Games in algebraic logic: axiomatisations and beyond

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Introduction

A classical problem in algebraic logic is to characterise classes of representable algebras. Taking the example of the representable Tarskian relation algebras, we will discuss how games can help with such problems, and how they lead to a deeper study of representability.

Outline

1. Algebras of relations: a quick introduction to relation algebras, representable relation algebras
2. Case study: atomic and finite relation algebras
   - atom structures; representations of finite relation algebras
   - two examples: McKenzie's algebra; the so-called 'anti-Monk algebra'
3. Games to characterise representability: the games, axioms from games, examples
4. Infinite relation algebras
5. Infinite atom structures; relation algebras from graphs
6. Games in algebraic logic: pros and cons

1 Algebras of relations

Algebraic formalisation of unary relations began with Boole in the 19th century. It was very successful. The boolean algebra axioms are sound and complete: every boolean algebra is isomorphic to a field of sets [30].

De Morgan proposed considering *binary* (and higher-arity) relations. Peirce and Schröder developed the theory and established many hundreds of laws of binary relations (see, e.g., [29]). [25] has an interesting discussion of the history. But Pierce lamented:

> The logic of relatives is highly multiform; it is characterized by innumerable immediate conclusions from the same set of premises. . . . The effect of these peculiarities is that this algebra cannot be subjected to hard and fast rules like those of the Boolian calculus; and all that can be done in this place is to give a general idea of the way of working with it.

In the 1940s, Tarski and his collaborators began to investigate binary relations with modern algebra. Tarski laid down the notion of a *field of binary relations,* by which he meant a subalgebra of a product of algebras of the form

$$\mathfrak{Re}(X) = (\wp(X \times X), \cup, \setminus, \emptyset, X \times X, Id_X, {}^{-1}, |\,),$$

for some set X, where

$$\begin{array}{rcl} Id_X & = & \{(x,x) : x \in X\}, \\ R^{-1} & = & \{(y,x) : (x,y) \in R\}, \\ R|S & = & \{(x,y) : \exists z((x,z) \in R \wedge (z,y) \in S)\}. \end{array}$$

He wanted to characterise the algebras isomorphic to fields of binary relations. Such algebras are called *representable relation algebras,* the class of them is denoted **RRA**, and the isomorphism is called a *representation.*

It's easily seen why Tarski wanted to admit *subalgebras* of $\mathfrak{Re}(X)$. They are simply obtained by omitting some of the relations in $\mathfrak{Re}(X)$, but they still contain $\emptyset$, $X \times X$, and Id_X, and are closed under the operations, so they can certainly be considered as algebras of binary relations. But why *products?* The answer is probably that under this definition, **RRA** is a *variety* — an equationally axiomatised class. This was proved by Tarski in [31]. It follows from Birkhoff's theorem [1] that **RRA** is closed under subalgebras, products, and homomorphic images.

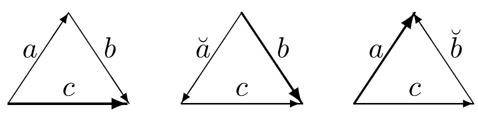
An algebra is *simple* if it has no non-trivial proper homomorphic images. We remark that all simple representable relation algebras are isomorphic to subalgebras of $\mathfrak{Re}(X)$ for some X: there is no need to consider products. For simplicity of exposition, we will generally restrict our attention here to simple algebras; but most of what we say is either true for arbitrary ones, or can easily be generalised to them. We also generally consider only *non-degenerate* relation algebras, satisfying $0 \neq 1$. (When $0 = 1$, the algebra has only one element; it is isomorphic to $\mathfrak{Re}(\emptyset)$ and so is representable. This case is not interesting.)

Relation algebras In 1940s, Tarski proposed axioms to capture **RRA**. These axioms defined the class **RA** of *'relation algebras'*.

Definition 1 A *relation algebra* is an algebra of the form $\mathcal{A} = (A, +, -, 0, 1, 1', \breve{}\,, ;)$ such that

- $(A, +, -, 0, 1)$ is a boolean algebra
- $(A, ;, 1')$ is a monoid
- 'Peircean law' (actually discovered by De Morgan):
  $(a\,;b) \cdot c \neq 0 \iff (\breve{a}\,;c) \cdot b \neq 0 \iff a \cdot (c\,;\breve{b}) \neq 0$ for all $a, b, c \in A$.

Considering *triangles* helps to make the point of the third axiom clear:



The above axioms are equivalent to Tarski's original ones, which were *equations.* The relation algebra axioms actually capture all equations valid in **RRA** that can be proved with 4 variables [24, 32].

Did Tarski's axioms capture **RRA**? Well, soundness (**RRA** $\subseteq$ **RA**) is easily seen. But completeness failed. In a rightly celebrated 1950 paper, Lyndon [21] gave an example of $\mathcal{A} \in$ **RA** $\setminus$ **RRA**. In 1964, Monk [27], building on work of Lyndon [23] and Jónsson [17], showed that **RRA** *is not finitely axiomatisable,* so proving the key 'negative' result in the field. Many more negative results about **RRA** are now known. One of the stronger ones is:

Theorem 2 (see [10, theorem 18.13]) *There is no algorithm that will tell whether an arbitrary finite relation algebra is representable.*

The following *problem* was stated for 'cylindric algebras' in [7], but the version for relation algebras is just as pertinent: *find a simple intrinsic characterisation of (the algebras in)* **RRA**. In the next sections, we will look into this question using games.

2 Case study: atomic and finite relation algebras

First, we try to cast relation algebras and representations in a more manageable form. This is quite useful for *atomic* relation algebras, and for representations of *finite* relation algebras. We will consider the general case later.

Atomic relation algebras An element a of a relation algebra $\mathcal{A}$ is said to be an *atom* if a is a minimal non-zero element with respect to the standard boolean algebra ordering '$\leq$', where $x \leq y \iff x + y = y$. $\mathcal{A}$ is said to be *atomic* if every non-zero element of it is $\geq$ an atom of it. All finite relation algebras are atomic, of course. We will say more about infinite atomic relation algebras in sections 4 and 5.

Atomic relation algebras can be quite easily specified. One can prove from the **RA** axioms that $\breve{}$ and ; are *additive.* That is, $(a+b)\breve{} = \breve{a} + \breve{b}$, $(a+b) \,;\, c = a \,;\, c + b \,;\, c$, and $a \,;(b+c) = a \,;\, b + a \,;\, c$ are valid laws in relation algebras. We can even prove from the **RA** axioms that $\breve{}$ and ; are additive over infinite sums. It follows that in an atomic relation algebra $\mathcal{A}$, the operations $\breve{}$ and ; are determined by their values on atoms, and we can specify $\mathcal{A}$ by stating:

- the set $\mathrm{At}\,\mathcal{A}$ of atoms of $\mathcal{A}$, and which elements of $\mathcal{A}$ are the sum of which atoms (this pins down the boolean structure of $\mathcal{A}$),
- which atoms are $\leq$ 1',
- $\breve{a}$, for each atom a (it turns out that $\breve{a}$ is also an atom),
- for each $a, b, c \in \mathrm{At}\,\mathcal{A}$, whether $a \,;\, b \geq c$ or not. In this case, we say that (a, b, c) is a *'consistent triple'*.

Remark: It follows from the Peircean law that (a, b, c) is consistent if and only if its *Peircean transforms* (a, b, c), $(\breve{a}, c, b)$, $(c, \breve{b}, a)$, $(b, \breve{c}, \breve{a})$, $(\breve{c}, a, \breve{b})$, $(\breve{b}, \breve{a}, \breve{c})$ are all consistent.

Ultrafilters Given a relation algebra $\mathcal{A}$, we'll write $\mathcal{A}$ for its domain as well. An *ultrafilter* of $\mathcal{A}$ is a subset $\alpha \subseteq \mathcal{A}$ such that

1. $a, b \in \alpha \Rightarrow a \cdot b \in \alpha$,
2. $a \geq b \in \alpha \Rightarrow a \in \alpha$,
3. α contains just one of a, $-a$, for every $a \in \mathcal{A}$.

Examples of ultrafilters are sets α of the form $\{b \in \mathcal{A} : b \geq a\}$, for any $a \in \operatorname{At}\mathcal{A}$. Such 'atom-generated' ultrafilters are called *principal.*

For simplicity of notation, we assume that $\mathcal{A}$ is simple. Suppose we are given a representation $h : \mathcal{A} \to \mathfrak{Re}(X)$ for some set X. For $x, y \in X$, let

$$h^{-1}(x, y) = \{a \in \mathcal{A} : (x, y) \in h(a)\}.$$

It is easy to check that

Lemma 3 *$h^{-1}(x, y)$ is always an ultrafilter of $\mathcal{A}$.*

Representations of finite simple relation algebras The following is well known and easily proved:

Lemma 4 *Any ultrafilter of a finite relation algebra is principal.*

Hence, a representation $h : \mathcal{A} \to \mathfrak{Re}(X)$ of a finite (simple) relation algebra $\mathcal{A}$ can be viewed in a simple way as a complete labelled directed graph $M = (X, \lambda)$, where X is a set and $\lambda : X \times X \to \operatorname{At}\mathcal{A}$ is a 'labelling function'. We just define $\lambda(x, y)$ to be the (unique) atom in $h^{-1}(x, y)$. It can be checked that for all $x, y, z \in X$,

- $\lambda(x, y) \leq 1' \iff x = y$.
- $\lambda(x, y) = \lambda(y, x)^{\smile}$.
- $\lambda(x, y) \leq \lambda(x, z) \,;\, \lambda(z, y)$. That is, *'all triangles are consistent'*.
- For all $a, b \in \operatorname{At}\mathcal{A}$, if $\lambda(x, y) \leq a \,;\, b$ then there is $w \in X$ with $\lambda(x, w) = a$ and $\lambda(w, y) = b$. *'All consistent triples are witnessed wherever possible.'*

Conversely, given a map $\lambda : X \times X \to \operatorname{At}\mathcal{A}$ satisfying these conditions, we can obtain a conventional representation $h : \mathcal{A} \to \mathfrak{Re}(X)$ by defining $h(a) = \{(x, y) \in X \times X : a \geq \lambda(x, y)\}$. The '$(X, \lambda)$' view of representations of finite relation algebras is very handy, as we will see.

Two finite relation algebras

1. *McKenzie's algebra $\mathcal{K}$.*

   4 atoms: 1', $<, >, \sharp$ (so 16 elements altogether).

   $\breve{1}' = 1'$, $\quad \breve{<} = >$, $\quad \breve{>} = <$, $\quad \breve{\sharp} = \sharp$.

   All triples are consistent except Peircean transforms of:
   $(1', a, a')$ for $a \neq a'$, $(<, <, >)$, $(<, <, \sharp)$, and $(\sharp, \sharp, \sharp)$.

2. *The 'anti-Monk algebra' $\mathcal{M}$.* We use this name not out of lack of respect, but because $\mathcal{M}$ is in some way the opposite of what are known as 'Monk algebras'. We believe $\mathcal{M}$ was discovered by Maddux.

   4 atoms: 1', $\mathsf{r}, \mathsf{b}, \mathsf{g}$.

   $\breve{a} = a$ for all atoms a ('symmetric algebra').

   All triples are consistent except Peircean transforms of: $(1', a, a')$ for $a \neq a'$, and $(\mathsf{r}, \mathsf{b}, \mathsf{g})$.

These are both relation algebras. Can you tell if they are in **RRA** or not?

3 Games and representability (finite relation algebras)

In [21], Lyndon characterised the *finite* representable relation algebras by a *'step by step' construction:*

1. Try to build 'step by step' a representation of a given finite relation algebra.
2. Write first-order axioms expressing that you can succeed.

Compare the Henkin construction of a model of a consistent first-order theory T as in [2, §2.1]. This could be used to test consistency of T: just see if the construction succeeds.

A minor variant of Lyndon's characterisation is quite easily done using *networks* and *games.*

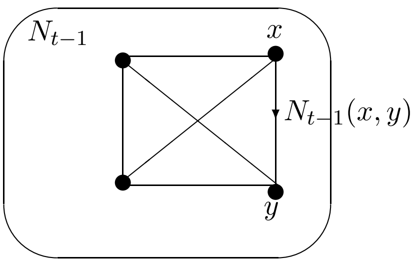
Networks (a.k.a. forcing conditions) A network is like a *piece of a representation* (if any!). It satisfies the *universal* conditions of 'representation'.

Definition 5 Let $\mathcal{A}$ be an atomic relation algebra. An $\mathcal{A}$*-network* is a complete labelled directed graph $N = (X, \lambda)$ where $X \neq \emptyset$ and $\lambda : X \times X \to \mathrm{At}\,\mathcal{A}$ is a labelling function satisfying, for all $x, y, z \in X$,

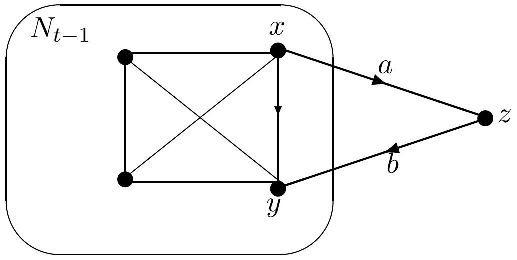
- $\lambda(x, y) \leq 1' \iff x = y$,
- $\lambda(x, y) = \lambda(y, x)^{\smile}$,
- $\lambda(x, y) \leq \lambda(x, z) ; \lambda(z, y)$ — *all triangles in* N *are consistent.*

We write N for any of N, X, λ. We rely on the context to tell which one is meant.

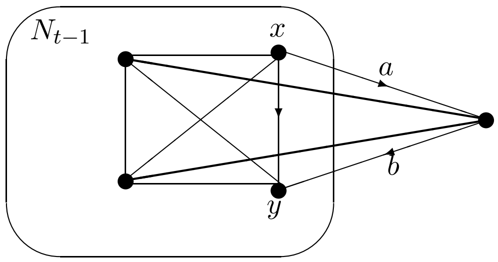
Games on $\mathcal{A}$-networks Let $\mathcal{A}$ be a non-degenerate atomic relation algebra — so $\mathrm{At}\,\mathcal{A} \neq \emptyset$ — and let $n \leq \omega$. The game $G_n(\mathcal{A})$ has two players — $\forall$ (male) and $\exists$ (female) — and n rounds. If $n = 0$, there are no rounds and we declare $\exists$ the winner. Assume $n > 0$. *In round* 0*,* $\forall$ picks $a_0 \in \mathrm{At}\,\mathcal{A}$, and $\exists$ plays an $\mathcal{A}$-network N_0 with a_0 occurring as a label in it. *In round t (*$1 \leq t < n$*),* suppose that the current network at the start of the round is N_{t-1}. Play goes as follows. First, $\forall$ picks $x, y \in N_{t-1}$ and $a, b \in At\mathcal{A}$ with $a ; b \geq N_{t-1}(x, y)$:



If there is already a node $z \in N_{t-1}$ with $N_{t-1}(x, z) = a$ and $N_{t-1}(z, y) = b$, then $\exists$ simply sets $N_t = N_{t-1}$. If not, she has more work to do. She begins by adding a new node z (say) to N_{t-1}, and labelling the edges (x, z) with a and (z, y) with b. This forms the basis of the new network N_t:



$\exists$ now has to *complete the labelling of N_t,* by defining $N_t(u, v)$ for all remaining pairs (u, v) of nodes. These are the ones other than (x, z), (z, y), and pairs of nodes of N_{t-1}, whose labels are already fixed:



It can be very hard for $\exists$ to complete the labelling. N_t must be a network, so all its triangles must be consistent. Moreover, N_t is then passed on to the next round (if any), in which $\forall$ can make new choices. So even if $\exists$ succeeds in creating a *network* N_t, she may have left herself open to a lethal attack by $\forall$ in a later round. If in some round she cannot manage to complete the labelling and create a network, she loses. Thus, $\exists$ wins the play of $G_n(\mathcal{A})$ if she always responds legally to $\forall$'s moves.

Note that it is in $\exists$'s interests to play as small a network (with as few nodes) as possible. Although she is permitted, by the rules of the game, to make arbitrarily large extensions to the networks played in the game, she only needs to include the nodes shown in the diagrams above. Additional nodes are superfluous and will only make it easier for $\forall$ to win, by giving him more rope to hang her with. We will always assume that she plays this way, so that N_0 has at most two nodes, and for each t, N_{t+1} has at most one more node than N_t.

The connection of the game to representability is given by the following theorem. It is more or less what Lyndon proved (but he didn't use games). The theorem is not restricted to simple relation algebras, but it only covers *finite* relation algebras; we will consider what to do about infinite relation algebras later.

Theorem 6 *Let $\mathcal{A}$ be a finite relation algebra.*

1. *$\mathcal{A} \in$ **RRA** if and only if $\exists$ has a winning strategy in $G_\omega(\mathcal{A})$.*
2. *$\exists$ has a winning strategy in $G_\omega(\mathcal{A})$ if and only if she has one in $G_n(\mathcal{A})$ for all finite n.*
3. *One can construct first-order sentences σ_n for $n < \omega$ (independently of $\mathcal{A}$) such that $\mathcal{A} \models \sigma_n$ if and only if $\exists$ has a winning strategy in $G_n(\mathcal{A})$.*

*Hence, for a finite relation algebra $\mathcal{A}$, we have $\mathcal{A} \in$ **RRA** $\iff \mathcal{A} \models \{\sigma_n : n < \omega\}$.*

Proof. We sketch the main ideas of the proof. For a rigorous treatment, see [10, chapter 11].

1. If $\mathcal{A} \in$ **RRA** then $\exists$ can use a representation as a guide in winning $G_\omega(\mathcal{A})$. Conversely, if she has a winning strategy in $G_\omega(\mathcal{A})$, then from plays of the game in which she uses her strategy and $\forall$ plays all possible moves at some stage, we can recover a representation of $\mathcal{A}$.

2. $\Rightarrow$ is clear. For the converse, we observe that because $\mathcal{A}$ is finite, $\exists$ has only finitely many possible responses to $\forall$'s move in any round. König's tree lemma can now be used to collimate her responses in the finite games into a single winning strategy in $G_\omega(\mathcal{A})$.

3. First, given an $\mathcal{A}$-network N, and $k < \omega$, we write an axiom $\tau_k(N)$ saying that $\exists$ *can win* $G_k(\mathcal{A})$ *starting from* N. We go by induction on k. All quantifiers are implicitly relativised to atoms.

$$\begin{aligned}
\tau_0(N) &= \bigwedge_{x \in N} \Big(N(x,x) \leq 1' \;\wedge \bigwedge_{y \in N \setminus \{x\}} N(x,y) \not\leq 1' \Big) \\
&\quad \wedge \bigwedge_{x,y \in N} N(x,y) = N(y,x)^{\smile} \;\wedge \bigwedge_{x,y,z \in N} N(x,y) \leq N(x,z) \,;\, N(z,y). \\
\tau_{k+1}(N) &= \bigwedge_{x,y \in N} \forall a,b \Big(N(x,y) \leq a \,;\, b \to \exists N' \supseteq N \\
&\qquad (\tau_k(N') \wedge \bigvee_{z \in N'} (N'(x,z) = a \wedge N'(z,y) = b)) \Big).
\end{aligned}$$

We then let $\sigma_n = \forall a_0 \exists N (\tau_{n-1}(N) \wedge \bigvee_{x,y \in N} N(x,y) = a_0)$ for $n > 0$, and $\sigma_0 = \top$.

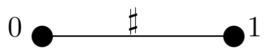
■

The axioms σ_n (plus the **RA** axioms) seem to give an *intrinsic* characterisation of the finite algebras in **RRA**. But is it a *simple* one? Can you tell whether McKenzie's algebra and the anti-Monk algebra satisfy the σ_n for all n?

It's easier to use the games G_n directly.

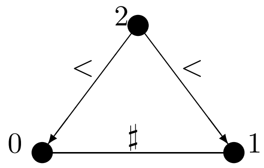
Example 7 (McKenzie's algebra $\mathcal{K}$) Recall that this relation algebra has 4 atoms: $1', <, >, \sharp$. We have $\breve{1'} = 1'$, $\breve{<} = >$, $\breve{>} = <$, $\breve{\sharp} = \sharp$. All triples of atoms are consistent except Peircean transforms of $(1', a, a')$ for $a \neq a'$, $(<, <, >)$, $(<, <, \sharp)$, and $(\sharp, \sharp, \sharp)$.

Consider the following play of $G_\omega(\mathcal{K})$. $\forall$ begins by picking the atom $\sharp$. $\exists$ responds with the network N_0 as shown below.



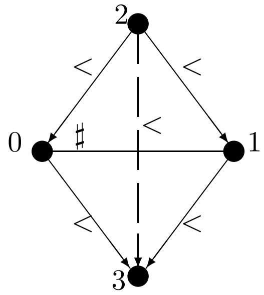
The edge $(0, 1)$ is labelled by $\sharp$. We know that in any $\mathcal{K}$-network N and nodes x, y of N, we have $N(x, y) = 1'$ if and only if $x = y$, and $N(y, x) = N(x, y)^{\smile}$. So $\exists$ has no choice over the labels of the remaining edges of N_0. We don't need an arrow on the edge in the diagram to indicate its direction, because $\breve{\sharp} = \sharp$, so the converse edge $(1, 0)$ will also be labelled $\sharp$.

$\forall$ continues by choosing the two nodes $0, 1$ of N_0, and the atoms $>, <$. $\exists$ has to add a new node, say 2, and label $(0, 2)$ with $>$ and $(2, 1)$ with $<$. She has no choice in labelling the remaining edges of her response, N_1:

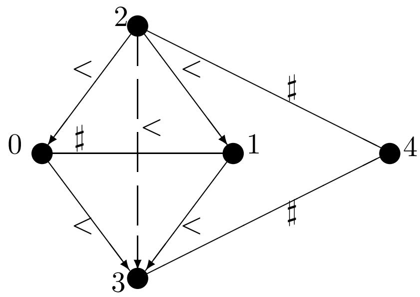


We prefer to show the edge $(2, 0)$, which will be labelled $\breve{>} = <$.

$\forall$ now picks the nodes $0, 1$ again, and the atoms $<, >$. $\exists$ now has to add a node 3, with $(0, 3)$ labelled $<$ and $(3, 1)$ labelled $>$. She has no choice over the remaining edges: in particular, she must label the edge $(2, 3)$ by $<$, since all other choices lead to inconsistency of the triangle $2, 0, 3$.



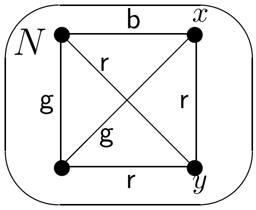
Now $\forall$ deals the killer blow, picking $2, 3$ and the atoms $\sharp, \sharp$. $\exists$ has to add a new node, say 4.



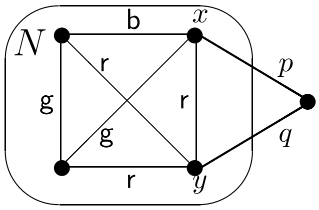
$\exists$ cannot consistently label the edge $(0, 4)$ by $<$ (because of the triangle $2, 0, 4$), nor by $>$ (because of the triangle $3, 0, 4$). She has to use $\sharp$. Similarly, she must label $(1, 4)$ with $\sharp$. But now, $0, 1, 4$ is an inconsistent triangle, and $\exists$ has lost. It is clear that she never had any real choice, so what we have described is a *winning strategy for* $\forall$ *in* $G_\omega(\mathcal{K})$ (and indeed in $G_4(\mathcal{K})$). $\exists$ has no winning strategy, so by theorem 6, $\mathcal{K}$ is not representable.

Example 8 (Anti-Monk algebra $\mathcal{M}$) Recall that $\mathcal{M}$ has 4 atoms: 1', $\mathsf{r}, \mathsf{b}, \mathsf{g}$. $\mathcal{M}$ is symmetric: we have $\breve{x} = x$ for all atoms x. All triples of atoms are consistent except Peircean transforms of $(1', a, a')$ for $a \neq a'$, and $(\mathsf{r}, \mathsf{b}, \mathsf{g})$.

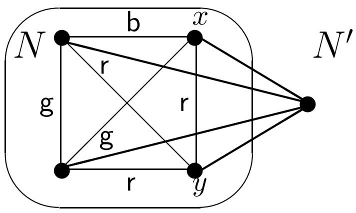
Consider a typical $\mathcal{M}$-network N as shown below. Observe that all triangles involve at most two colours from $\mathsf{r}, \mathsf{b}, \mathsf{g}$, as required for consistency. We don't need any arrows at all on edges this time, since $\breve{a} = a$ for all atoms a, so the labels on an edge (u, v) and the converse edge (v, u) are always the same.



Suppose that N is in play in some round of the game $G_\omega(\mathcal{M})$. A typical move of $\forall$ will be to pick two nodes and some atoms or other. We assume by way of example that he picks the two right-hand nodes x, y in the diagram, and the atoms p, q, say. If there is a suitable node in N, as in the game rules, then $\exists$ has an easy job. We'll assume there isn't; it follows that $p, q \neq 1$'. $\exists$ must now add a new node on the right as shown:

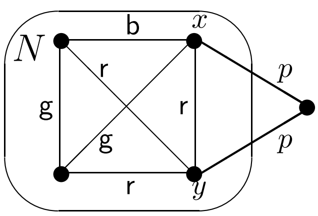


Then, she must fill in the remaining labels, to give a network N', say:

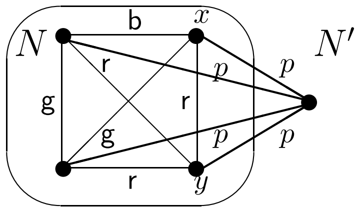


In this example, the edge (x, y) that $\forall$ picked in N is labelled r. His chosen atoms p, q, combined with r, must not all be different, or his choice would be illegal because $\mathsf{r} \not\leq p\,;q$. So two of p, q, r must be equal. There are two possibilities.

Case 1: $p = q$**.** So N looks like:

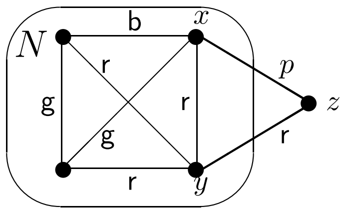


In this case, $\exists$ simply uses p to label all remaining edges:

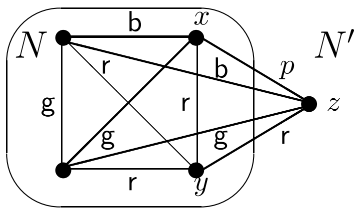


It is clear that all triangles have at least two edges of the same colour, so are consistent.

Case 2: $\mathsf{r} = p \neq q$ **or** $\mathsf{r} = q \neq p$**.** Let's suppose that $\mathsf{r} = q \neq p$ (the other case is similar):



Observe that x *and* z *look the same as seen from* y*:* the labels on the edges (y, x) and (y, z) are the same. $\exists$ *tries to make this true for the other nodes, as well as* y*.* That is, she defines $N'(t, z) = N(t, x)$ for all nodes t of N other than x, y:



Now, there are three kinds of triangle in N':

1. Triangles consisting of nodes of N. These are certainly consistent, because N is a network.
2. Triangles of the form t, x, z, involving x, z. These have two edges with identical colours, because $N'(t, z) = N'(t, x)$. So they are consistent.
3. Triangles of the form t, u, z, involving z but not x. The sides of such a triangle are coloured the same as in the triangle t, u, x of N (because z looks the same as x from t, and from u). But the triangle t, u, x is consistent, by case 1, and hence, so is triangle t, u, z.

So all triangles of N' are consistent, and N' is a $\mathcal{M}$-network.

This can be elaborated into a winning strategy for $\exists$ in $G_\omega(\mathcal{M})$, showing that $\mathcal{M}$ is representable. This elegant strategy is due to Maddux (personal communication).

Summary

1. McKenzie's algebra $\mathcal{K} \notin$ **RRA**. So **RRA** $\subset$ **RA**, as Lyndon (1950) showed. In fact, $\mathcal{K}$ is one of the smallest non-representable relation algebras. There are other 4-atom non-representable relation algebras, but all relation algebras with at most 3 atoms are representable.

2. The anti-Monk algebra $\mathcal{M} \in$ **RRA**.

   Exercise: show that if (X, λ) is any representation of $\mathcal{M}$, then X is infinite. This is perhaps surprising, given that $\mathcal{M}$ is symmetric.

4 Infinite relation algebras

Games can still be used to characterise representability of infinite relation algebras. But there are some issues that need dealing with first.

4.1 Complete representations

Recall that a relation algebra is *atomic* if every non-zero element of it lies above an atom. All finite relation algebras are atomic, but not all infinite relation algebras are — indeed, some have no atoms at all. Even the atomic ones need care. Lemma 3 holds for infinite algebras, but lemma 4 does not: not all ultrafilters of an infinite relation algebra, even an atomic one, are principal. So we cannot assume that in a representation of such an algebra, we can associate an atom with every edge in the representation.

Let us start by picking out the representations where we *can* associate atoms to edges.

Definition 9 A representation h of a relation algebra $\mathcal{A}$ is said to be a *complete representation* if $h^{-1}(x, y)$ is a principal ultrafilter of $\mathcal{A}$ — it contains an atom of $\mathcal{A}$ — for every $x, y \in X$.

Complete representations are special kinds of representations. It is not hard to show that in the above notation,

Theorem 10 [10, theorem 2.21] *h is a complete representation just in case h preserves all existing infima and suprema in $\mathcal{A}$: that is, if $S \subseteq \mathcal{A}$, and S has a least upper bound $a \in \mathcal{A}$ (with respect to $\geq$), then*

$$h(a) = \bigcup_{s \in S} h(s) \subseteq X \times X,$$

and similarly for greatest lower bounds.

This property gave rise to the name 'complete representation'.

Any representation of a finite relation algebra is complete. A model-theoretic saturation argument will easily show that any infinite representable relation algebra has incomplete representations. So for infinite relation algebras, the question of interest is whether they have *any* complete representation at all.

Definition 11 A relation algebra is said to be *completely representable* if it has a complete representation. We write **CRA** for the class of completely representable relation algebras.

It is not hard to see that any completely representable relation algebra must be atomic. It's easy to find non-atomic representable relation algebras, and these cannot have any complete representation. But in fact, there are even *atomic* relation algebras that have a representation but don't have a complete representation. They are representable, but not completely representable. The first such relation algebra was given by Lyndon in [21], though it was not recognised as such at the time.

Games can help to analyse complete representations. We can generalise the game $G_n(\mathcal{A})$ seen earlier to a game $G_\kappa(\mathcal{A})$ with κ rounds, where κ is any cardinal. Then we can prove

Theorem 12 *Let $\mathcal{A}$ be any atomic relation algebra $\mathcal{A}$. If $\mathcal{A}$ is completely representable, then $\exists$ has a winning strategy in $G_\kappa(\mathcal{A})$ for any κ. If $\exists$ has a winning strategy in $G_\kappa(\mathcal{A})$ for $\kappa = |\operatorname{At}\mathcal{A}| + \aleph_0$, then $\mathcal{A}$ is completely representable.*

There is also an approximate characterisation of complete representability, generalising theorem 6:

Theorem 13 ([9, 10]) *For any atomic relation algebra $\mathcal{A}$, the following are equivalent:*

1. *$\exists$ has a winning strategy in $G_n(\mathcal{A})$ for all finite n,*
2. *$\mathcal{A}$ is elementarily equivalent to (i.e., satisfies the same first-order sentences as) some completely representable relation algebra.*

It is easily seen that the class **CRA** of completely representable relation algebras is pseudo-elementary (see [2, exercise 4.1.17] for the definition). However, there are many negative results about it. [9, 10] used game-inspired relation algebras to show that **CRA** is not elementary (it is not definable by any set of first-order sentences). By theorem 2, it is not definable by a second-order (or higher-order) sentence, or a sentence of fixed-point logic. The completely representable relation algebras with countably many atoms can be characterised using the infinitary logic $L_{\infty\omega}$, using theorem 12 (this was observed by Väänänen at the meeting). But the countability assumption is necessary: there are atomic relation algebras $\mathcal{A}, \mathcal{B}$, the former with uncountably many atoms, that agree on all $L_{\infty\omega}$-sentences, with $\mathcal{B}$ completely representable and $\mathcal{A}$ not.[1] So **CRA** is not definable by a sentence of $L_{\infty\omega}$.

4.2 Games and representations for infinite relation algebras

So much for *complete* representations. What about arbitrary ones? Can we use games to test whether an infinite relation algebra is representable?

Our game characterisation of the finite representable relation algebras in theorem 6 relied on every edge in a representation being labelled by an atom — that is, on completeness of the representation. For infinite relation algebras, which may not have complete representations, this is not going to work.

There are two ways out of this difficulty. We can modify the games to handle arbitrary (possibly incomplete) representations. One of the changes is that player $\forall$ will choose arbitrary elements of the algebra, not just atoms. Then, we can use universal algebra to turn the σ_n of theorem 6 into equations. This gives an equational axiomatisation of **RRA**. The method is very close to one of Lyndon from 1956 [22]. See [8, 10] for details.

Alternatively, we can take advantage of *canonical extensions*.

Definition 14 The *canonical extension* $\mathcal{A}^\sigma$ of a relation algebra $\mathcal{A}$ is the relation algebra formed from the set of all sets of ultrafilters of $\mathcal{A}$. We will identify a set $\{\alpha\}$, consisting of a single ultrafilter α, with the ultrafilter α itself. So the atoms of $\mathcal{A}^\sigma$ are essentially the ultrafilters of $\mathcal{A}$. Then:

- The atoms $\leq$ 1' (in the sense of $\mathcal{A}^\sigma$) are precisely the ultrafilters containing 1' (in the sense of $\mathcal{A}$).
- The converse of an atom (ultrafilter) α is the ultrafilter consisting of the converses of all the elements of α: in symbols, $\breve{\alpha} = \{\breve{a} : a \in \alpha\}$.
- A triple (α, β, γ) of ultrafilters is consistent just when every triple (a, b, c) of elements of $\mathcal{A}$ taken from them (i.e., $a \in \alpha$, $b \in \beta$, $c \in \gamma$) satisfies the consistency condition $(a\,;b) \cdot c \neq 0$ (this generalises the consistency condition for atoms given in §2).

[1] In the notation of [10, theorem 17.25], take $\mathcal{A} = \mathcal{A}_{\mathsf{K}_{\omega_1}, \mathsf{K}_\omega}$ and $\mathcal{B} = \mathcal{A}_{\mathsf{K}_\omega, \mathsf{K}_\omega}$

Apart from some changes in notation, this definition is due to Jónsson and Tarski [19, 20], and it generalises Stone's related construction for boolean algebras [30]. Any relation algebra $\mathcal{A}$ has a canonical extension $\mathcal{A}^\sigma$, and $\mathcal{A}$ embeds in $\mathcal{A}^\sigma$ via $a \mapsto \{\alpha : \alpha$ an ultrafilter of $\mathcal{A},\ a \in \alpha\}$. For finite $\mathcal{A}$, we have $\mathcal{A} \cong \mathcal{A}^\sigma$. Thus, the following generalises theorem 6:

Theorem 15 *A relation algebra $\mathcal{A}$ is representable if and only if $\exists$ has a winning strategy in $G_n(\mathcal{A}^\sigma)$ for all finite n.*

Proof. $\Rightarrow$: In an important result, Monk proved that if $\mathcal{A}$ is representable then $\mathcal{A}^\sigma$ is representable. (Monk did not publish it; his result is reported in McKenzie's Ph.D. dissertation [26].) In fact, it can even be shown that if $\mathcal{A}$ is representable then $\mathcal{A}^\sigma$ is *completely* representable [10, theorem 3.36]. So by theorem 12, $\exists$ has a winning strategy in $G_n(\mathcal{A}^\sigma)$ for all finite n.

$\Leftarrow$: Assume that $\exists$ has a winning strategy in $G_n(\mathcal{A}^\sigma)$ for all finite n. By theorem 13, $\mathcal{A}^\sigma$ is elementarily equivalent to some (completely) representable relation algebra $\mathcal{B}$. Up to isomorphism, $\mathcal{A}$ is a subalgebra of $\mathcal{A}^\sigma$. We saw in section 1 that **RRA** is a variety, and so is closed under elementary equivalence and under taking isomorphic copies of subalgebras. So we obtain $\mathcal{A} \in$ **RRA** as required. ∎

This means that we can still use the games G_n to characterise representability. We just need to play on the canonical extension, not the relation algebra itself. (For finite algebras, this makes no difference.)

5 Infinite atom structures

Recall from section 2 that for an atomic relation algebra, if we know the value of the relation algebra operators applied to atoms, then we can determine these operators on arbitrary elements. For an atomic relation algebra $\mathcal{A}$, we call

$$\mathfrak{At}\,\mathcal{A} = (\mathrm{At}\,\mathcal{A},\ \{a \in \mathrm{At}\,\mathcal{A} : a \leq 1'\}, \{(a, \breve{a}) : a \in \mathrm{At}\,\mathcal{A}\}, \{(a,b,c) : a,b,c \in \mathrm{At}\,\mathcal{A},\ a\,;b \geq c\})$$

the *atom structure* of $\mathcal{A}$. A tuple (S, I, f, C) is called an *atom structure* if it is the atom structure of some atomic relation algebra. We used atom structures in section 2 as a kind of notational device to allow us to present finite relation algebras more concisely. They certainly serve this function, but in some ways it is with infinite atomic relation algebras that connections between the representability of an algebra and the properties of its atom structure become most interesting.

Any atomic relation algebra uniquely determines its atom structure, but once we move away from finite relation algebras, we see that there can be many relation algebras possessing the same atom structure but with different (non-isomorphic) boolean structures. The boolean structure of $\mathcal{A}$ (i.e., which suprema of sets of atoms exist in $\mathcal{A}$), together with the atom structure, determine $\mathcal{A}$ up to isomorphism. Informally, we have

$$\text{atomic relation algebra} = \text{atomic boolean algebra} + \text{atom structure}.$$

Now all boolean algebras are representable, but the representability problem for relation algebras is highly non-trivial. So we might surmise that the difficulties in representing an (atomic) relation algebra reside in its atom structure. More precisely, we might guess that whether an atomic relation algebra is representable or not is determined by its atom structure. For *complete* representations, in which all edges are labelled by atoms, this is of course true (though the 'completely representable atom structures' are at least as hard to characterise as the completely representable relation algebras). But for arbitrary representations, it is not so clear.

What are the possible atomic relation algebras with a given atom structure? At one end of the spectrum we can define the *complex algebra* $\mathrm{Cm}\,\mathcal{S}$ of an atom structure $\mathcal{S}$. Its domain is the full power set of the domain of $\mathcal{S}$, and the relation algebra operations are determined by the atom structure. If the cardinality of the atom structure $\mathcal{S}$ is λ then $\mathrm{Cm}\,\mathcal{S}$ has cardinality 2^λ. At the other end of the spectrum, the *term algebra* $\mathrm{Tm}\,\mathcal{S}$ is the smallest relation algebra whose atom structure is $\mathcal{S}$. It is the subalgebra of $\mathrm{Cm}\,\mathcal{S}$ generated, using the relation algebra operations, by the atoms. The cardinality of the term algebra is λ, for infinite atom structures. It is easily seen that if $\mathcal{A}$ is an atomic relation algebra with $\mathfrak{At}\,\mathcal{A} = \mathcal{S}$, then up to isomorphism, $\mathcal{A}$ is a subalgebra of $\mathrm{Cm}\,\mathcal{S}$ and $\mathrm{Tm}\,\mathcal{S}$ is a subalgebra of $\mathcal{A}$.

So we may distinguish two types of representability for atom structures. An atom structure is *weakly representable* if it is the atom structure of some representable relation algebra. An atom structure is *strongly representable* if every relation algebra with that atom structure is representable. Since **RRA** is closed under subalgebras, we can easily see that:

Theorem 16

1. *An atom structure is weakly representable if and only if its term algebra is representable.*
2. *An atom structure is strongly representable if and only if its complex algebra is representable.*

For finite atom structures, the term algebra is the same as the complex algebra, so weak and strong representability coincide.

Several questions immediately present themselves:

- Is representability of an atomic relation algebra determined by its atom structure? That is, could an (infinite) atom structure be weakly representable but not strongly representable?
- Is the class of weakly representable atom structures elementary?
- What about the class of strongly representable atoms structures?
- Can we define either class with finitely many axioms?

The last question is easily dealt with: by theorem 2, there can be no finite axiomatisation of either class. Also, since **RRA** is a variety, a result of [33] shows that the class of weakly representable atom structures is also elementary.

The other questions are more tricky. To help us answer them, we look at a class of interesting atom structures obtained from graphs.

Graphs and relation algebras By a *graph,* we mean an irreflexive symmetric 'edge' relation on a finite or infinite set of 'nodes'. A set I of nodes of a graph is said to be *independent* if no two nodes in I are connected by a graph edge. For finite k, a *k-colouring* of a graph is a partition of its nodes into at most k independent sets. The *chromatic number* of a graph is the least finite k for which it has a k-colouring, and if there is no such k then the chromatic number is ∞.

Given a graph Γ, we can make an atom structure $\mathcal{S}(\Gamma) = (S, I, f, C)$ whose atoms are red, blue, and green copies of each node of Γ, plus 1' as an extra atom. That is, the set of atoms is

$$S = \{\mathsf{r}_x, \mathsf{g}_x, \mathsf{b}_x : x \in \Gamma\} \cup \{1'\}. \tag{1}$$

(Here and below, if Γ is a graph, we also let Γ denote its set of nodes.) The set I of sub-identity atoms is just $\{1'\}$. The converse function f leaves each atom fixed — $\mathcal{S}(\Gamma)$ is symmetric. To define C, we stipulate that all triples of atoms are consistent (included in C) except Peircean transforms of $(1', a, a')$ for $a \neq a'$, and monochromatic triples of nodes forming an independent set in Γ — that is, triples $(\mathsf{r}_x, \mathsf{r}_y, \mathsf{r}_z)$ where $\{x, y, z\} \subseteq \Gamma$ is independent — and similarly for green and blue.

It turns out, for any graph Γ, that $\mathrm{Cm}(\mathcal{S}(\Gamma))$ is a relation algebra (to prove associativity of composition we need to take advantage of the three colours) and so $\mathcal{S}(\Gamma)$ is a genuine relation algebra atom structure. Surprisingly, perhaps, strong representability of $\mathcal{S}(\Gamma)$ is entirely determined by the chromatic number of Γ, in the case where Γ is infinite:

Theorem 17 ([11, 10]) *For infinite Γ, the relation algebra $\mathrm{Cm}(\mathcal{S}(\Gamma))$ is representable if and only if Γ has chromatic number ∞.*

Proof. First, some notation: if $Z \subseteq \Gamma$, we let $\mathsf{r}_Z = \{\mathsf{r}_z : z \in Z\}$, and similarly we define $\mathsf{g}_Z, \mathsf{b}_Z$. Note that these are all in $\mathrm{Cm}\,\mathcal{S}(\Gamma)$, since the domain of the complex algebra is the full power set of the set of atoms.

$\Rightarrow$: Suppose that $h : \mathrm{Cm}\,\mathcal{S}(\Gamma) \to \mathfrak{Re}(X)$ is a representation. Supposing, for contradiction, that Γ has finite chromatic number, its set of nodes can be partitioned into independent sets $I_0, \ldots, I_{n-1}$ for some finite n.

Clearly, in $\mathrm{Cm}\,\mathcal{S}(\Gamma)$ we have

$$1' + \mathsf{r}_{I_0} + \mathsf{g}_{I_0} + \mathsf{b}_{I_0} \cdots + \mathsf{r}_{I_{n-1}} + \mathsf{g}_{I_{n-1}} + \mathsf{b}_{I_{n-1}} = 1.$$

Now h respects $+$: we have $h(a+b) = h(a) \cup h(b)$, for any $a, b \in \mathrm{Cm}\,\mathcal{S}(\Gamma)$. So for any distinct $x, y \in X$, since $(x, y) \notin h(1')$, we know that $(x, y) \in h(\mathsf{c}_{I_k})$ for some $k < n$ and some colour $\mathsf{c} \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$. Clearly, X is infinite (since $\mathcal{S}(\Gamma)$ is). Ramsey's theorem [28] will now show that there are distinct $x_i \in X$ for $i < \omega$, some $k < n$, and a colour $\mathsf{c} \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$, such that letting $a = \mathsf{c}_{I_k}$, we have $(x_i, x_j) \in h(a)$ for all $i < j < \omega$. In particular, $(x_0, x_1), (x_1, x_2), (x_0, x_2) \in h(a)$, so that $(x_0, x_2) \in (h(a)\,; h(a)) \cdot h(a) = h((a\,; a) \cdot a)$. But for any nodes $p, q, s \in I_k$ we know that $\{p, q, s\}$ is independent (since I_k is), and so $(\mathsf{c}_p, \mathsf{c}_q, \mathsf{c}_s)$ is not a consistent triple of atoms. It follows that $(a\,; a) \cdot a = 0$, and $h(0) = \emptyset$, so this is impossible.

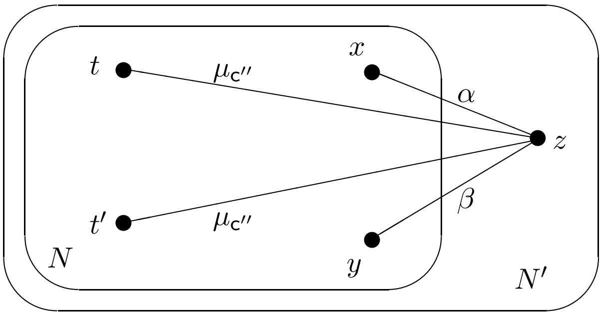
$\Leftarrow$: Assume Γ has infinite chromatic number. Call a set X of nodes of Γ *small* if the induced subgraph of Γ on the set of nodes X has finite chromatic number. Call a set *large* if its complement is small. Then the set of all nodes is large, any superset of a large set is large, and the intersection of two large sets is still large (because the union of two small sets is small). Using Zorn's lemma or the (weaker) boolean prime ideal theorem, for each colour $\mathsf{c} \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$ the set $\{\mathsf{c}_L : L \subseteq \Gamma,\ L \text{ large}\}$ of c-coloured copies of large sets can be extended to an ultrafilter μ_c of $\mathrm{Cm}\,\mathcal{S}(\Gamma)$ — that is, an atom of the canonical extension $(\mathrm{Cm}\,\mathcal{S}(\Gamma))^\sigma$ (see definition 14). The underlying set of nodes of any element of this ultrafilter is not small, and so in particular, not independent.

The three atoms $\mu_\mathsf{r}, \mu_\mathsf{g}, \mu_\mathsf{b}$ are very useful for $\exists$ when playing the game $G_\omega((\mathrm{Cm}\,\mathcal{S}(\Gamma))^\sigma)$. In fact, they allow her to win it. First, a little calculation will establish the following facts:

1. Since in $\mathrm{Cm}\,\mathcal{S}(\Gamma)$ we have $\{1'\} + \mathsf{r}_\Gamma + \mathsf{g}_\Gamma + \mathsf{b}_\Gamma = 1$, any ultrafilter must contain one of these four sets. So for any ultrafilters α, β of $\mathrm{Cm}\,\mathcal{S}(\Gamma)$ that do not contain $\{1'\}$, there are $\mathsf{c}, \mathsf{c}' \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$ such that $\mathsf{c}_\Gamma \in \alpha$ and $\mathsf{c}'_\Gamma \in \beta$. Since we have three colours, we can pick a colour $\mathsf{c}'' \neq \mathsf{c}, \mathsf{c}'$ (this is why we introduced three colours). Then it can be checked that $(\alpha, \beta, \mu_{\mathsf{c}''})$ is a consistent triple of atoms of $(\mathrm{Cm}\,\mathcal{S}(\Gamma))^\sigma$.

2. For any ultrafilter α of $\mathrm{Cm}\,\mathcal{S}(\Gamma)$ and any $\mathsf{c} \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$, the triple $(\mu_\mathsf{c}, \mu_\mathsf{c}, \alpha)$ is a consistent triple of atoms of $(\mathrm{Cm}\,\mathcal{S}(\Gamma))^\sigma$.

In the game $G_\omega((\mathrm{Cm}\,\mathcal{S}(\Gamma))^\sigma)$, suppose that in some round, the current network is N, and that $\forall$ picks nodes $x, y \in N$ and atoms (ultrafilters) α, β. If $\exists$ has to extend the network, we will have $\{1'\} \notin \alpha, \beta$, so she can choose c'' as in (1) above. Then letting the new network be N' with new node z, she labels $N'(z,t) = N'(t,z) = \mu_{\mathsf{c}''}$ for each node t of N with $t \neq x, y$.



By facts 1 and 2 above, N' is a network. So this gives a winning strategy for her in the game, showing that $(\mathrm{Cm}\,\mathcal{S}(\Gamma))^\sigma$ and hence its subalgebra $\mathrm{Cm}\,\mathcal{S}(\Gamma)$ are representable. By theorem 16, $\mathcal{S}(\Gamma)$ is strongly representable. ■

The theorem allows us to translate problems about atom structures into problems about graphs. Graphs seem easier to work with, and far more is known about them.

If we replace $\mathrm{Cm}\,\mathcal{S}(\Gamma)$ by a subalgebra (e.g., $\mathrm{Tm}\,\mathcal{S}(\Gamma)$), the left-to right implication in theorem 17 can fail. Even if the nodes of Γ can be partitioned into independent sets $I_0, \ldots, I_{n-1}$ for some finite n, it might be that the element $\{\mathsf{c}_x : x \in I_k\}$ does not belong to the algebra, for some $k < n$ and some colour c. Indeed, taking the graph Z with nodes $\mathbb{Z}$ and edges between consecutive integers only, a not too difficult exercise shows that the term algebra $\mathrm{Tm}\,\mathcal{S}(Z)$ is indeed representable, though the chromatic number of Z is just two. (The first part of the exercise is to calculate exactly which sets of atoms are generated using the relation algebra operations.) Thus, $\mathcal{S}(Z)$ is weakly but (by theorem 17) not strongly representable, and we conclude:

Theorem 18 *There exist weakly but not strongly representable atom structures.*

A more complicated sequence of graphs Γ_k ($k < \omega$) is derived from graphs invented by Erdős [3]. Each Γ_k has infinite chromatic number, but an ultraproduct Γ of the Γ_k has chromatic number just two. We can use this wonderful construction in graph theory to answer the last remaining question from those listed above. It follows from theorem 17 that every $\mathcal{S}(\Gamma_k)$ is strongly representable, but an ultraproduct $\mathcal{S}(\Gamma)$ of them is not strongly representable. ($\mathcal{S}(-)$ commutes with taking ultraproducts.) By Łoś's theorem (see [2, theorem 4.1.9]), any first-order sentence true in all the $\mathcal{S}(\Gamma_k)$ must also be true in $\mathcal{S}(\Gamma)$. We conclude that:

Theorem 19 *The class of strongly representable atom structures is not elementary: it cannot be defined by* any *set of first-order axioms.*

Probabilistic constructions of graphs have been useful on several other occasions. For example, in theorem 15 we mentioned Monk's result that if $\mathcal{A}$ is a representable relation algebra then so also is its canonical extension $\mathcal{A}^\sigma$. We say that **RRA** is a *canonical* variety. But does it have an axiomatisation by equations ε that are *individually* canonical, in the sense that for any relation algebra $\mathcal{A}$, if $\mathcal{A} \models \varepsilon$ then $\mathcal{A}^\sigma \models \varepsilon$? The answer is 'no': [16] uses a probabilistic graph construction to show that:

Theorem 20 *Any axiomatisation of* **RRA** *must have infinitely many non-canonical equations.*

Similar considerations led to a proof that not every canonical variety is generated by an elementary class of frames [5, 6], solving a problem of Fine in modal logic [4].

More details of these and other related results can be found in [11, 16] or in [10, chapter 14].

6 Games in algebraic logic: pros and cons

Games have made a substantial contribution to our understanding of relation algebras. The idea has many precursors, notably in the seminal paper of Lyndon [21]. Let us end with a rundown of the pros and cons of using games in relation algebras and algebraic logic generally.

6.1 Pros

1. Games provide a simple practical test for representability. (They are also very useful for theoretical purposes.)
2. Games can be used to produce axioms as well (with care, they sometimes even yield finite axiomatisations).
3. Sometimes, a winning strategy can be extracted and used for other things, such as decidability, complexity, finite model property.
4. Games on relation algebras generalise to games for other kinds of algebras of relations, such as complex algebras (see, e.g., [15]).
5. Most importantly in our view, games can suggest some fairly sophisticated constructions of relation algebras. These can be used to prove:
   (a) **RRA** is not finitely axiomatisable (first proved by Monk in [27], not using games).
   (b) **RRA** is not axiomatisable by equations using finitely many variables altogether (stated by Tarski in a video made in 1974 and published by Jónsson in [18]).
   (c) **RRA** is not closed under Monk completions [14, 10]: the example $\mathrm{Tm}\,\mathcal{S}(Z)$ above shows this, since its completion is isomorphic to $\mathrm{Cm}\,\mathcal{S}(Z)$. Hence, **RRA** is not Sahlqvist-axiomatisable [34].
   (d) In first-order logic, more 3-variable sentences are provable with $n+1$ variables than with n variables, for all $n \geq 3$ ([12], motivated by games and relation algebras [13]).
   (e) For a finite relation algebra $\mathcal{A}$, it is undecidable whether $\mathcal{A} \in$ **RRA** [10].
   (f) **RRA** is canonical (Monk), but any first-order axiomatisation of it has infinitely many non-canonical axioms [16].

6.2 Cons

We use games as a *construction method,* essentially forcing, to build representations of relation algebras. In general, the representations so obtained are *infinite.* These games are not good at building *finite representations.*

For example, suppose that $\mathcal{A}$ is a finite relation algebra with a *'flexible atom', f*, say. This means that (a, b, f) is consistent for all atoms $a, b \neq 1'$. The game $G_\omega(\mathcal{A})$ shows that $\mathcal{A}$ is representable: $\exists$ can win by using f to label network edges wherever needed, and it will always be consistent to do so.

Problem 21 (Maddux) *Must such an $\mathcal{A}$ have a* finite *representation?*

There is a general issue here: *find ways of constructing finite representations.* Can we combine games with, e.g., probabilistic constructions?

Some algebraic logicians avoid games and prefer the traditional 'step by step' approach, enumerating the requirements of a construction and dealing with them one by one. Certainly, games are not needed in simple cases, but when the going gets tougher we believe that they are invaluable, and they bring their own insights. The feeling that games are in some way undignified is addressed by Hodges [10, page vii], who comments:

> 'The notion of a game has to do with people acting together, setting themselves and each other tasks. As a result, game-theoretic versions of mathematical ideas often have a direct intuitive appeal when compared with more formalistic treatments. In the period 1900–1950 logic was fighting to establish itself as a serious branch of mathematics, and if you want your mathematics to be serious you don't start by talking about people setting up competitions or exercise sessions. Today logic has won its battle for recognition, and [we] can afford to make intuitiveness one of [our] chief aims.'

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