcal{C} \to Q$ with $f \sqsubset g$. So there are $b \in \mathcal{B}$, $c \in \mathcal{C}$ with $b \vartriangleleft c$. Write k for the greatest element of b. Fixing f, choose $\mathcal{C}, g, b, c$ to make k minimal. We will write this least value of k as $k(f)$.*

Remark 5.7 It can be checked that both $\sqsubseteq$s are reflexive and transitive. (Take care with condition 4b.)

If $\mathcal{C} \subseteq \mathcal{B}$ are barriers and $\mathcal{C} \sqsubseteq \mathcal{D}$, then $\mathcal{B} \sqsubseteq \mathcal{D}$. If $f : \mathcal{B} \to Q$, $g : \mathcal{D} \to Q$, and $f{\restriction}\mathcal{C} \sqsubseteq g$, then $f \sqsubseteq g$. So if $f : \mathcal{B} \to Q$ is minimal bad, then so is $f{\restriction}\mathcal{C}$.

5.4 Nash-Williams/Laver theorem

Lemma 5.8 *Given $f : \mathcal{B} \to Q$ which is bad but not minimal bad, there are $\mathcal{D} \sqsupset \mathcal{B}$ and a bad $h : \mathcal{D} \to Q$ with $h \sqsupset f$, such that*

1. *for some $b \in \mathcal{B}$ with maximal element $k(f)$, we have $b \notin \mathcal{D}$,*

2. *every $m \le k(f)$ in the base $\bigcup \mathcal{B}$ of $\mathcal{B}$ is in the base of $\mathcal{D}$.*

Proof. Take bad $f : \mathcal{B} \to Q$, not minimal bad, and $g \sqsupset f$ bad, such that $g : \mathcal{C} \to Q$ for some $\mathcal{C} \sqsupset \mathcal{B}$, and for some $s \in \mathcal{B}$ and $t \in \mathcal{C}$ we have $s \lessdot t$ and $\max(s) = k(f) = n$, say. Put

$$\begin{aligned} S &= (\{0, 1, \ldots, n\} \cap \textstyle\bigcup \mathcal{B}) \setminus \bigcup \mathcal{C}, \\ \mathcal{D} &= \mathcal{C} \cup \{b \in \mathcal{B} : b \subseteq \textstyle\bigcup \mathcal{C} \cup S \text{ and } b \cap S \neq \emptyset\}. \end{aligned}$$

Claim: $\mathcal{D}$ is a barrier on $\bigcup \mathcal{C} \cup S$.

Proof of Claim: Certainly, $\mathcal{D} \subseteq [\bigcup \mathcal{C} \cup S]^{<\omega}$. We check that $\mathcal{D}$ is an $\subseteq$-antichain. Since this is true for $\mathcal{B}$ and for $\mathcal{C}$, it suffices to take $b \in \mathcal{B}$ with $b \subseteq \bigcup \mathcal{C} \cup S$ and $b \cap S \neq \emptyset$, and $c \in \mathcal{C}$, and check that $b \not\subseteq c \not\subseteq b$. Since $c \cap S = \emptyset$, we have $b \not\subseteq c$. If $c \subseteq b$, then $c \subset b$. But c has an initial segment $b' \in \mathcal{B}$, so $b' \subset b$, contradicting that $\mathcal{B}$ is a barrier.

Now let $X \subseteq \bigcup \mathcal{C} \cup S$ be infinite; we want to find an initial segment of X in $\mathcal{D}$. If X has an initial segment in $\mathcal{C}$, we are done. Assume it doesn't. Let $b = X \restriction \mathcal{B}$ (see definition 4.1). Certainly, $b \subseteq \bigcup \mathcal{C} \cup S$. We claim that $b \cap S \neq \emptyset$, so that $b \in \mathcal{D}$.

Assume for contradiction that $b \cap S = \emptyset$. As X has no initial segment in $\mathcal{C}$, a barrier on $\bigcup \mathcal{C}$, we have $X \not\subseteq \bigcup \mathcal{C}$, and so $X \cap S \neq \emptyset$. Take $s \in X \cap S$. Now $b \cap S = \emptyset$, and $b \lessdot X$. So $\max(b) < s \le n$.

Let $Y = X \cap \bigcup \mathcal{C}$. Then Y is infinite, so has an initial segment c in $\mathcal{C}$. Since $b \lessdot X$ and $b \subseteq \bigcup \mathcal{C}$, we have $b \lessdot Y$ as well. So b, c are $\trianglelefteq$-comparable. If $c \lessdot b$, then c has an initial segment b' in $\mathcal{B}$, and $b' \subset b$, which is impossible as $\mathcal{B}$ is a barrier. So $b \trianglelefteq c$. But $\max(b) < n$, so by minimality of n we must have $b = c$. So X has an initial segment c in $\mathcal{C}$, a contradiction. This proves the claim.

We now show that $\mathcal{D}$ has the two properties cited in the lemma. If $b \in \mathcal{B}$, $c \in \mathcal{C}$, and $b \lessdot c$, then $b \subset c \in \mathcal{D}$; so by the claim, $b \notin \mathcal{D}$. Since $\mathcal{B}$ contains such a b with $\max(b) = n$, the first property is established. Also, every $x \in \{0, 1, \ldots, n\} \cap \bigcup \mathcal{B}$ is the least element of some infinite subset of $\bigcup \mathcal{C} \cup S$, which by the claim has an initial segment $d \in \mathcal{D}$. So $x \in d$, whence $\{0, 1, \ldots, n\} \cap \bigcup \mathcal{B} \subseteq \bigcup \mathcal{D}$.

It is clear that $\mathcal{B} \sqsubset \mathcal{D}$. Define $h : \mathcal{D} \to Q$ by

$$h(d) = \begin{cases} f(d), & \text{if } d \in \mathcal{B}, \\ g(d), & \text{if } d \in \mathcal{C}. \end{cases}$$

This is well-defined, because $f \sqsubseteq g$. Trivially, if $d \in \mathcal{B} \cap \mathcal{D}$, $h(d) = f(d)$. Also, if $b \in \mathcal{B}$, $d \in \mathcal{D}$, and $b \lessdot d$, then $d \notin \mathcal{B}$. So $h(d) = g(d) \prec f(b)$ since $f \sqsubseteq g$. So $f \sqsubset h$.

Moreover, h is bad. For let $s \lhd t$ in $\mathcal{D}$. There are four cases.

1. If $s, t \in \mathcal{B}$, then as f is bad, $h(s) = f(s) \not\leq f(t) = h(t)$.

2. The case where $s, t \in \mathcal{C}$ is similar, using badness of g.

3. Assume that $s \in \mathcal{B}$ and $t \in \mathcal{C}$. Let $b \in \mathcal{B}$ be an initial segment of t. Since $f \sqsubseteq g$, we have $g(t) \preccurlyeq f(b)$, and so $g(t) \leq f(b)$.

   Since $s \lhd t$, $\min(s) \notin b$, so $s \neq b$, and hence as $\mathcal{B}$ is a barrier, $b \not\subseteq s$. It follows that $s \lhd b$. If $h(s) \leq h(t)$ — i.e., $f(s) \leq g(t)$ — then by transitivity, $f(s) \leq f(b)$, contradicting badness of f. So $h(s) \not\leq h(t)$ as required.

4. Finally assume for contradiction that $s \in \mathcal{C} \setminus \mathcal{B}$ and $t \in \mathcal{B} \setminus \mathcal{C}$. Then $t \cap S \neq \emptyset$. As $s \cap S = \emptyset$, and $s \lhd t$, there is $x \in S \cap t$ larger than all elements of s. So $\max(s) < n$. But s has an initial segment b in $\mathcal{B}$, which must also have maximum $< n$. By minimality of n we have $s = b \in \mathcal{B}$, a contradiction.

□

The main theorem is next. The idea is that $\mathcal{B}$ is to be refined in a step-by-step manner to $\mathcal{C} \sqsupset \mathcal{B}$ by reducing values of f. We may need to extend sequences to do this.

Theorem 5.9 (Nash-Williams, interpreted by Laver) *For every barrier $\mathcal{B}$ and bad map $f : \mathcal{B} \to Q$, there are a barrier $\mathcal{C} \sqsupseteq \mathcal{B}$ and a minimal bad $g : \mathcal{C} \to Q$ with $f \sqsubseteq g$.*

Proof. If $\mathcal{B}_0$ is a barrier and $f_0 : \mathcal{B}_0 \to Q$ is bad but not minimal bad, we can find a barrier $\mathcal{B}_1$ with $\mathcal{B}_0 \sqsubseteq \mathcal{B}_1$, and a bad map $f_1 : \mathcal{B}_1 \to Q$ with $f_1 \sqsupset f_0$ as in lemma 5.8. If f_1 is not minimal bad, we can repeat to get $\mathcal{B}_2$ and $f_2 : \mathcal{B}_2 \to Q$, etc. If this process stops after a finite number of iterations, we're done.

Otherwise, it goes on infinitely many times. In this case, we claim that $k(f_0) \leq k(f_1) \leq \cdots$ and the sequence rises infinitely often. Certainly the sequence cannot fall, as $k(f_i)$ is minimal. Each application of the lemma removes from $\mathcal{B}_i$ some set b with maximum $k(f_i)$, and $\mathcal{B}_{i+1}$ contains some c with $b \lessdot c$. No later $\mathcal{B}_j$ contains b, because $\mathcal{B}_{i+1} \sqsubseteq \mathcal{B}_j$, so there would be $b' \trianglelefteq b$ in $\mathcal{B}_{i+1}$, so $b' \subset c$, contradicting that $\mathcal{B}_{i+1}$ is a barrier. There are finitely many such sets b, so this cannot happen infinitely often. The claim is proved.

Let X_i be the base of $\mathcal{B}_i$ $(i < \omega)$. Since $\mathcal{B}_0 \sqsubseteq \mathcal{B}_1 \sqsubseteq \cdots$, we have $X_0 \supseteq X_1 \supseteq \cdots$. Define $X = \bigcap_{n<\omega} X_n$. Since $k(f_i) \in X$ for all i (see lemma 5.8), we see by the claim that X is infinite. Let $\mathcal{D} = \{b : b \text{ is in cofinitely many } \mathcal{B}_i\}$.

Claim: $\mathcal{D}$ is a barrier on X.
Proof of Claim: Certainly, if $b \in \mathcal{D}$ then $b \subseteq X_i$ for cofinitely many $i < \omega$, so $b \subseteq X$. Also, if $b, c \in \mathcal{D}$ and $b \subseteq c$, then $b = c$ (as this holds in cofinitely many $\mathcal{B}_i$).

Consider any infinite $Y \subseteq X$. For each $\mathcal{B}_i$, we have $Y \subseteq X_i$, so some $b_i \in \mathcal{B}_i$ is an initial segment of Y. Clearly, $b_i \trianglelefteq b_{i+1}$ or $b_{i+1} \lessdot b_i$. If $b_{i+1} \lessdot b_i$, then as $\mathcal{B}_i \sqsubseteq \mathcal{B}_{i+1}$, b_{i+1} has an initial segment $c \in \mathcal{B}_i$. So $c \subset b_i$, which is impossible since $\mathcal{B}_i$ is a barrier. So $b_i \trianglelefteq b_{i+1}$. If $b_i \lessdot b_{i+1}$, then $f_i(b_i) \succ f_{i+1}(b_{i+1})$. Otherwise, $b_i = b_{i+1}$, so since $f_{i+1} \sqsupset f_i$, we have $f_i(b_i) = f_{i+1}(b_{i+1})$. But $\prec$ is well-founded, so the sequence $(b_i : i < \omega)$ becomes constant at j, say. Then $b_j \in \mathcal{D}$ is an initial segment of Y. The claim is proved.

We note that $\mathcal{D} \sqsupset \mathcal{B}_i$ for each i.

We now define $g : \mathcal{D} \to Q$ to agree eventually with all f_i — i.e., for all $d \in \mathcal{D}$, $g(d) = f_i(d)$ for all i with $d \in \mathcal{B}_i$. This is well-defined since $f_0 \sqsubseteq f_1 \sqsubseteq \cdots$. We note that $g \sqsupset f_i$ for each i.

Claim: g is minimal bad.
Proof of Claim: If $d \lhd e$ in $\mathcal{D}$ then $d \lhd e$ in some $\mathcal{B}_i$. So $g(d) = f_i(d) \not\leq f_i(e) = g(e)$. So g is bad.

If there are $\mathcal{E} \sqsupset \mathcal{D}$ and bad $h : \mathcal{E} \to Q$ with $h \sqsupset g$, then there are $s \in \mathcal{D}$ and $t \in \mathcal{E}$ with $s \lessdot t$. We may choose $i < \omega$ with $s \in \mathcal{B}_i$ and $\max(s) < k(f_i)$ (as the $k(f_i)$ rise arbitrarily high). Then (cf. remark 5.7) $\mathcal{E} \sqsupset \mathcal{B}_i$ and $h \sqsupset f_i$, so h contradicts the value of $k(f_i)$. This proves theorem 5.9. □

5.5 Sequences

Now for some practical applications. Fix a set Q and a pre-order $\leq$ on it.

Definition 5.10

1. *We can extend* $\leq$ *to* $\bigcup_{\alpha \text{ ordinal}} Q^\alpha$ *as follows. For* $x \in Q^\alpha$, $y \in Q^\beta$, *put* $x \leq y$ *iff there is a one-one order-preserving map* $\varphi : \alpha \to \beta$ *such that for all* $i < \alpha$ *we have* $x_i \leq y_{\varphi(i)}$. *(Regarding* $Q = Q^1$, *this is an extension of* $\leq$ *on* Q.*) As usual,* $x < y$ *means* $x \leq y \not\leq x$.

2. *For* $x, y \in \bigcup_\alpha Q^\alpha$, *define* $x \prec y$ *to mean* x *is isomorphic to a subsequence of* y *and* $\mathrm{dom}(x) < \mathrm{dom}(y)$ *as ordinals. Clearly,* $\prec$ *is transitive and well-founded (as ordinals are), and* $x \prec y \Rightarrow x < y$ *(*$<$ *as above).*

Theorem 5.11 *If* $(Q, \leq)$ *is a bqo then so is* $(\bigcup_\alpha Q^\alpha, \leq)$.

Proof. Suppose for contradiction that there is a barrier $\mathcal{B}$ and a bad map $f : \mathcal{B} \to \bigcup_\alpha Q^\alpha$. By theorem 5.9, we can assume that f is minimal bad. By the Nash-Williams Ramsey theorem (theorem 4.3), we can also assume that either all $f(b)$ have successor length, or they all have limit length. By remark 5.7, this does not affect minimality.

Case I: the $f(b)$ have successor length. Write $f(b) = f_1(b)\hat{\ } f_2(b)$ with $f_2(b) \in Q$ (we 'remove' the last element). By corollary 4.11, we can assume that each of f_1, f_2 is either bad or perfect. Now f_2 can't be bad, as Q is a bqo; hence it is perfect. But f is bad, so f_1 must be bad.

Define $g : \mathcal{B}(2) \to \bigcup_\alpha Q^\alpha$ by $g(s \cup t) = f_1(s)$. Then $\mathcal{B}(2) \sqsupset \mathcal{B}$; and because $f_1(s) \prec f(s)$ for all s, we have $g \sqsupset f$. Also, g is bad: for if $s \cup t \lhd t \cup v$ in $\mathcal{B}(2)$ (see remark 4.9), then because $s \lhd t$ and f_1 is bad, $g(s \cup t) = f_1(s) \not\leq f_1(t) = g(t \cup v)$. This contradicts the minimality of f.

Case II: each $f(b)$ has limit length. Consider $s \lhd t$ in $\mathcal{B}$. Because of the poor quality of f we have $f(s) \not\leq f(t)$. Define an order-preserving map $\varphi : \mathrm{dom}(f(s)) \to \mathrm{dom}(f(t))$ by induction: at each stage, φ maps the next element $i < \mathrm{dom}(f(s))$ to the least $j < \mathrm{dom}(f(t))$ such that $f(t)_j \geq f(s)_i$ and $j > \varphi(k)$ for all $k < i$. If this succeeded, we'd have $f(s) \leq f(t)$; so there must be a proper initial segment of $f(s)$ that's $\not\leq f(t)$. Take (say) the shortest such — $f(s)_t$, say.

Define $g : \mathcal{B}(2) \to \bigcup_\alpha Q^\alpha$ by $g(s \cup t) = f(s)_t$. This is a non-trivial use of $\mathcal{B}(2)$. As $\mathcal{B} \sqsubset \mathcal{B}(2)$, and $f(s)_t \prec f(s)$ whenever $s \lhd t$, we have $f \sqsubset g$. But g is bad, for if $s \cup t \lhd t \cup v$ then $g(s \cup t) = f(s)_t \not\leq f(t)$, so $g(s \cup t) \not\leq f(t)_v = g(t \cup v)$. This is a contradiction. □

5.6 Matrices

Fix $1 \leq d < \omega$, and write Q for ω^d. Recall from example 5.2 that $(\omega, \leq)$ is a bqo. Extend $\leq$ to $\bigcup_\alpha \omega^\alpha$, as in definition 5.10. By theorem 5.11, $(\bigcup_\alpha \omega^\alpha, \leq)$

is a bqo, and hence so is its subset $(Q, \leq)$. Note that $x \leq y$ in Q iff $x_i \leq y_i$ for all $i < d$.

By definition 5.10 and theorem 5.11 again, we may extend the order $\leq$ on Q to an order $\leq$ on $\bigcup_\alpha Q^\alpha$, and it is a bqo.

Definition 5.12 *Define* $\sqsubseteq$ *on* $\bigcup_\alpha Q^\alpha$ *by: if* $\xi \in Q^\alpha$, $\eta \in Q^\beta$, *then* $\xi \sqsubseteq \eta$ *iff* $\forall j < \beta\, \exists i < \alpha\, (\xi_i \leq \eta_j)$

Lemma 5.13 $(\bigcup_\alpha Q^\alpha, \sqsubseteq)$ *is a bqo.*

Proof. For a sequence $\xi = (\xi_i : i < \alpha) \in Q^\alpha$, write $\mathrm{rng}(\xi)$ for the set $\{\xi_i : i < \alpha\} \in \wp(Q)$. Given a barrier $\mathcal{B}$ on ω and a map $f : \mathcal{B} \to \bigcup_\alpha Q^\alpha$, define $g : \mathcal{B} \to \wp(Q)$ by $g(s) = \mathrm{rng}(f(s))$. By proposition 5.5, there are $s \lhd t$ in $\mathcal{B}$ such that for all $q \in g(t)$ there is $p \in g(s)$ with $p \leq q$. So clearly, $f(s) \sqsubseteq f(t)$. □

Definition 5.14 *Define* $\preccurlyeq$ *on* $\bigcup_\alpha Q^\alpha$ *by:* $\xi \preccurlyeq \eta$ *iff* $\xi \leq \eta$ *and* $\xi \sqsubseteq \eta$.

By lemma 5.3, $\preccurlyeq$ is a bqo on $\bigcup_\alpha Q^\alpha$.

Definition 5.15

1. *For an ordinal* α, *let* $\mathcal{M}_\alpha$ *be the set of all maps* $m : \alpha \times d \to \omega$.
2. *Define a bijection* $\theta_\alpha : Q^\alpha \to \mathcal{M}_\alpha$ *by* $\theta_\alpha(\xi) = m$, *where* $m(i, j) = (\xi_i)_j$ *for* $i < \alpha$, $j < d$.
3. *Let* $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$. *Define a bijection* $\theta = (\bigcup_\alpha \theta_\alpha) : \bigcup_\alpha Q^\alpha \to \mathcal{M}$.
4. *For* $m, n \in \mathcal{M}$, *write* $m \leq n$ *if, given that* $m \in \mathcal{M}_\alpha$ *and* $n \in \mathcal{M}_\beta$, *there is surjective* $f : \beta \to \alpha$ *such that* $m(f(i), j) \leq n(i, j)$ *for all* $i < \beta$, $j < d$.

Theorem 5.16 *For any sequence* $m_0, m_1, \ldots$ *in* $\mathcal{M}$, *there are* $n < k < \omega$ *with* $m_n \leq m_k$. *Indeed,* $(\mathcal{M}, \leq)$ *is a (class) bqo.*

Proof. Let $\xi^0, \xi^1, \ldots \in \bigcup_\alpha Q^\alpha$ be such that $\theta(\xi^i) = m_i$ for $i < \omega$. Since $(\bigcup_\alpha Q^\alpha, \preccurlyeq)$ is a bqo and $\mathcal{B} = \{\{n\} : n < \omega\}$ is a barrier on ω, the map $(\{n\} \mapsto \xi^n) : \mathcal{B} \to \bigcup_\alpha Q^\alpha$ is not $\preccurlyeq$-bad, and so there are $n < k < \omega$ with $\xi^n \preccurlyeq \xi^k$.

Write ξ for ξ^n and η for ξ^k; suppose that $\xi \in Q^\alpha$ and $\eta \in Q^\beta$. By $\xi \leq \eta$, there is a one-one order-preserving map $\varphi : \alpha \to \beta$ with $\xi_i \leq \eta_{\varphi(i)}$ for all

$i < \alpha$. By $\xi \sqsubseteq \eta$, for all $i < \beta$ there is $\psi(i) < \alpha$ with $\xi_{\psi(i)} \leq \eta_i$. Define $f : \beta \to \alpha$ by

$$f(i) = \begin{cases} \varphi^{-1}(i), & \text{if } i \in \text{rng}(\varphi), \\ \psi(i), & \text{otherwise.} \end{cases}$$

Then f is surjective, and $\xi_{f(i)} \leq \eta_i$ in Q for all $i < \beta$. Since $\theta(\xi) = m_n$ and $\theta(\eta) = m_k$, that is

$$m_n(f(i), j) = (\xi_{f(i)})_j \leq (\eta_i)_j = m_k(i, j) \text{ for all } i < \beta,\ j < d.$$

So $m_n \leq m_k$. The bqo part is an exercise. □

Probably one can generalise to higher dimensions.

6 Countable linear orderings

Notation 6.1 We will write $\alpha, \beta, \gamma, \ldots$ for linear orderings. $i, j, k, \ldots$ will denote ordinals. We write $\alpha \leq \beta$ if α embeds into β in an order-preserving fashion. We write $\alpha < \beta$ if $\alpha \leq \beta \not\leq \alpha$.

Theorem 6.2 (Laver) *The class of countable linear orderings is a bqo under* $<$.

(Transcription of notes is incomplete. Please see [1, 3, 4] for the proofs.)

References

[1] R. Laver, *Better-quasi-orderings and a class of trees*, Studies in Foundations and Combinatorics (Gian-Carlo Rota, ed.), Advances in Mathematics Supplementary Studies, vol. 1, Academic Press, 1978, pp. 31–48.

[2] R. Mansfield and G Weitkamp, *Recursive aspects of descriptive set theory*, Oxford Logic Guides, vol. 11, Oxford University Press, 1985.

[3] C. St.J. A. Nash-Williams, *On well-quasi-ordering infinite trees*, Proc. Cambr. Philos. Soc. **61** (1965), 697–720.

[4] ———, *On better-quasi-ordering transfinite sequences*, Proc. Cambr. Philos. Soc. **64** (1968), 273–290.

[5] S. G. Simpson, *Bqo theory and Fraïssé's conjecture*, 1985, chapter 9 of [2].