Counterexamples for hybrid bisimulation theorem: (i) with nominals, (ii) in the finite

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The 'bisimulation theorem' in [3] says that when there are no nominals, every first-order $\theta(x)$ in the correspondence language that is invariant under quasi-injective bisimulations over all models is equivalent to a hybrid $\mathcal{H}(\downarrow)$-sentence over all models. Here are examples showing that this fails with nominals, and fails over finite models even without nominals. I hope to put them in a paper soon.

Notation summary. We let $\textsc{Prop}, \textsc{Nom}$ be the sets of propositional atoms and nominals, respectively. A *Kripke model* is a triple $A = (W, R, V)$, where $W \neq \emptyset$ is the *domain* of A, $R \subseteq W^2$, and $V : \textsc{Prop} \,\dot\cup\, \textsc{Nom} \to \wp(W)$, such that $V(c)$ is a singleton $\{c^A\}$ for each $c \in \textsc{Nom}$. We identify (notationally) A with W. A *pointed Kripke model* is a pair (A, a) where A is a Kripke model and $a \in A$. When needed, we write R^A instead of R.

For $U \subseteq W$ we write $R \restriction U = R \cap U^2$. For a binary relation S on W and $a \in W$, we write $S(a)$ for $\{b \in A : aSb\}$.

We let $R^0 = \{(w, w) : w \in W\}$, $R^{n+1} = R^n \mid R = \{(w, u) \in W^2 : \exists v(wR^n vRu)\}$ for $n < \omega$, $R^{\leq n} = \bigcup_{k \leq n} R^k$, and $R^* = \bigcup_{n<\omega} R^n$, the reflexive transitive closure of R. We say that A is *rooted (at a)* if $R^*(a) = W$ for some $a \in W$; such an a is called a *root* of A. A *cluster* is an equivalence class of the equivalence relation $R^* \cap (R^{-1})^*$ on W, where R^{-1} is the converse of R.

The *submodel of A generated by $a \in A$* is the submodel $(R^*(a), R \restriction R^*(a))$ of A with domain $R^*(a)$. This is a well-defined Kripke model only when $c^A \in R^*(a)$ for each $c \in \textsc{Nom}$ — this needs to be checked each time.

For Kripke models A, B, a bisimulation $Z : A \to B$ is a relation $Z \subseteq A \times B$ such that for each $(a, b) \in Z$: $A, a \models p$ iff $B, b \models p$ for each $p \in \textsc{Prop} \cup \textsc{Nom}$; if $aR^A a'$ then there is $b' \in B$ with $bR^B b'$ and $a'Zb'$ ('Forth'); and a similar condition swapping A, B ('Back'). Z is *quasi-injective* if aZb, aZb', and $b(R^B)^* b'$ imply $b = b'$, and similarly swapping A, B.

1 Two general lemmas

LEMMA 1.1 *Let (J, j) and (K, k) be pointed Kripke models and $Z : J \to K$ a quasi-injective bisimulation with jZk. Let J', K' be their submodels generated by j, k, respectively, and assume that J' is a well-defined Kripke model. Then so is K', and $Z' = Z \cap (J' \times K') : J' \to K'$ is also a quasi-injective bisimulation with $jZ'k$.*

Proof. Let $c \in \textsc{Nom}$ be arbitrary. As J' is well defined, $c^J \in J' = (R^J)^*(j)$. By (possibly iterated) Forth for Z, there is $y \in (R^K)^*(k) = K'$ with $c^J Z y$. As $J, c^J \models c$ and Z preserves c, we have $K, y \models c$, so $y = c^K$. Hence, $c^K \in K'$ for each nominal c, and K' is well defined.

Since $j \in J'$ and $k \in K'$, we have $jZ'k$. Suppose $xZ'y$, $x' \in J'$, and $xR^{J'}x'$. So xZy and $xR^J x'$. As Z is a bisimulation, there is $y' \in K$ with $yR^K y'$ and $x'Zy'$. Since $xZ'y$, we have $y \in K'$, so $y' \in R(y) \subseteq K'$. So $x'Z'y'$, proving 'Forth' for Z'. 'Back' is similarly proved.

Since $Z' \subseteq Z$, it is plain that Z' is quasi-injective. □

DEFINITION 1.2 We say that a relation $Z \subseteq X \times Y$ is *functional at* $x \in X$ if there exists a unique $y \in Y$ with xZy, and *functional at* $y \in Y$ if there exists a unique $x \in X$ with xZy.

LEMMA 1.3 *Let M, N be rooted Kripke models with roots m, n, respectively, and let $Z : M \to N$ be a quasi-injective bisimulation with mZn. Then:*

1. *Z is functional at m.*
2. *Z is functional at c^M for each nominal c.*
3. *Let C be a cluster in M, and suppose that Z is functional at some point $c \in C$. Then Z is functional at every point in C.*
4. *If $M \models \forall xy(xR^*y \vee yR^*x)$ then Z is functional at every point in N.*

Proof. In the proof, we write R for the accessibility relation in both M, N.

1. By assumption, mZn. Suppose $y \in N$ and mZy. Since n is a root of N, we have $y \in R^*(n)$, so $y = n$ as Z is a quasi-injective bisimulation.
2. We are given that mZn. Since m is a root of M, we have $c^M \in R^*(m)$, so by (possibly iterated) Forth there is $y \in N$ with $c^M Z y$. Since Z preserves nominals, $y = c^N$.
3. Fix the unique $d \in N$ with cZd. Let $x \in C$ be arbitrary. Then $x \in R^*(c)$, so by (always possibly iterated) Forth there is $y \in R^*(d)$ with xZy.

   To show uniqueness, let $y' \in N$ with xZy' be given. Since $c \in R^*(x)$, by Forth there is $d' \in R^*(y')$ with cZd'. But Z is functional at c, so $d' = d$. Now $y \in R^*(d)$ and $d = d' \in R^*(y')$, so $y \in R^*(y')$ — and xZy, xZy'. As Z is quasi-injective, $y' = y$.
4. Take $y \in N$. As n is a root of N and mZn, by Back we have xZy for some $x \in M$. If $x' \in M$ and $x'Zy$, then by assumption, $x' \in R^*(x)$ or $x \in R^*(x')$. Either way, as Z is a quasi-injective bisimulation, $x = x'$.

□

2 Bisimulation theorem fails with nominals

This was left open in [3].

All Kripke models in this section are for a hybrid signature $\textsc{Prop} \uplus \textsc{Nom}$ with a single nominal, c, and no propositional atoms. Figure 1 shows two such pointed models, (M, m) and (N, n).

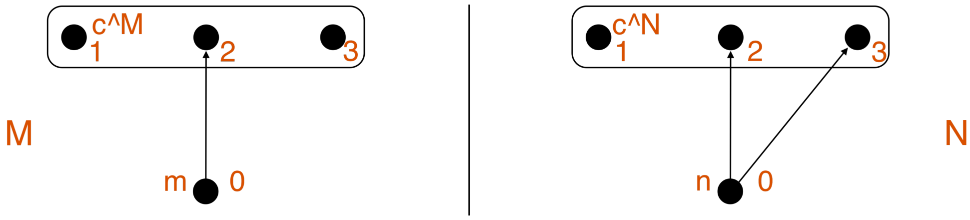

Figure 1: pointed Kripke models (M, m) and (N, n)

Formally, M and N have the same domain, $\{0, \ldots, 3\}$; $R^M = \{1, 2, 3\}^2 \cup \{(0, 2)\}$, and $R^N = R^M \cup \{(0, 3)\}$. Finally, $c^M = c^N = 1$ and $m = n = 0$. The ovals in the figure are R-cliques, and clusters.

Let $\varphi(x)$ be a first-order formula in the correspondence language, saying that the submodel generated by x is isomorphic to M by an isomorphism taking x to m. This is first-order definable because M is finite and (so) has finite depth.

Formally, let $\Delta(x_0, \ldots, x_3)$ comprise $\{x_i \neq x_j : i < j \leq 3\}$, $\{x_i R x_j : i R^M j\}$, $\{\neg(x_i R x_j) : i, j \leq 3,\ \neg(i R^M j)\}$, and $\{c = x_1\}$. This is essentially the (basic) diagram of M (see [2, p.16]), with x_i assigned to i for each $i \leq 3$. Then define

$$\varphi(x_0) = \exists x_1 x_2 x_3 \big(\bigwedge \Delta \wedge \forall y (x_0\, R^{\leq 3}\, y \to \bigvee_{i \leq 3} y = x_i) \big).$$

LEMMA 2.1 *Let A be a Kripke model and suppose that $a_0 \in A$ is a root of A. Then $A \models \varphi(a_0)$ iff $(A, a_0) \cong (M, m)$, where (M, m) is as on the left of figure 1.*

Proof. Clearly, $M \models \varphi(m)$, and $\Leftarrow$ follows as φ is preserved under isomorphism.

For $\Rightarrow$, suppose that $A \models \varphi(a_0)$. Let $a_1, a_2, a_3 \in A$ witness the $\exists x_1 x_2 x_3$ in φ, where x_0 is assigned to a_0. Define a map $f : M \to A$ by $f(i) = a_i$ for each $i \leq 3$. Then $f(m) = a_0$, and f is an embedding by definition of Δ.

We show that f is surjective, hence an isomorphism. Suppose for contradiction that there is $b \in A \setminus \{a_0, \ldots, a_3\}$. Since a_0 is a root of A, we have $b \in R^n(a_0)$ for some $n < \omega$. Choose b so that n is least possible. Since $b \notin \{a_0\} = R^0(a_0)$, we have $n > 0$, so there is $b' \in R^{n-1}(a_0)$ with $b' R b$. As n is minimal, $b' \in \{a_0, \ldots, a_3\}$.

Now $M \subseteq (R^M)^{\leq 2}(0)$ by inspection of figure 1. Because $f(0) = f(m) = a_0$ and f preserves R, we have $f(M) = \{a_0, \ldots, a_3\} \subseteq R^{\leq 2}(a_0)$. So $a_0\, R^{\leq 2}\, b'$, and hence $a_0\, R^{\leq 3}\, b$. But $A \models \varphi(a_0)$, so $A \models \bigvee_{i \leq 3} b = a_i$, a contradiction. □

We now observe that φ is invariant under well-defined generated submodels.

LEMMA 2.2 *Let (J, j) be a pointed Kripke model and let J' be its submodel generated by j. Assume that $c^J \in J'$, so that J' is a well-defined Kripke model. Then $J \models \varphi(j)$ iff $J' \models \varphi(j)$.*

Proof. All quantifiers in φ are effectively relativised to the subset $R^{\leq 3}(j)$ of J'. □

The following fails if we delete the nominal c from M, N, φ.

PROPOSITION 2.3 *$\varphi(x)$ is invariant on all models under quasi-injective bisimulations.*

Proof. Let (J, j) and (K, k) be pointed Kripke models and $Z : J \to K$ a quasi-injective bisimulation with jZk. Suppose that $J \models \varphi(j)$. We show that $K \models \varphi(k)$.

Let J', K' be the generated submodels of J, K generated by j, k, respectively, and $Z' = Z \cap (J' \times K')$. Since $J \models \varphi(j)$, it follows that $c^J \in J'$, so J' is a well-defined Kripke model, and j is a root of it. Then by lemma 1.1, K' is also a well-defined Kripke model (ie. $c^K \in K'$) and $Z' : J' \to K'$ is a quasi-injective bisimulation with $jZ'k$.

By lemma 2.2, $J' \models \varphi(j)$, so by lemma 2.1, $(J', j) \cong (M, m)$. By inspection of figure 1, J' comprises the root j and a cluster containing $c^{J'}$, and $J' \models \forall xy(xR^*y \vee yR^*x)$. So by lemma 1.3, $Z' : J' \to K'$ is a bijection (this may fail without c). Hence, it is an isomorphism, and $(J', j) \cong (K', k)$. Since $J' \models \varphi(j)$, we obtain $K' \models \varphi(k)$. By lemma 2.2 again, $K \models \varphi(k)$. □

PROPOSITION 2.4 *$\varphi(x)$ is not equivalent even over finite models to any $\mathcal{H}(\downarrow)$-sentence.*

Proof. It's an exercise to show that the pointed Kripke models (M, m) and (N, n) in figure 1 agree on all $\mathcal{H}(\downarrow)$-sentences. By lemma 2.1, $M \models \varphi(m)$ and $N \not\models \varphi(n)$, the latter because (N, n) is rooted and not isomorphic to (M, m). So φ cannot be equivalent to a $\mathcal{H}(\downarrow)$-sentence even over models with ≤ 4 points. □

Combining these propositions gives:

THEOREM 2.5 *The bisimulation theorem in [3] fails, both classically (over all models) and over finite models, with a single nominal and no propositional atoms.*

[1] recovers a positive result classically, using a broader notion of bisimulation under which (M, m) and (N, n) are bisimilar and φ is not invariant.

3 Bisimulation theorem fails in the finite

We give an example of a first-order formula $\theta(x)$ that is invariant under quasi-injective bisimulations on finite models (in fact we show more), but is not equivalent over finite models to any $\mathcal{H}(\downarrow)$-sentence. No propositional atoms or nominals are needed.

We take $\textsc{Prop} = \textsc{Nom} = \emptyset$ and omit assignments in Kripke models — they are just frames $A = (W, R)$. A *predecessor* of a point $x \in W$ is a point $y \in W$ with yRx — possibly $y = x$.

Let $\theta(x)$ be a first-order formula saying:

1. $R^2(x) \subseteq R^{\leq 1}(x)$
2. every $y \in R(x)$ satisfies $yRx \vee |R(y)| \geq 2$
3. every $y \in R(x)$ has at most one predecessor in $R(x)$ (not $R^{\leq 1}(x)$, mind).

This is easy to write up more formally — θ is the conjunction of:

1. $\forall yz(xRyRz \to z = x \vee xRz)$
2. $\forall y(xRy \to yRx \vee \exists zt(yRz \wedge yRt \wedge z \neq t))$
3. $\forall yzt(xRy \wedge xRzRy \wedge xRtRy \to z = t)$.

LEMMA 3.1 *Let (A, a) be a finite pointed Kripke model, rooted at a, and with $A \models \theta(a)$. Then A is a cluster.*

Proof. As a is a root, $A = R^*(a)$. By clause 1 and induction on path lengths, $A = R^{\leq 1}(a)$. So it suffices to take arbitrary $b \in R(a)$ and show that $a \in R^*(b)$.

For contradiction, suppose that $a \notin R^*(b)$. Then $R^*(b) \subseteq A \setminus \{a\} \subseteq R(a)$.

Let $D = (R^*(b), R \restriction R^*(b))$ be the submodel of A generated by b. It is a finite directed graph. By clause 3 of θ and because $R^*(b) \subseteq R(a)$, each node of D has in-degree ≤ 1. By clause 2 and because $a \notin R^*(b) \subseteq R(a)$, it follows that each node of D has out-degree ≥ 2.

But since D is finite, the sum of the in-degrees of nodes in D must equal the sum of their out-degrees — both are equal to $|R \restriction R^*(b)|$. This is a contradiction. So indeed, $a \in R^*(b)$. □

PROPOSITION 3.2 *Let (M, m) and (N, n) be finite pointed Kripke models that agree on $\mathcal{H}(\downarrow)$-sentences. Then they agree on θ. Hence, θ is invariant under quasi-injective bisimulations on finite models — and indeed under any relation that preserves $\mathcal{H}(\downarrow)$-sentences in the finite.*

Proof. For the first part, suppose $M \models \theta(m)$. We show that $N \models \theta(n)$.

Let M', N' be the generated submodels of M, N generated by m, n, respectively. Since all quantifiers in θ are relativised to $R^{\leq 2}(x)$, it is invariant under generated submodels, so $M' \models \theta(m)$. By lemma 3.1, M' is a cluster.

Now (M, m) and (N, n) are finite and agree on $\mathcal{H}(\downarrow)$-sentences. Since such sentences are invariant under generated submodels, (M', m) and (N', n) also agree on $\mathcal{H}(\downarrow)$-sentences. Since M' is a finite cluster, we can write an $\mathcal{H}(\downarrow)$-sentence expressing its isomorphism type (exercise), and as n is a root of N', it follows that $(M', m) \cong (N', n)$. So certainly, $N' \models \theta(n)$.

Again as θ is invariant under generated submodels, $N \models \theta(n)$ as required.

The second part follows since quasi-injective bisimulations preserve $\mathcal{H}(\downarrow)$-sentences. □

Below, for an ordinal $n = \{m : m < n\}$, we write ${}^n2$ for the set of all functions from n into $2 = \{0, 1\}$; ${}^{<n}2 = \bigcup_{m<n} {}^m2$; and ${}^{\leq n}2 = {}^{<n+1}2$. For $t \in {}^n2$ and $i < 2$, we write $t^\frown i \in {}^{n+1}2$ for the map extending t by $t^\frown i(n) = i$.

PROPOSITION 3.3 *$\theta(x)$ is not equivalent over finite models to any $\mathcal{H}(\downarrow)$-sentence.*

Proof. Suppose for contradiction that θ is equivalent in finite models to (the standard translation of) an $\mathcal{H}(\downarrow)$-sentence ψ, of $\diamond$-depth n, say.

Let $M = (W, R)$, where $W = \{a\} \cup {}^{\leq n}2$ for some point $a \notin {}^{\leq n}2$, and with

$$R = (\{a\} \times {}^{\leq n}2) \cup \{(t, t^\frown i) : t \in {}^{<n}2,\ i < 2\} \cup ({}^n2 \times \{a\}).$$

Then $M \models \theta(a)$. It is important here that $a \notin R(a)$ — a has many predecessors in $R(a)$ (the elements of ${}^n2$ at least), so if $a \in R(a)$ then clause 3 would fail.

Let $N = (W \cup \{e\}, S)$, where $e \notin W$ is a new point (a 'copy' of $\emptyset \in {}^02$) and

$$S = R \cup \{(a, e)\} \cup (\{e\} \times {}^12).$$

Then $N \models \neg\theta(a)$ because the points in ^{12} now have two predecessors ($\emptyset$ and e) in $S(a)$, so clause 3 fails.

But it can be shown (exercise; remark 3.5 may help) that (M, a) and (N, a) agree on all $\mathcal{H}(\downarrow)$-sentences of depth $\leq n$, including ψ. Since they are finite, they agree on θ too. As $M \models \theta(a)$ and $N \models \neg\theta(a)$, this is a contradiction. □

The two propositions combine to give:

THEOREM 3.4 *The bisimulation theorem in [3] fails in the finite (with no nominals or propositional atoms).*

REMARK 3.5 The sentence $\downarrow x\Diamond\downarrow y\Diamond\downarrow z\Diamond^n(x\wedge\Diamond(\neg y\wedge\Diamond z))$ of $\Diamond$-depth $n+4$ holds in (N,a) but not (M,a), but we really need to go right round the circuit to distinguish them, and this takes $\Diamond$-depth $> n$. If we allow @ as well, we can use $\downarrow x\Diamond\downarrow y\Diamond\downarrow z@_x\Diamond(\neg y\wedge\Diamond z)$, of $\Diamond$-depth 4. So the example fails for $\downarrow, @$.

EXAMPLE 3.6 By proposition 3.3 and the bisimulation theorem in [3], θ cannot be invariant under quasi-injective bisimulations on arbitrary models. To see an explicit example, pick a point $a\notin\omega\cup{}^{<\omega}2$ and let

$$\begin{aligned} M &= \big(\{a\}\cup{}^{<\omega}2,\ (\{a\}\times{}^{<\omega}2)\cup\{(t,t^\frown i): t\in{}^{<\omega}2,\ i<2\}\big).\\ N &= \big(\{a\}\cup\omega,\ (\{a\}\times\omega)\cup\{(n,n+1): n<\omega\}\big).\end{aligned}$$

Unlike in proposition 3.3, neither model is a single cluster, and indeed they are acyclic, so no two distinct points in them lie in the same cluster. Then $M\models\theta(a)$, and $N\models\neg\theta(a)$ because clause 2 fails in N; yet $Z=\{(a,a)\}\cup\bigcup_{n<\omega}({}^n2\times\{n\}):(M,a)\to(N,a)$ is a quasi-injective bisimulation. One could get another example by adjoining e to M as in the proof of proposition 3.3, but we cannot use a 'linear' (N,a) like the above in the proposition, because it would differ from (M,a) on the $\mathcal{H}(\downarrow)$-sentence $\downarrow x\Diamond\downarrow y\Diamond\downarrow z\Diamond(x\wedge\Diamond(y\wedge\Diamond\neg z))$ of $\Diamond$-depth 5.

By proposition 3.2, weaker notions of bisimulation are unlikely to recover a positive result here. One may ask whether it might be done by adding suitable operators to the hybrid language able to express θ, while maintaining existence of first-order standard translations and invariance of hybrid sentences under quasi-injective bisimulations in the finite.

References

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