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Commutativity of Quantifiers in Varying-Domain Kripke Models

Abstract. A possible-worlds semantics is defined that validates the main axioms of Kripke's original system for first-order modal logic over varying-domain structures. The novelty of this semantics is that it does not validate the commutative quantification schema $\forall x \forall y \varphi \to \forall y \forall x \varphi$, as we show by constructing a counter-model.

Keywords: possible-worlds semantics, commutative quantification, premodel, model, Kripkean model.

Introduction and Overview

Kripke's model theory for first-order modal logic [3] assigns to each world w a set Dw thought of as the domain of individuals that exist in w. The quantifier $\forall x$ is interpreted at a world as meaning "for all existing x". This semantics does not validate the Universal Instantiation schema

UI $\forall x\varphi \to \varphi(y/x)$, where y is free for x in φ,[1]

because the value of variable y may not exist in a particular world. It does however validate the variant

UI° $\forall y(\forall x\varphi \to \varphi(y/x))$, where y is free for x in φ,

along with the schemata

UD $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$,

VQ $\varphi \to \forall x\varphi$, where x is not free in φ,

of Universal Distribution, and Vacuous Quantification, as well as being sound for the Universal Generalisation rule

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[1] $\varphi(\tau/x)$ is the formula obtained by uniform substitution of term τ in place of free x in φ; the side condition is the usual proviso that no variable of τ becomes bound in $\varphi(\tau/x)$ as a result.

UG from φ infer $\forall x\varphi$.

In addition this semantics validates the schema

CQ $\forall x\forall y\varphi \to \forall y\forall x\varphi$

of Commutative Quantification, which was shown by Fine [1] not to be derivable from UI°, UD and VQ by using UG and valid Boolean reasoning. This raises the question of whether there is some plausible, "possible-worlds style", structural model theory for systems that have the axioms UI°, UD and VQ, but perhaps not CQ.[2]

In this paper such a semantics is presented, and a model constructed that falsifies CQ while validating the other three quantificational axioms, along with the axioms for any specified normal propositional modal logic. The approach has been used previously in [5] and [2] to give a complete semantics for the quantified relevant logic RQ and for a range of first-order modal logics that are incomplete for their standard possible-worlds models.

There are two basic ideas involved. The first, already long exploited in propositional modal logic, is that not every set of worlds need count as a proposition. Instead we take a collection *Prop* of sets of worlds, the *admissible propositions*, that forms a Boolean set algebra closed also under the operation that interprets the modality $\Box$. The "truth value" of any formula must then be a member of *Prop*.

The second notion has long been exploited in algebraic logic: the universal quantifier $\forall x$ is interpreted as a *greatest lower bound* in the lattice of propositions, this being the natural interpretation of arbitrary *conjunctions*. To illustrate this, suppose we have the set W of worlds, and a universe U of individuals that serves as the range of the quantifier $\forall x$. If φ is a formula in which x is only the free variable, let $\varphi(a)$ be the result of replacing free x in φ by the individual a, viewed as a constant. Let $|\forall x\varphi|$ and $|\varphi(a)|$ be the sets of worlds (subsets of W) at which these sentences are true, respectively. Intuitively, $\forall x\varphi$ is semantically equivalent to the conjunction of the $\varphi(a)$'s for all $a \in U$. So

$$|\forall x\varphi| = \bigcap_{a\in U} |\varphi(a)|,$$

where $\bigcap$ is set-theoretic intersection. This makes $|\forall x\varphi|$ the greatest lower bound of the $|\varphi(a)|$'s in the lattice of *all* subsets of W, i.e. the largest/weakest

[2] The axiomatisation of [3] took as axioms the *closures* of all instances of UI°, UD, VQ, tautologies and appropriate modal schemata, with detachment for material implication as the only inference rule. UG and Necessitation (from φ infer $\Box\varphi$) are then derivable rules. Here a closure of φ is any sentence obtained by prefixing universal quantifiers and copies of $\Box$ to φ in any order.

proposition that implies all of the propositions $|\varphi(a)|$. But if we are constrained to use the set *Prop* of *admissible* propositions, which may not be the full powerset $\wp W$ of W, then instead we should take

$$|\forall x\varphi| = \bigsqcap_{a\in U} |\varphi(a)|,$$

where $\bigsqcap$ is the greatest lower bound operation in the ordered set $(\mathit{Prop}, \subseteq)$. The definition of "model" should require that $\bigsqcap_{a\in U} |\varphi(a)|$ always exists in *Prop*. It will be the weakest *admissible* proposition that implies all of the $|\varphi(a)|$'s. *But it may not be equal to* $\bigcap_{a\in U} |\varphi(a)|$!

This interpretation, as developed in [2], has the quantifiers ranging over a fixed domain of possible individuals. But here we have the varying domains $Dw \subseteq U$ of existing individuals, with $\forall x\varphi$ being equivalent to the conjunction of the assertions "if a exists then $\varphi(a)$" for all $a \in U$. To formalise this, let $Ea = \{w \in W : a \in Dw\}$, so that Ea represents the proposition "a exists". Then we want

$$|\forall x\varphi| = \bigsqcap_{a\in U} Ea \Rightarrow |\varphi(a)|, \tag{0.1}$$

where $\Rightarrow$ is the Boolean set implication operation: $X \Rightarrow Y = (W \setminus X) \cup Y$. When $\bigsqcap = \bigcap$, equation (0.1) reproduces the Kripkean semantics of [3] for the quantifier $\forall x$.

In working with greatest lower bounds we put

$$\bigsqcap S = \bigcup\{X \in \mathit{Prop} : X \subseteq \bigcap S\},$$

so that $\bigsqcap S$ is defined for an arbitrary $S \subseteq \wp W$. When $S \subseteq \mathit{Prop}$ and $\bigsqcap S \in \mathit{Prop}$, then $\bigsqcap S$ is indeed the greatest lower bound of S in *Prop*. Also, if $\bigcap S \in \mathit{Prop}$, then $\bigsqcap S = \bigcap S$. But by making $\bigsqcap$ a totally defined operation we ensure that $|\forall x\varphi|$ is always defined, regardless of whether it is admissible. We will see that admissibility of $|\forall x\varphi|$ is not required for the validity of a number of principles, including UI°, UD and UG, but is required for VQ.

We will show that if all of the Ea's are admissible (i.e. $Ea \in \mathit{Prop}$), then the definition (0.1) of $|\forall x\varphi|$ validates CQ. The same conclusion holds if U is finite, or if the Boolean algebra *Prop* is atomic, hence if *Prop* is finite, and hence if W is finite. Moreover, validity of CQ follows if equality is definable in the model in the sense that there is a formula "$x \approx y$" such that

$$|a \approx b| = \begin{cases} W, & \text{if } a = b, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus the construction of a falsifying model for CQ is not a simple matter.

In Sections 1–3 we define *model structures, premodels* (in which $|\forall x\varphi|$ need not be admissible) and *models* (in which it is), and prove several soundness results. Section 4 gives sufficient criteria for validity of CQ, and Section 5 constructs its falsifying model. The final Section 6 briefly states completeness results for various logics relative to the given semantics, and points out some interesting relationships between CQ and the Barcan formula.

1. Model Structures

A *model structure* is a system $\mathcal{S} = (W, R, \mathit{Prop}, U, D)$ such that

- W is a set, and R is a binary relation on W;
- Prop is a Boolean subalgebra of the powerset algebra $\wp W$;
- Prop is closed under the operation $[R]$ defined by

$$[R]X = \{w \in W : \forall v \in W(wRv \text{ implies } v \in X)\};$$

- U is a set, and D is a function assigning to each $w \in W$ a subset $Dw \subseteq U$.

Members of Prop are called the *admissible* sets of $\mathcal{S}$. For each $a \in U$ we define $Ea = \{w \in W : a \in Dw\}$. Sets of the form Ea may be referred to as "existence sets". They are not required to be admissible.

Using Prop we define, for each $X \subseteq W$,

$$X{\downarrow} = \bigcup\{Y \in \mathit{Prop} : Y \subseteq X\},$$
$$X{\uparrow} = \bigcap\{Y \in \mathit{Prop} : X \subseteq Y\},$$

giving $X{\downarrow} \subseteq X \subseteq X{\uparrow}$. The sets $X{\downarrow}$ and $X{\uparrow}$ need not belong to Prop, but if they do, then $X{\downarrow}$ is the largest admissible subset of X, and $X{\uparrow}$ the smallest admissible superset. So if $X \in \mathit{Prop}$, then $X{\downarrow} = X{\uparrow} = X$. Operations $\bigsqcap$ and $\bigsqcup$ on $\wp\wp W$ are defined by putting, for all $S \subseteq \wp W$,

$$\bigsqcap S = (\bigcap S){\downarrow}, \qquad \bigsqcup S = (\bigcup S){\uparrow}.$$

Then any *admissible* X has $X \subseteq \bigsqcap S$ iff $X \subseteq \bigcap S$. If $S \subseteq \mathit{Prop}$ and $\bigsqcap S \in \mathit{Prop}$, then $\bigsqcap S$ is the *greatest lower bound* of S in the partially-ordered set $(\mathit{Prop}, \subseteq)$, i.e. the largest admissible set included in every member of S. Dual statements hold concerning the role of $\bigsqcup S$ as the *least upper bound* of $S \subseteq \mathit{Prop}$.

It is quite possible that $\bigsqcap S$ is admissible while $\bigcap S$ is not. However, if $\bigcap S \in \mathit{Prop}$ then $\bigsqcap S = \bigcap S$.

We now record some useful facts about $\sqcap$, some of which involve the Boolean set "implication" operation $\Rightarrow$, defined by $X \Rightarrow Y = (W \setminus X) \cup Y$. Its main property is that $Z \subseteq X \Rightarrow Y$ iff $Z \cap X \subseteq Y$.

In the following Lemma, X_i, Y_i, X_{ij} are subsets of W, S is a subset of $\wp W$, and $\bigsqcap_{i \in I} X_i$ is $\bigsqcap\{X_i : i \in I\}$.

LEMMA 1.1.

(1) *If* $X_i \subseteq Y_i$ *for all* $i \in I$, *then* $\bigsqcap_{i \in I} X_i \subseteq \bigsqcap_{i \in I} Y_i$.

(2) $\bigsqcap_{i \in I} \bigsqcap_{j \in J} X_{ij} = \bigsqcap_{j \in J} \bigsqcap_{i \in I} X_{ij}$, *provided that both sides of this equation belong to Prop.*

(3) *If* $X \in \mathit{Prop}$, *then* $X \Rightarrow \bigsqcap S = \bigsqcap_{Y \in S}(X \Rightarrow Y)$.

(4) *If* $\{Y_i : i \in I\} \subseteq \mathit{Prop}$, *then* $\bigsqcap_{i \in I}(X_i \Rightarrow Y_i) = \bigsqcap_{i \in I}(X_i{\uparrow} \Rightarrow Y_i)$.

PROOF.

(1) $\bigcap_{i \in I} X_i \subseteq \bigcap_{i \in I} Y_i$, and the operation $\downarrow$ is $\subseteq$-monotonic.

(2) (N.B: the X_{ij}'s need not be admissible here.)
Let $X = \bigsqcap_{i \in I} \bigsqcap_{j \in J} X_{ij}$. Then $X \subseteq X_{ij}$ for all $(i, j) \in I \times J$. So, for a given $j_0 \in J$ we have $X \subseteq X_{ij_0}$ for all $i \in I$, hence $X \subseteq \bigsqcap_{i \in I} X_{ij_0}$ because $X \in \mathit{Prop}$. Since this holds for every $j_0 \in J$, $X \subseteq \bigsqcap_{j \in J} \bigsqcap_{i \in I} X_{ij}$, again as X is admissible. The converse inclusion holds by a symmetric argument.

(3) (N.B: the members of S need not be admissible.)
Since $Y \subseteq (X \Rightarrow Y)$, $\bigsqcap S \subseteq \bigsqcap_{Y \in S}(X \Rightarrow Y)$ by (1). Also, as $W \setminus X \subseteq (X \Rightarrow Y)$, and $W \setminus X \in \mathit{Prop}$ because $X \in \mathit{Prop}$, we have $W \setminus X \subseteq \bigsqcap_{Y \in S}(X \Rightarrow Y)$. Altogether then,

$$X \Rightarrow \bigsqcap S = W \setminus X \cup \bigsqcap S \subseteq \bigsqcap_{Y \in S}(X \Rightarrow Y).$$

For the converse inclusion it is enough to show that any admissible subset of $\bigcap_{Y \in S}(X \Rightarrow Y)$ is a subset of $X \Rightarrow \bigsqcap S$. But if $Z \in \mathit{Prop}$ has $Z \subseteq \bigcap_{Y \in S}(X \Rightarrow Y)$, then for all $Y \in S$, $Z \subseteq (X \Rightarrow Y)$, so $Z \cap X \subseteq Y$. Hence $Z \cap X \subseteq \bigsqcap S$ as $Z \cap X \in \mathit{Prop}$. Therefore $Z \subseteq X \Rightarrow \bigsqcap S$.

(4) (N.B: the X_i need not be admissible.)
First, since $X_i \subseteq X_i{\uparrow}$, we have $(X_i{\uparrow} \Rightarrow Y_i) \subseteq (X_i \Rightarrow Y_i)$, for all $i \in I$. Hence $\bigsqcap_{i \in I}(X_i{\uparrow} \Rightarrow Y_i) \subseteq \bigsqcap_{i \in I}(X_i \Rightarrow Y_i)$ by (1).
For the converse inclusion, let Z be any admissible subset of $\bigsqcap_{i \in I}(X_i \Rightarrow Y_i)$. Then for all $i \in I$, $Z \subseteq X_i \Rightarrow Y_i$, hence $X_i \subseteq Z \Rightarrow Y_i$. But $Z \Rightarrow Y_i$ is admissible (by admissibility of Z and Y_i), and so $X_i{\uparrow} \subseteq Z \Rightarrow Y_i$, implying that $Z \subseteq X_i{\uparrow} \Rightarrow Y_i$. Hence $Z \subseteq \bigsqcap_{i \in I}(X_i{\uparrow} \Rightarrow Y_i)$.

■

2. Premodels and Models

Let $\mathcal{L}$ be a set of relation and function symbols and individual constants. A *premodel* $\mathcal{M} = (\mathcal{S}, |\cdot|^{\mathcal{M}})$ for $\mathcal{L}$, based on a model structure $\mathcal{S}$, is given by an interpretation function $|\cdot|^{\mathcal{M}}$ on $\mathcal{L}$ that assigns

- to each n-ary relation symbol P a function $|P|^{\mathcal{M}} : U^n \to \mathit{Prop}$,
- to each individual constant c an element $|\mathsf{c}|^{\mathcal{M}} \in U$, and
- to each n-ary function symbol F a function $|F|^{\mathcal{M}} : U^n \to U$.

We deal with first-order modal $\mathcal{L}$-formulas generated using a set $\{x_n : n < \omega\}$ of first-order variables, but often regard this set simply as ω by identifying x_n with n. A variable-assignment is then a map $f \in {}^{\omega}U$. Any $\mathcal{L}$-term τ can be interpreted via f as an element $\tau^{\mathcal{M}} f \in U$ in the usual way. We use the letters $x, y, z, \cdots$ for variables, and define $f[a/x]$ to be the function that "updates" f by assigning the value $a \in U$ to x and otherwise acting as f.

A premodel gives an interpretation $|\varphi|^{\mathcal{M}} : {}^{\omega}U \to \wp W$ to each $\mathcal{L}$-formula. For each assignment f, $|\varphi|^{\mathcal{M}} f$ is thought of as the set of worlds at which φ is true under f. This is defined by induction on the formation of φ:

- $|P\tau_1 \cdots \tau_n|^{\mathcal{M}} f = |P|^{\mathcal{M}}(\tau_1^{\mathcal{M}} f, \ldots, \tau_n^{\mathcal{M}} f) \in \mathit{Prop}$,
- $|\top|^{\mathcal{M}} f = W$ and $|\bot|^{\mathcal{M}} f = \emptyset$,
- $|\neg\varphi|^{\mathcal{M}} f = W \setminus |\varphi|^{\mathcal{M}} f$, and $|\varphi \wedge \psi|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f \cap |\psi|^{\mathcal{M}} f$,
- $|\Box\varphi|^{\mathcal{M}} f = [R]|\varphi|^{\mathcal{M}} f$,
- $|\forall x\varphi|^{\mathcal{M}} f = \bigsqcap_{a \in U} \big(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big)$.

Thus if $X \in \mathit{Prop}$, then $X \subseteq |\forall x\varphi|^{\mathcal{M}} f$ iff $X \subseteq Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]$ for all $a \in U$. We have

$$\begin{aligned} |\forall x\varphi|^{\mathcal{M}} f &= \Big[\bigcap_{a \in U} Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\Big]{\downarrow}. \\ &= \Big[\bigcap_{a \in U} (W \setminus Ea) \cup |\varphi|^{\mathcal{M}} f[a/x]\Big]{\downarrow}. \end{aligned}$$

Identifying $\exists$ with $\neg\forall\neg$ gives

$$\begin{aligned} |\exists x\varphi|^{\mathcal{M}} f &= \bigsqcup_{a \in U} Ea \cap |\varphi|^{\mathcal{M}} f[a/x] \\ &= \Big[\bigcup_{a \in U} Ea \cap |\varphi|^{\mathcal{M}} f[a/x]\Big]{\uparrow}. \end{aligned}$$

REMARK 2.1. The semantics of [3] interprets an n-ary relation symbol P as a function

$$\Phi(P,\cdot) : W \to \wp(U^n)$$

assigning to each world w an n-ary relation $\Phi(P,w) \subseteq U^n$. From such a Φ we can define $|P| : U^n \to \wp W$ by

$$w \in |P|(a_1,\ldots,a_n) \quad \text{iff} \quad \langle a_1,\ldots,a_n\rangle \in \Phi(P,w).$$

Alternatively, this can be viewed as a definition of Φ, given $|P|$, so the two methods are equivalent. We find that use of the "proposition-valued" functions $|\varphi|$ provides a convenient way of handling the restriction to admissible propositions.

It is worth emphasising that this kind of model theory allows relations and properties to hold of non-existent objects (e.g. Pegasus has wings). Thus it is not required that $\Phi(P,w) \subseteq (Dw)^n$; equivalently, it is not required that

$$|P|(a_1,\ldots,a_n) \subseteq Ea_1 \cap \cdots \cap Ea_n. \qquad \blacksquare$$

Writing $\mathcal{M}, w, f \models \varphi$ to mean that $w \in |\varphi|^{\mathcal{M}} f$, we get the following clauses for this satisfaction relation $\models$, with all except that for $\forall$ being familiar:

- $\mathcal{M}, w, f \models P\tau_1 \cdots \tau_n$ iff $w \in |P\tau_1 \ldots \tau_n|^{\mathcal{M}} f$,
- $\mathcal{M}, w, f \models \top$ and $\mathcal{M}, w, f \not\models \bot$,
- $\mathcal{M}, w, f \models \neg\varphi$ iff $\mathcal{M}, w, f \not\models \varphi$,
- $\mathcal{M}, w, f \models \varphi \wedge \psi$ iff $\mathcal{M}, w, f \models \varphi$ and $\mathcal{M}, w, f \models \psi$,
- $\mathcal{M}, w, f \models \Box\varphi$ iff for all $v \in W(wRv$ implies $\mathcal{M}, v, f \models \varphi)$.
- $\mathcal{M}, w, f \models \forall x\varphi$ iff there is an $X \in \mathit{Prop}$ such that $w \in X$ and $X \subseteq \bigcap_{a\in U}\big(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big)$.

A formula φ is *valid in* premodel $\mathcal{M}$, written $\mathcal{M} \models \varphi$, if $|\varphi|^{\mathcal{M}} f = W$ for all f, i.e. if $\mathcal{M}, w, f \models \varphi$ for all $w \in W$ and $f \in {}^{\omega}U$.

As with standard semantics, satisfaction of a formula depends only on value-assignment to *free* variables:

LEMMA 2.2. *In any premodel $\mathcal{M}$, for any formula φ, if assignments $f, g \in {}^{\omega}U$ agree on all free variables of φ, then $|\varphi|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} g$.*

PROOF. The only departure from the standard proof is the inductive case that φ is $\forall x\psi$. Then if f and g agree on all free variables of φ, then for each $a \in U$, $f[a/x]$ and $g[a/x]$ agree on all free variables of ψ, so $|\psi|^{\mathcal{M}} f[a/x] = |\psi|^{\mathcal{M}} g[a/x]$ by induction hypothesis. Hence

$$|\varphi|^{\mathcal{M}} f = \bigsqcap_{a\in U} \big(Ea \Rightarrow |\psi|^{\mathcal{M}} f[a/x]\big) = \bigsqcap_{a\in U} \big(Ea \Rightarrow |\psi|^{\mathcal{M}} g[a/x]\big) = |\varphi|^{\mathcal{M}} g.$$

■

This result can be used to establish the usual relationship between syntactic substitution of terms for variables and updating of evaluations:

LEMMA 2.3. *Let φ be any formula, and τ a term that is free for x in φ. Then in any premodel $\mathcal{M}$, for any $f \in {}^{\omega}U$, $|\varphi(\tau/x)|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x]$.*

PROOF. Again the only nonstandard case is when φ is of the form $\forall y\psi$. First, when x is not free in φ then f and $f[\tau^{\mathcal{M}} f/x]$ agree on all free variables of φ, and $\varphi(\tau/x)$ is just φ, so the result is given by Lemma 2.2.

Otherwise, x is free in φ, so $x \neq y$ and $\varphi(\tau/x) = \forall y(\psi(\tau/x))$ with τ free for x in ψ, so y does not occur in τ. Then

$$|\varphi(\tau/x)|^{\mathcal{M}} f = \bigsqcap_{a\in U} Ea \Rightarrow |\psi(\tau/x)|^{\mathcal{M}} f[a/y], \quad \text{and}$$
$$|\varphi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x] = \bigsqcap_{a\in U} Ea \Rightarrow |\psi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x][a/y].$$

But for any $a \in U$, the induction hypothesis on ψ gives

$$|\psi(\tau/x)|^{\mathcal{M}} f[a/y] = |\psi|^{\mathcal{M}} f[a/y][\tau^{\mathcal{M}} f[a/y]/x],$$

and $\tau^{\mathcal{M}} f[a/y] = \tau^{\mathcal{M}} f$ because y is not in τ, while

$$f[a/y][\tau^{\mathcal{M}} f/x] = f[\tau^{\mathcal{M}} f/x][a/y]$$

as $y \neq x$. So altogether

$$|\psi(\tau/x)|^{\mathcal{M}} f[a/y] = |\psi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x][a/y],$$

and hence $|\varphi(\tau/x)|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x]$ in this case. ■

COROLLARY 2.4. *If $\mathcal{M} \models \varphi$, then $\mathcal{M} \models \varphi(\tau/x)$ whenever τ is free for x in φ.*

PROOF. If $\mathcal{M} \models \varphi$, then for any f, $|\varphi(\tau/x)|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x] = W$. ■

We will say that a formula φ is *admissible in* $\mathcal{M}$ if the function $|\varphi|^{\mathcal{M}}$ has the form ${}^{\omega}U \to \mathit{Prop}$, i.e. $|\varphi|^{\mathcal{M}} f \in \mathit{Prop}$ for all $f \in {}^{\omega}U$. Every atomic formula $P\tau_1 \cdots \tau_n$ is admissible. Given the closure properties of *Prop* it is evident that the set of admissible formulas is closed under the Boolean connectives and $\Box$. In particular, every *quantifier-free* formula is admissible.

A *model* for $\mathcal{L}$ is a premodel in which every $\mathcal{L}$-formula is admissible.

LEMMA 2.5. *In any model* $\mathcal{M}$, $|\forall x\varphi|^{\mathcal{M}} f = \bigsqcap_{a\in U} \big(Ea{\uparrow} \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big)$.

PROOF. As φ is admissible in $\mathcal{M}$, $\{|\varphi|^{\mathcal{M}} f[a/x] : a \in U\} \subseteq \mathit{Prop}$. Hence by Lemma 1.1(4),

$$\bigsqcap_{a\in U} \big(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big) = \bigsqcap_{a\in U} \big(Ea{\uparrow} \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big).$$

■

3. Soundness and $\mathcal{M}$-Equivalence

We now fix a a premodel $\mathcal{M}$, and examine the validity of various principles in it, identifying some whose validity requires $\mathcal{M}$ to be a model. From now on, the $\mathcal{M}$-superscript will often be dropped from the notation $|\varphi|^{\mathcal{M}} f$.

PROPOSITION 3.1. *The schemata* UI° *and* UD *are valid in* $\mathcal{M}$, *and the rule* UG *is sound for validity in* $\mathcal{M}$.

PROOF. UG is dealt with first, as it is simplest. If $\mathcal{M} \models \varphi$, then for any f and a, $Ea \Rightarrow |\varphi| f[a/x] = Ea \Rightarrow W = W$, so $|\forall x\varphi| f = \bigsqcap\{W\} = W$. Hence $\mathcal{M} \models \forall x\varphi$.

For UD, suppose that $\mathcal{M}, w, f \models \forall x(\varphi \to \psi)$ and $\mathcal{M}, w, f \models \forall x\varphi$. Then there exist $X, Y \in \mathit{Prop}$ such that

$$w \in X \subseteq \bigcap_{a\in U} Ea \Rightarrow |\varphi \to \psi| f[a/x], \quad \text{and}$$
$$w \in Y \subseteq \bigcap_{a\in U} Ea \Rightarrow |\varphi| f[a/x].$$

Then $w \in X \cap Y \in \mathit{Prop}$, and for all a,

$$X \cap Y \cap Ea \subseteq |\varphi \to \psi| f[a/x] \cap |\varphi| f[a/x] \subseteq |\psi| f[a/x],$$

hence $X \cap Y \subseteq Ea \Rightarrow |\psi| f[a/x]$. This shows $\mathcal{M}, w, f \models \forall x\psi$.

For UI°, let y be free for x in φ. It suffices to show that for any f and a,

$$Ea \subseteq |\forall x\varphi \to \varphi(y/x)| f[a/y]. \tag{3.1}$$

For then $Ea \Rightarrow |\forall x\varphi \to \varphi(y/x)| f[a/y] = W$ for all $a \in U$, so

$$|\forall y(\forall x\varphi \to \varphi(y/x))| f = \bigsqcap\{W\} = W,$$

and hence $\mathcal{M} \models \forall y(\forall x\varphi \to \varphi(y/x))$.

To prove (3.1), let $w \in Ea$. Then if $w \in |\forall x\varphi| f[a/y]$, there exists $X \in \mathit{Prop}$ with

$$w \in X \subseteq \bigcap_{b \in U} Eb \Rightarrow |\varphi| f[a/y][b/x].$$

In particular, when $b = a$, since $w \in Ea$ we get $w \in |\varphi| f[a/y][a/x]$. But by Lemma 2.3, $|\varphi| f[a/y][a/x] = |\varphi(y/x)| f[a/y]$ because $y^{\mathcal{M}} f[a/y] = a$. Thus

$$w \in |\forall x\varphi| f[a/y] \Rightarrow |\varphi(y/x)| f[a/y] = |\forall x\varphi \to \varphi(y/x)| f[a/y].$$

■

Next we consider the validity of VQ:

Proposition 3.2. *Suppose that x has no free occurrence in φ. If φ is admissible in $\mathcal{M}$, then $\mathcal{M} \models \varphi \to \forall x\varphi$.*

Proof. For any $f \in {}^{\omega}U$ and $a \in U$, the assignments f and $f[a/x]$ agree on all free variables of φ, so by Lemma 2.2,

$$|\varphi| f = |\varphi| f[a/x] \subseteq Ea \Rightarrow |\varphi| f[a/x].$$

But $|\varphi| f \in \mathit{Prop}$ by $\mathcal{M}$-admissibility of φ, so

$$|\varphi| f \subseteq \bigsqcap_{a \in U} \big(Ea \Rightarrow |\varphi| f[a/x]\big) = |\forall x\varphi| f.$$

Hence $|\varphi| f \Rightarrow |\forall x\varphi| f = W$ for all f. ■

Corollary 3.3. *Every model validates VQ.*

Proof. In a model, every φ is admissible. ■

We say that formulas φ and ψ are *$\mathcal{M}$-equivalent* if $|\varphi|^{\mathcal{M}} = |\psi|^{\mathcal{M}}$. The following properties of this equivalence relation are left to the reader to check.

Proposition 3.4. *In any premodel $\mathcal{M}$:*

(1) *φ is $\mathcal{M}$-equivalent to ψ iff $\mathcal{M} \models \varphi \leftrightarrow \psi$.*

(2) *If φ is tautologically equivalent to ψ (i.e. $\varphi \leftrightarrow \psi$ is a tautology), then φ and ψ are $\mathcal{M}$-equivalent.*

(3) *$\mathcal{M}$-equivalence is a congruence on the algebra of $\mathcal{L}$-formulas, i.e. if the pair φ, ψ are $\mathcal{M}$-equivalent, then so are the pairs $\neg\varphi, \neg\psi$ and $\varphi \wedge \theta, \psi \wedge \theta$ and $\Box\varphi, \Box\psi$ and $\forall x\varphi, \forall x\psi$ and $\exists x\varphi, \exists x\psi$ etc.*

(4) *If ψ is obtained from φ by replacing some subformula by an $\mathcal{M}$-equivalent formula, then ψ is $\mathcal{M}$-equivalent to φ.* ■

The next result will be used in a model construction in Section 5.

Proposition 3.5. *In any premodel $\mathcal{M}$:*

(1) *$\exists x(\varphi \vee \psi)$ and $\exists x\varphi \vee \exists x\psi$ are $\mathcal{M}$-equivalent.*

(2) *$\exists x(\varphi \wedge \psi)$ and $\varphi \wedge \exists x\psi$ are $\mathcal{M}$-equivalent if φ is admissible in $\mathcal{M}$ and has no free occurrences of x.*

Proof. (1) It is enough to show that the formula

$$\exists x(\varphi \vee \psi) \leftrightarrow \exists x\varphi \vee \exists x\psi$$

is valid in $\mathcal{M}$. But, as the reader can check, this formula is derivable from tautologies and instances of UD using the rule UG and valid Boolean reasoning. Hence it is valid in $\mathcal{M}$ by Proposition 3.1.

(2) If φ is $\mathcal{M}$-admissible and without free x, then $\neg\varphi$ is $\mathcal{M}$-admissible and without free x, so by Lemma 3.2 the formulas $\varphi \to \forall x\varphi$ and $\neg\varphi \to \forall x\neg\varphi$ are valid in $\mathcal{M}$. But from these two, using tautologies, UD, UG and valid Boolean reasoning we can derive

$$\exists x(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \exists x\psi,$$

which is therefore valid in $\mathcal{M}$.

■

4. Validating CQ

We now give some conditions under which the formulas $\forall x\forall y\varphi$ and $\forall y\forall x\varphi$ are $\mathcal{M}$-equivalent in a model. Of course we can assume $x \neq y$ here, for otherwise there is no work to do. Then assignments $f[a/x][b/y]$ and $f[b/y][a/x]$ are identical, and may be written $f[a/x, b/y]$ or $f[b/y, a/x]$.

Lemma 4.1. *In a premodel $\mathcal{M}$, let $f \in {}^\omega U$ and let $\mathcal{B}$ be any Boolean subalgebra of Prop that contains $|\varphi|^{\mathcal{M}}f[a/x, b/y]$, $|\forall x\varphi|f[b/y]$, and $|\forall y\varphi|f[a/x]$ for all $a, b \in U$. Then exactly the same atoms of $\mathcal{B}$ are included in the sets $|\forall x\forall y\varphi|^{\mathcal{M}}f$ and $|\forall y\forall x\varphi|^{\mathcal{M}}f$.*

Proof. Let X be an atom of $\mathcal{B}$ with $X \not\subseteq |\forall x\forall y\varphi|f$. Then as $X \in \mathit{Prop}$, there exists $a_0 \in U$ such that

$$X \not\subseteq Ea_0 \Rightarrow |\forall y\varphi|f[a_0/x]. \tag{4.1}$$

Hence $X \not\subseteq |\forall y\varphi|f[a_0/x]$, so again as $X \in \mathit{Prop}$ there exists $b_0 \in U$ such that

$$X \not\subseteq Eb_0 \Rightarrow |\varphi|f[a_0/x, b_0/y]. \tag{4.2}$$

Hence $X \not\subseteq |\varphi|f[a_0/x, b_0/y]$. But X is a $\mathcal{B}$-atom and $|\varphi|f[a_0/x, b_0/y] \in \mathcal{B}$ as given, so X must be *disjoint* from $|\varphi|f[a_0/x, b_0/y] = |\varphi|f[b_0/y, a_0/x]$. Since $X \cap Ea_0 \neq \emptyset$ by (4.1), this implies

$$X \not\subseteq Ea_0 \Rightarrow |\varphi|f[b_0/y, a_0/x].$$

Hence

$$X \not\subseteq \bigsqcap_{a \in U} Ea \Rightarrow |\varphi|f[b_0/y, a/x] = |\forall x\varphi|f[b_0/y].$$

Again the atomicity of X then makes X disjoint from $|\forall x\varphi|f[b_0/y] \in \mathcal{B}$. Since $X \cap Eb_0 \neq \emptyset$ by (4.2),

$$X \not\subseteq Eb_0 \Rightarrow |\forall x\varphi|f[b_0/y].$$

Hence

$$X \not\subseteq \bigsqcap_{b \in U} Eb \Rightarrow |\varphi|f[b/y] = |\forall y\forall x\varphi|f.$$

Conversely, interchanging x and y in this argument shows that if $X \not\subseteq |\forall y\forall x\varphi|f$, then $X \not\subseteq |\forall x\forall y\varphi|f$. ■

Proposition 4.2. *A **model** validates CQ if any of the following hold:*

(1) *Prop is an atomic Boolean algebra.*

(2) *Prop is finite.*

(3) *The universe U is finite.*

Proof. (1) Put $\mathcal{B} = \mathit{Prop}$. For any f, all sets $|\varphi|f[a/x, b/y]$, $|\forall x\varphi|f[b/y]$, $|\forall y\varphi|f[a/x]$ are in $\mathcal{B}$ by admissibility. But likewise the sets $|\forall x\forall y\varphi|f$ and $|\forall y\forall x\varphi|f$ are in $\mathcal{B}$, and include the same atoms of $\mathcal{B}$ by Lemma 4.1, hence as $\mathcal{B}$ is atomic this makes $|\forall x\forall y\varphi|f = |\forall y\forall x\varphi|f$.

(2) By (1), as any finite Boolean algebra is atomic.

(3) If U is finite, then for any f,

$$\{|\forall x \forall y \varphi| f, |\forall y \forall x \varphi| f\}$$
$$\cup \{|\varphi| f[a/x, b/y], |\forall x \varphi| f[b/y], |\forall y \varphi| f[a/x] : a, b \in U\}$$

is a finite subset of *Prop*, so it generates a Boolean subalgebra $\mathcal{B}$ of *Prop* that is finite, hence atomic. The proof that $|\forall x \forall y \varphi| f = |\forall y \forall x \varphi| f$ in $\mathcal{B}$ then follows by the argument of (1).

■

Next we consider consequences of admissibility of the "existence sets" Ea and $Ea{\uparrow}$.

PROPOSITION 4.3. *If a model has $Ea{\uparrow} \in Prop$ for all $a \in U$, then it validates CQ.*

PROOF. Since we are working in a model, we can use Lemma 2.5 to replace Ea by $Ea{\uparrow}$ in the definition of $|\forall x \varphi|$. Thus

$$|\forall x \forall y \varphi| f$$
$$= \textstyle\bigsqcap_{a \in U} \big(Ea{\uparrow} \Rightarrow \bigsqcap_{b \in U}(Eb{\uparrow} \Rightarrow |\varphi| f[a/x, b/y])\big)$$
$$= \textstyle\bigsqcap_{a \in U} \bigsqcap_{b \in U} \big(Ea{\uparrow} \Rightarrow (Eb{\uparrow} \Rightarrow |\varphi| f[a/x, b/y])\big) \quad \text{by Lemma 1.1(3) as } Ea{\uparrow} \in Prop,$$
$$= \textstyle\bigsqcap_{a \in U} \bigsqcap_{b \in U} \big(Ea{\uparrow} \cap Eb{\uparrow} \Rightarrow |\varphi| f[a/x, b/y]\big) \quad \text{by set theory.}$$

Similarly, $|\forall y \forall x \varphi| f = \bigsqcap_{b \in U} \bigsqcap_{a \in U} \big(Eb{\uparrow} \cap Ea{\uparrow} \Rightarrow |\varphi| f[b/y, a/x]\big)$.

But $Eb{\uparrow} \cap Ea{\uparrow} \Rightarrow |\varphi| f[b/y, a/x] = Ea{\uparrow} \cap Eb{\uparrow} \Rightarrow |\varphi| f[a/x, b/y]$, so the $\bigsqcap$-commutation result of Lemma 1.1(2) applies to give $|\forall x \forall y \varphi| f = |\forall y \forall x \varphi| f$.

■

COROLLARY 4.4. *If a model has $Ea \in Prop$ for all $a \in U$, then it validates CQ.*

PROOF. If $Ea \in Prop$, then $Ea = Ea{\uparrow}$. ■

We say that *equality is definable in* $\mathcal{M}$ if for any distinct variables x, y, there is an $\mathcal{L}$-formula "$x \approx y$" such that

$$|x \approx y|^{\mathcal{M}} f = \begin{cases} W, & \text{if } fx = fy, \\ \emptyset, & \text{otherwise.} \end{cases}$$

COROLLARY 4.5. *If equality is definable in a model, then it validates CQ.*

PROOF. Let $a \in U$ be arbitrary, and suppose $f \in {}^\omega I$ satisfies $fx = a$. Then $|\exists y(x \approx y)|f = [\bigcup_{b \in U} Eb \cap |x \approx y|f[b/y]]\!\uparrow = Ea\!\uparrow$. Hence $Ea\!\uparrow \in \mathit{Prop}$ as every formula is admissible in $\mathcal{M}$. By Proposition 4.3, CQ is valid in $\mathcal{M}$.[3] ■

A premodel $\mathcal{M}$ will be called *Kripkean* if it always has

$$|\forall x\varphi|^{\mathcal{M}} f = \bigcap_{a \in U} \bigl(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\bigr).$$

This means that $\forall$ gets the varying-domain semantics of Kripke [3]:

$$\mathcal{M}, w, f \models \forall x\varphi \ \text{ iff } \ \text{for all } a \in Dw,\ \ \mathcal{M}, w, f[a/x] \models \varphi. \tag{4.3}$$

A Kripkean *model* has

$$\Bigl[\bigcap_{a \in U} Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\Bigr] \in \mathit{Prop}$$

by admissibility of formula $\forall x\varphi$, and conversely this last condition implies that a model is Kripkean.

PROPOSITION 4.6. *Every Kripkean premodel validates CQ.*

PROOF. This is straightforward, essentially because the quantifiers *for all existing* ... commute in the metalanguage. A more formal proof can be given by repeating the proof of Proposition 4.3 with $\bigcap$ in place of $\bigsqcap$ (and Ea in place of $Ea\!\uparrow$). Instead of parts (2) and (3) of Lemma 1.1, the results

$$\bigcap_{i \in I} \bigcap_{j \in J} X_{ij} = \bigcap_{j \in J} \bigcap_{i \in I} X_{ij}, \qquad X \Rightarrow \bigcap S = \bigcap_{Y \in S} (X \Rightarrow Y),$$

are used. These are laws of set theory that hold independently of any admissibility constraints. ■

5. A Countermodel to CQ

This section exhibits a model that falsifies an instance of CQ. It is not so hard to construct a premodel that does this, but we wish to ensure that every formula is admissible in $\mathcal{M}$, so that it validates VQ as well as UI° and

[3] For this proof to work it suffices in fact that $|x \approx y|^{\mathcal{M}} f \supseteq Efx$ when $fx = fy$, and $|x \approx y|^{\mathcal{M}} f = \emptyset$ otherwise.

UD. From what has been shown in the last Section, our model must have infinite sets for U and *Prop*, and hence for W. Also *Prop* cannot be atomic, and cannot contain every Ea, or every $Ea{\uparrow}$. Moreover, the model cannot be Kripkean, or permit the definability of equality.

Let $\sim$ denote a fixed (but arbitrary) equivalence relation on $\mathbb{Q}$ (the rationals) with infinitely many equivalence classes, each of which is dense in $\mathbb{Q}$: so each interval (a, b) for $a < b$ in $\mathbb{Q}$ contains a point from each equivalence class. Such a relation is easy to construct. Let $b/{\sim}$ denote the $\sim$-equivalence class containing b.

We define a model structure $\mathcal{S} = (W, R, \mathit{Prop}, U, D)$, where

- $W = U = \mathbb{Q}$;
- either $R = \emptyset$, or $R = \{(a, a) : a \in \mathbb{Q}\}$;
- *Prop* is the Boolean subalgebra of $\wp(\mathbb{Q})$ generated by the set of all half-open intervals $[a, b) = \{x \in \mathbb{Q} : a \leq x < b\}$, where $a, b \in \mathbb{Q}$ and $a < b$;
- $Da = \{a\}$ for each $a \in \mathbb{Q}$. Hence $Ea = \{a\}$.

We have actually defined two model structures, depending on the choice of R. In the first case with $R = \emptyset$, $[R]X = W$ for all $X \subseteq W$. In the second case with R the identity relation, $[R]X = X$. Hence in both cases *Prop* is $[R]$-closed. In the first case (W, R) (and hence (W, R, Prop)) validates the smallest normal propositional modal logic containing $\Box\bot$, while in the second case it validates the smallest normal logic containing the schema $\Box\varphi \leftrightarrow \varphi$. But each normal propositional modal logic is a sublogic of one of these two [4], so is validated by one of these structures. We will make use of that fact in Section 6.

Each non-empty $X \in \mathit{Prop}$ is a finite union of intervals of the form $(-\infty, a)$, $[b, c)$, and $[d, +\infty)$. *Prop* is atomless, and $Ea{\uparrow} = Ea = \{a\} \notin \mathit{Prop}$ for all $a \in \mathbb{Q}$.

LEMMA 5.1. *Write $\mathbb{Q}/{\sim}$ for the set of all $\sim$-classes, and let $\mathcal{E} \subseteq \mathbb{Q}/{\sim}$. Then $(\bigcup\mathcal{E}){\uparrow}$ and $(\bigcup\mathcal{E}){\downarrow}$ are admissible, with*

$$(\textstyle\bigcup\mathcal{E}){\uparrow} = \begin{cases} \emptyset, & \text{if } \mathcal{E} = \emptyset, \\ \mathbb{Q}, & \text{otherwise,} \end{cases} \qquad (\textstyle\bigcup\mathcal{E}){\downarrow} = \begin{cases} \mathbb{Q}, & \text{if } \mathcal{E} = \mathbb{Q}/{\sim}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

PROOF. If $\mathcal{E} = \emptyset$ then $\bigcup\mathcal{E} = \emptyset$, and clearly $\emptyset{\uparrow} = \emptyset$. Otherwise, by density, any non-empty $X \in \mathit{Prop}$ intersects $\bigcup\mathcal{E}$, and so $(\bigcup\mathcal{E}){\uparrow} = \mathbb{Q}$. The case of $\downarrow$ is similar (or it can be derived from the $\uparrow$ case, using the equation $S{\downarrow} = \mathbb{Q} \setminus ((\mathbb{Q} \setminus S){\uparrow})$ for $S \subseteq \mathbb{Q}$). ■

Now let $\mathcal{L}$ consist of two binary relation symbols, P and $\sim$. (The two uses of $\sim$ will be distinguished by context.) We define an $\mathcal{L}$-premodel on $\mathcal{S}$ by putting, for each $a, b \in \mathbb{Q}$,

- $|{\sim}|^{\mathcal{M}}(a,b) = \begin{cases} \mathbb{Q}, & \text{if } a \sim b, \\ \emptyset, & \text{otherwise;} \end{cases}$
- $|P|^{\mathcal{M}}(a,b) = \begin{cases} \mathbb{Q}, & \text{if } a \sim b, \\ \text{some non-empty interval} & \\ [b,c) \text{ not containing } a, & \text{otherwise.} \end{cases}$

Note that *Prop* contains $|{\sim}|^{\mathcal{M}}(a,b)$ and $|P|^{\mathcal{M}}(a,b)$ for all $a, b \in \mathbb{Q}$, as required. The definition ensures that $b \in |P|^{\mathcal{M}}(a,b)$ for all b, while $a \in |P|^{\mathcal{M}}(a,b)$ iff $a \sim b$.

PROPOSITION 5.2. *$\mathcal{M}$ does not validate $\forall x \forall y Pxy \to \forall y \forall x Pyx$.*

PROOF. We show that for any $f \in {}^{\omega}U$,

$$|\forall x \forall y Pxy| f = \mathbb{Q} \quad \text{while} \quad |\forall y \forall x Pxy| f = \emptyset.$$

Now $|\forall y Pxy| f = \big[\bigcap_{b \in \mathbb{Q}} Eb \Rightarrow |P|(fx, b)\big]{\downarrow}$. But for any b,

$$Eb \Rightarrow |P|(fx,b) = \{b\} \Rightarrow |P|(fx,b) = \mathbb{Q},$$

since $b \in |P|(fx,b)$. Hence $|\forall y Pxy| f = \mathbb{Q}{\downarrow} = \mathbb{Q}$. It follows that for any f, $|\forall x \forall y Pxy| f = [\bigcap_{a \in \mathbb{Q}} Ea \Rightarrow \mathbb{Q}]{\downarrow} = \mathbb{Q}$ as well.

On the other hand, $|\forall x Pxy| f = \big[\bigcap_{a \in \mathbb{Q}} Ea \Rightarrow |P|(a, fy)\big]{\downarrow}$. But

$$Ea \Rightarrow |P|(a,fy) = \mathbb{Q} \setminus \{a\} \cup |P|(a,fy) = \begin{cases} \mathbb{Q}, & \text{if } a \sim fy, \\ \mathbb{Q} \setminus \{a\}, & \text{otherwise} \end{cases},$$

so $|\forall x Pxy| f = \big[\bigcap_{a \not\sim fy} \mathbb{Q} \setminus \{a\}\big]{\downarrow} = (fy/{\sim}){\downarrow} = \emptyset$ by Lemma 5.1.

It follows that for any f, $|\forall y \forall x Pxy| f = [\bigcap_{b \in \mathbb{Q}} \mathbb{Q} \setminus \{b\} \cup \emptyset]{\downarrow} = \emptyset{\downarrow} = \emptyset$ as well. ∎

Notice that this proof shows that $\mathcal{M}$ is *non-Kripkean*: since $\emptyset \neq fy/{\sim}$, we have

$$|\forall x Pxy| f \neq \bigcap_{a \in \mathbb{Q}} Ea \Rightarrow |P|(a, fy).$$

We now have to show that the premodel $\mathcal{M}$ is actually a *model*, i.e. $|\varphi|^{\mathcal{M}} f$ is always admissible. This is done as follows. As before, we say that formulas φ, ψ are *$\mathcal{M}$-equivalent* if $|\varphi| = |\psi|$ in this $\mathcal{M}$.

PROPOSITION 5.3. *Let φ be any formula. Then*

(1) *φ is $\mathcal{M}$-equivalent to a quantifier-free formula.*

(2) *$|\varphi|^{\mathcal{M}} f \in Prop$ for all $f \in {}^{\omega}I$.*

PROOF. We prove both parts simultaneously by induction on φ. In the proof, we write '$\mathcal{M}$-equivalent' simply as 'equivalent'. Let us say that a formula φ is *coherent* if it satisfies the two conditions of the proposition. *Any formula that is equivalent to a coherent one is itself coherent*, a fact that will be used repeatedly. To begin with, any formula is equivalent to one formed from atomic formulas by the propositional connectives and the quantifier $\exists$, so we can suppose without loss of generality that φ has this form.

If φ is atomic, we are given the coherence. The set of coherent formulas is clearly closed under the Boolean connectives. It is also closed under $\Box$, since $\Box\varphi$ is equivalent to the coherent $\top$ when $R = \emptyset$, and equivalent to φ itself when R is the identity relation.

Assume that φ is coherent. We will prove that $\exists x\varphi$ is coherent. Inductively, there is a quantifier-free formula ψ equivalent to φ, and so $\exists x\varphi$ is coherent if the equivalent $\exists x\psi$ is coherent. Thus we can suppose that φ is quantifier-free. But then there is a quantifier-free ψ in disjunctive normal form that is tautologically equivalent to φ, and hence equivalent to φ in $\mathcal{M}$. Again, $\exists x\varphi$ will be coherent if the equivalent $\exists x\psi$ is. Thus we can suppose that φ is in disjunctive normal form.

So, suppose that φ is $\varphi_1 \vee \cdots \vee \varphi_n$, where each φ_i is a conjunction of *literals*, i.e. atomic and negated-atomic formulas. If each $\exists x\varphi_i$ is coherent, then so is $\exists x\varphi_1 \vee \cdots \vee \exists x\varphi_n$, which is equivalent to $\exists x(\varphi_1 \vee \cdots \vee \varphi_n)$ by Lemma 3.5(1), so $\exists x\varphi$ will be coherent. Hence we can suppose that φ is a conjunction of literals.

Next we can split off the conjuncts of φ in which x does not occur. For, if φ is equivalent to $\psi \wedge \theta$ with ψ a literal not containing x, and $\exists x\theta$ is coherent, then so is $\psi \wedge \exists x\theta$, which is equivalent to $\exists x(\psi \wedge \theta)$ by Lemma 3.5(2), hence equivalent to $\exists x\varphi$. So we can suppose that x occurs in each conjunct of φ.

Similarly, we can delete $P(x, x)$ and $x \sim x$ if they occur as conjuncts of φ, since each is equivalent to $\top$ by the definitions of $|{\sim}|^{\mathcal{M}}$ and $|P|^{\mathcal{M}}$, and $\exists x(\top \wedge \theta)$ is equivalent to $\exists x\theta$. Moreover, if the negation of $P(x, x)$ or $x \sim x$ occurs in φ then we are done, since $\exists x(\bot \wedge \theta)$ is equivalent to the coherent $\bot$. Finally, $y \sim x$ with y different to x can be replaced by the equivalent $x \sim y$. So altogether we can suppose that we are dealing with a formula of the form $\exists x\varphi$, where

$$\begin{aligned}\varphi \quad = \quad & \bigwedge_i P(x,y_i) \wedge \bigwedge_j P(z_j,x) \wedge \bigwedge_k \neg P(x,u_k) \wedge \bigwedge_l \neg P(v_l,x) \\ & \qquad \wedge \bigwedge_m (x \sim s_m) \wedge \bigwedge_n \neg(x \sim t_n),\end{aligned}$$

all variables y_i, z_j, etc are distinct from x, and each $\bigwedge$ could be empty. Now for any $f \in {}^{\omega}I$, we have

$$\begin{aligned}|\exists x\varphi|f \quad = \quad & \Big[\bigcup_{a\in\mathbb{Q}}\Big(Ea \cap \bigcap_i |P|(a,fy_i) \cap \bigcap_j |P|(fz_j,a) \\ & \cap \bigcap_k \big(\mathbb{Q}\setminus|P|(a,fu_k)\big) \cap \bigcap_l \big(\mathbb{Q}\setminus|P|(fv_l,a)\big) \\ & \cap \bigcap_m |{\sim}|(a,fs_m) \cap \bigcap_n \big(\mathbb{Q}\setminus|{\sim}|(a,ft_n)\big)\Big)\Big]{\uparrow}.\end{aligned}$$

Any empty intersection here is interpreted as $\mathbb{Q}$. Now $Ea = \{a\}$ for any $a \in \mathbb{Q}$. So

$$\begin{aligned}|\exists x\varphi|f = \Big\{a \in \mathbb{Q}: \quad & a \in \bigcap_i |P|(a,fy_i) \cap \bigcap_j |P|(fz_j,a) \\ & \cap \bigcap_k \big(\mathbb{Q}\setminus|P|(a,fu_k)\big) \cap \bigcap_l \big(\mathbb{Q}\setminus|P|(fv_l,a)\big) \\ & \cap \bigcap_m |{\sim}|(a,fs_m) \cap \bigcap_n \big(\mathbb{Q}\setminus|{\sim}|(a,ft_n)\big)\Big\}{\uparrow}.\end{aligned}$$

Observe now that

- $\{a \in \mathbb{Q} : a \in |P|^{\mathcal{M}}(a,b)\} = \{a \in \mathbb{Q} : a \in |{\sim}|^{\mathcal{M}}(a,b)\} = b/{\sim}$ for any $b \in \mathbb{Q}$,
- $\{b \in \mathbb{Q} : b \in |P|^{\mathcal{M}}(a,b)\} = \mathbb{Q}$ for any $a \in \mathbb{Q}$.

So the set $|\exists x\varphi|f$ above is

$$\begin{aligned}\Big[\bigcap_i (fy_i/{\sim}) \ \cap \bigcap_j \mathbb{Q} \ \cap \bigcap_k (\mathbb{Q}\setminus(fu_k/{\sim})) \ \cap \bigcap_l \emptyset \\ \cap \bigcap_m (fs_m/{\sim}) \ \cap \bigcap_n (\mathbb{Q}\setminus(ft_n/{\sim}))\Big]{\uparrow}.\end{aligned}$$

If the l-conjunction is non-empty — a condition determined by φ and independent of f — this set is $\emptyset$, and so $\exists x\varphi$ is equivalent to $\bot$. We are done.

Otherwise, write Y for the set of all variables y_i, s_m above, and write Z for the set of all variables u_k, t_n. Then

$$\begin{aligned}|\exists x\varphi|f &= \Big[\bigcap_{y\in Y}(fy/{\sim}) \,\cap\, \bigcap_{z\in Z}(\mathbb{Q}\setminus(fz/{\sim}))\Big]{\uparrow}\\ &= \Big[\bigcap_{y\in Y}(fy/{\sim}) \,\setminus\, \bigcup_{z\in Z}(fz/{\sim})\Big]{\uparrow}\end{aligned}$$

The set in square brackets here is a Boolean combination of $\sim$-equivalence classes. It is therefore of the form $\bigcup\mathcal{E}$ for some set $\mathcal{E}$ of $\sim$-classes. So by Lemma 5.1, the $\uparrow$ of the set belongs to *Prop*. This proves part (2) of the Proposition.

For part (1), there are two cases, syntactically determined by φ.

- If $Y=\emptyset$, then $|\exists x\varphi|f=\mathbb{Q}$ for all f, because there are infinitely many $\sim$-classes in $\mathbb{Q}$ and only finitely many of them are eliminated by the Z-term. So $\exists x\varphi$ is equivalent to $\top$ in this case.
- if $Y\neq\emptyset$, then $|\exists x\varphi|f$ is $\mathbb{Q}$ if all the fy are $\sim$-equivalent and no fz is $\sim$-equivalent to them: for then, the set inside the square brackets is a single $\sim$-equivalence class, so its $\uparrow$ is $\mathbb{Q}$. Otherwise, $|\exists x\varphi|f$ is $\emptyset$. Thus, for any $f\in{}^{\omega}I$,

$$|\exists x\varphi|f = \Big|\bigwedge_{y,y'\in Y} y\sim y' \wedge \bigwedge_{y\in Y, z\in Z}\neg(y\sim z)\Big|f.$$

  So $\exists x\varphi$ is equivalent to this (quantifier-free) formula if $Y\neq\emptyset$ (and, as one can see, if $Y=\emptyset$ as well).

This completes the proof of Proposition 5.3. ■

6. Completeness and the Barcan Formulas

Let L be any (consistent) normal propositional modal logic. For a given signature $\mathcal{L}$, let $\mathrm{Q}^{-}\mathrm{L}$ be the smallest set of $\mathcal{L}$-formulas that includes

- all tautologies,
- all $\mathcal{L}$-substitution-instances of L-theorems,
- the schema $\Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)$,
- the schemata UI°, UD and VQ,

and is closed under

- detachment for material implication,
- the rule of Necessitation: from φ infer $\Box\varphi$, and
- the rule UG.

Now in the last section we defined two models for $\mathcal{L} = \{P, \sim\}$, call them $\mathcal{M}_0$ and $\mathcal{M}_1$, with $R = \emptyset$ and $R =$ the identity relation, respectively. We noted that the underlying propositional frame (W, R) of one of these models validates L, by the result of [4]. But then this model itself validates all $\mathcal{L}$-substitution-instances of L-theorems, by an argument given in the proof of [2, Theorem 2]. From the soundness results we have proved, and the evident soundness of Necessitation in any premodel, it then follows that this model validates Q^{-}L, while falsifying CQ.

It is notable that both the "Barcan formula"

BF $\forall x\Box\varphi \to \Box\forall x\varphi$

and its converse

CBF $\Box\forall x\varphi \to \forall x\Box\varphi$

are valid in $\mathcal{M}_0$ and $\mathcal{M}_1$. This follows from the fact that $\Box\psi$ is equivalent to $\top$ in $\mathcal{M}_0$, and to ψ in $\mathcal{M}_1$.

It turns out that for any $\mathcal{L}$, the logic Q^{-}L is complete for the class of all $\mathcal{L}$-models validating L (i.e. validating all $\mathcal{L}$-substitution-instances of L-theorems). This can be shown by a Henkin-model construction which reveals that the axioms UI°, UD and VQ, together with the rule UG, exactly capture the $\forall$-semantics

$$|\forall x\varphi| = \bigsqcap_{a\in U} Ea \Rightarrow |\varphi(a)|$$

of the $\mathcal{L}$-models we have used.

The converse Barcan formula is valid in any $\mathcal{L}$-model satisfying the *expanding domains* condition

$$wRv \quad \text{implies} \quad Dw \subseteq Dv, \tag{6.1}$$

equivalent to the requirement that $Ea \subseteq [R]Ea$ for all $a \in U$.

The logic Q^{-}L+CBF is complete for the class of its expanding domain models. But it is also complete for the class of its models that have *constant domains*:

$$wRv \quad \text{implies} \quad Dw = Dv. \tag{6.2}$$

This last claim may raise the eyebrows of some readers who are used to thinking of (6.2) as a condition that also validates the Barcan formula, which is typically not derivable in $Q^{-}L$+CBF. But the point is that BF can only be shown to be valid in the presence of (6.2) when the model is *Kripkean* in the sense of (4.3), in which case it also validates CQ.

The schema CQ is not a theorem of $Q^{-}L$+CBF+BF, as the models $\mathcal{M}_0$ and $\mathcal{M}_1$ show. The logic $Q^{-}L$+CBF+BF+CQ can be shown to be complete for its class of constant-domain *Kripkean* models. These results indicate that the main role of the Barcan formula in possible-worlds model theory is not to provide models that have constant domains, but rather to ensure that in a Henkin-style construction, the quantifier $\forall$ can be given the Kripkean interpretation via $\bigcap$.

Justification of all these claims will be presented elsewhere.

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