The small index property — a thank-you note to Dov Gabbay

Ian Hodkinson
Department of Computing, Imperial College London, London SW7 2AZ, UK
i.hodkinson@imperial.ac.uk

Abstract

This note is to thank Dov Gabbay on his 80th birthday for some help he gave me, over 35 years ago. It contributed to a solution of a prominent problem in model theory and permutation groups, on whether the random graph has the small index property. Surprisingly, this work helped to prove that some modal fragments of first-order logic have the finite model property, topics that were of interest to Dov at the time.

1 Introduction

When I was small, I had to write 'thank-you letters' to people who had kindly given me birthday presents. Sad to say, it was not something I enjoyed. But on this occasion, I am writing a thank-you note going in the other direction, as it were, and it is a happier task for me.

I joined Dov Gabbay's research group in the Department of Computing at Imperial College London in November 1988, and stayed for several years. Much of our work was in temporal logic, a prominent part of logic in computer science, and one that Dov was already well known for: e.g., [7, 8, 9, 10, 11, 12, 13, 14]. The work led to a number of publications and took up most of my time, but I am not going to reminisce about it here. It would be of gravely limited interest, risk falling into sentimentality, and my memories after over 35 years are partial and unreliable and I would probably fail to give credit to many who deserve it.

Although all the same risks apply, I am going to reminisce instead about one curious outlying piece of work that I became involved with at that time. It seemed to have no relation to temporal logic or computing at all, though that impression was wrong. I, and as we will see, others too, owe considerable thanks to Dov in regard to it.

I arrived in Dov's group having recently done a PhD and postdoc with Wilfrid Hodges in mathematical logic (more precisely, model theory), and one particular open problem from that field came with me and lay on my desk. It asked *whether the random graph has the small index property.*

This question concerns the topological structure of the automorphism group of a certain infinite graph. It seemed about as far from any possible application as one could easily get. Nonetheless, I was preoccupied by it. I do not quite remember why, but it may have been because at least two of the very few papers that I had already written or co-written involved the small index property, and I was still attending some model theory seminars in Wilfrid's group at Queen Mary and Westfield College (now 'Queen Mary University of London'). The small index property and the random-graph instance of it were of wide interest in these circles and had attracted people who I had actually met, such as Peter Cameron, David Evans, Dugald Macpherson, Πeter Neumann, Simon Thomas, and John Truss, as well as Wilfrid and his associates at QMW and visitors including Ehud Hrushovski.

So I said to Dov that I would like to spend some 'work time' looking at this problem — not to the exclusion of my day job, but still seriously. (I might have been rather forceful in my request: I was a difficult person to work with.)

Not only did Dov permit this, *when he absolutely didn't have to,* but he actively encouraged me and indeed got involved in the technicalities of the work itself.

As a result of this generosity and enthusiasm, I had a small role in a collaboration of people working on this problem that started when Saharon Shelah visited Wilfrid in London in summer 1989. Daniel Lascar also became involved, in early 1991. Eventually we showed that the random graph does in fact have the small index property [21].[1]

This paper laid out a new method of proving the small index property, and also stimulated interest in another area that was involved in the proof: 'EPPA', the 'extension property for partial automorphisms'.

In 1991, EPPA was a nascent field, with (as I recall) only one result to its name — by Truss [30] — if indeed it had a name at all. A strengthening of Truss's result, wanted for [21], was proved by Hrushovski at a conference in Banff, Alberta, in spring 1991 [23]. This really got the EPPA ball rolling, and the field advanced quite rapidly, initially through work of Bernhard Herwig [16, 17] and Herwig–Lascar [18], with contributions by many others later. The study of EPPA is now very extensive, perhaps more so than the study of the small index property itself. See, e.g., [24] for recent references.

Within ten years, EPPA became the crucial ingredient in proving the 'finite

[1] We also showed the same for totally categorical structures — not discussed here.

algebra on finite base property' in algebraic logic [19, 2], and in Grädel's proof [15] of the finite model property for the *guarded fragment.* This is a 'modal fragment' of first-order logic. Such fragments were of interest to modal logicians at the time, including Dov himself; and the finite model property was also an interest of Dov's [7, 8]. EPPA was used later to prove the finite model property for other algebras and modal fragments, and these attempts in turn probably stimulated more work in EPPA.

So the story came full circle. A seemingly utterly abstract theoretical problem led unpredictably but rather directly and within a decade to contributions to modal logic, which is one of Dov's core interests and is used in computer science, philosophy, economics, and even mathematical logic. People who use the phrase 'blue sky research' often claim as a justification for it that this kind of thing can happen, though they are probably hoping for lucrative applications. The finite model property of modal fragments is not yet among these, but I still find this example rather striking — even salutory.

Of course the random graph would have been shown in time to have the small index property without my involvement, and perhaps better and more quickly. But the fact is that I *was* involved, and I will always be grateful to Dov for allowing me time to work on this beautiful problem. I, and I believe other workers in the affected areas too, owe him great gratitude for this.

So: *thank you Dov, and I wish you a very happy birthday!*

In the rest of this note, I will flesh out the picture with a little more technical detail for those interested. Any experts who have read this far need not read further. Mostly I will not give full proofs, as they were published over 30 years ago and would take too much space, but I will at least try to indicate how each topic involves the next. Sections 2 and 3 discuss the the random graph and the small index property. Section 4 concerns EPPA, and section 5 its applications to the finite model property.

2 The random graph

By a *graph,* we will mean an undirected loopfree graph, which is to say a set of 'nodes' endowed with an irreflexive symmetric binary 'edge' relation.

The so-called *random graph,* also known as the *Rado graph,* the *Erdős–Renyi graph,* and the *countable universal homogeneous graph,* is a particular countably infinite graph. It is generally defined only up to isomorphism. There are several equivalent characterisations of it:

1. It is the unique (up to isomorphism) graph on a countably infinite set N such

that for each pair X, Y of disjoint finite subsets of N, there is a node in N that is connected by an edge to every node in X but to no node in Y.

2. (Erdős–Renyi) It is (up to isomorphism) the graph with the same set of nodes N that is obtained with probability 1 when a coin is tossed for each unordered pair of distinct nodes, and an edge placed between them just when the coin comes up 'heads'. Probably the name 'random graph' comes from this.

3. It is the unique (up to isomorphism) countable graph that is:

   - *universal:* it embeds all finite graphs as induced subgraphs,
   - *homogeneous:* every finite partial isomorphism (isomorphism between finite induced subgraphs) of it extends to an automorphism (an isomorphism from the whole graph onto itself).

   It plays the role for graphs that the rational order $(\mathbb{Q}, <)$ plays for linear orders: they are both in a certain sense the 'limit' of a class of finite structures (the class of all finite graphs and the class of all finite linear orders, respectively).

   This notion of limit can be made precise using a general theorem of Fraïssé [20, theorem 7.1.2]. The *Fraïssé limits* so obtained are countable universal homogeneous structures, unique up to isomorphism, and include the random graph, $(\mathbb{Q}, <)$, partial orders, tournaments, boolean algebras, and so on.

4. There are weird characterisations by binary expansions and quadratic residues [4, exercise 2.10(1)]. As Cameron once said, since the random graph has probability 1, any construction is likely to give it!

5. There is a connection to 0–1 laws in finite model theory, of relevance to computing. A graph can be seen as a structure for a relational signature L comprising a single binary relation symbol, interpreted as the edge relation. A first-order L-sentence is true in the random graph just when it holds in almost all finite graphs, which means that the proportion of graphs with nodes $\{0, 1, \ldots, n\}$ in which the sentence is true tends to 1 as $n \to \infty$. Up to isomorphism, the random graph is the only countable graph for which this holds.

The random graph is an attractive structure of great interest in several areas. But does it have the small index property?

3 The small index property

Let M be a (model-theoretic) structure. As we said, an *automorphism* of M is an isomorphism from M onto itself. For example, the automorphisms of a graph are the permutations of its set of nodes that take edges to edges and non-edges to non-edges.

The identity map on the domain of M is certainly an automorphism, and the inverse of an automorphism is an automorphism. Also, the composition $f \circ g$ of automorphisms f, g is an automorphism too. So the set of automorphisms of M forms a *group,* written $\operatorname{Aut} M$. Fixing M, I will write it as just G.

The group G carries a topology called the *topology of pointwise convergence.* For a finite sequence or 'tuple' $\bar{a}$ of elements of M, write $G_{\bar{a}}$ for the set of automorphisms of M that fix each element of $\bar{a}$. This is a subgroup of G. The basic open subsets of G are now defined to be the left cosets of the subgroups $G_{\bar{a}}$, taken over all tuples $\bar{a}$ of elements of M. Arbitrary open sets are the unions of such cosets — these are closed under finite intersections. Group inverse and composition are continuous with respect to this topology, so G becomes a *topological group.* Its open subgroups are precisely those that contain some $G_{\bar{a}}$.

The basic open subsets of G, together perhaps with $\emptyset$, are precisely the sets $\{g \in G : g \supseteq p\}$, taken over all finite partial isomorphisms p of M, and so these p determine the topology. The left cosets of any $G_{\bar{a}}$ are in one-one correspondence with the images $g(\bar{a})$ of $\bar{a}$ under automorphisms $g \in G$. For, if $g, h \in G$, then $gG_{\bar{a}} = hG_{\bar{a}}$ iff $h^{-1}g \in G_{\bar{a}}$, iff $h^{-1}g(\bar{a}) = \bar{a}$, iff $g(\bar{a}) = h(\bar{a})$.

Now if M is countable, there are only countably many such images $g(\bar{a})$, and it follows that the index $|G : G_{\bar{a}}|$ of any $G_{\bar{a}}$ in G — the number of cosets it has — is countable. Every open subgroup of G contains some $G_{\bar{a}}$, so its index is no bigger. Hence, when M is countable, *every open subgroup has countable index in* G. (Hence also, G has at most 2^{ω} open subgroups.[2])

DEFINITION 3.1. We say that (countable) M *has the small index property* if a strong converse holds: every subgroup of $\operatorname{Aut} M$ of index $< 2^{\omega}$ (we say 'of small index') is open in $\operatorname{Aut} M$.

This 'small index property' is good to have because if M has it, then the abstract group $\operatorname{Aut} M$ determines its topology: the open subsets are the unions of cosets of subgroups of countable index. That is useful because in many cases, we can recover M up to bi-interpretability from $\operatorname{Aut} M$ as a topological group [1].

When I was working on this topic, I was mainly interested in whether (certain) Fraïssé limits have the small index property. A few of them were known to have

[2] This can often be improved to 'countably many' by a result of Evans [20, exercise 7.3.15].

it, including the infinite set without structure — unsurprisingly the first case to be established, by Semmes and Dixon–Neumann–Thomas [28, 5] — and the rational order $(\mathbb{Q}, <)$, by Truss [29]. (For a longer list of examples known at the time, see [21, example 1.1].) But for the random graph, in spite of some effort, it was open, and no general method of establishing the small index property was known.

It should be noted that not every Fraïssé limit has the small index property. A counterexample was given by Hrushovski.[3] Its automorphism group has 2^{2^ω} subgroups of index 2, which is too many for them all to be open.

However, a weaker small index property does hold generally: for every countable structure M, every *closed* subgroup of $\operatorname{Aut} M$ of small index is open. This was proved by Evans [6], extending a result of Kueker [25].

And Dov proved it too, from scratch, in discussions with me on the problem.

A generalisation of this was involved in the proof in [21] that the random graph has the small index property. I will write Γ for the random graph, and G for $\operatorname{Aut}\Gamma$. The useful topological notions of meagre and comeagre sets are applicable in this context. A subset of G is *comeagre* (think 'large') if it contains a countable intersection of dense open sets, and *meagre* (or *of first category*) if its complement is comeagre. It turns out that:

- every subgroup of G whose intersection with some $G_{\bar{a}}$ is meagre in $G_{\bar{a}}$ (in the subspace topology) has large index (2^ω) in G,
- every subgroup of G whose intersection with some $G_{\bar{a}}$ is comeagre in $G_{\bar{a}}$ must contain that $G_{\bar{a}}$, and is therefore open.

So we can restrict attention to subgroups H of G that are neither meagre nor comeagre within every subgroup $G_{\bar{a}}$. *Every comeagre subset of every $G_{\bar{a}}$ overlaps both H and its complement.* We complete the proof by showing that *H has large index in G.*

To indicate why this is, I will take a shortcut suggested by Hrushovski [23, p.412]. Even so, I can give only a scandalously bare outline of the argument. Given EPPA for graphs, defined in definition 4.1 below and established in [23], we can choose a sequence $f_0, f_1, \ldots \in G$ of 'mutually generic automorphisms'[4] of Γ such that:

G1. Each permutation of the f_n is induced by conjugation by some element of G. Formally, for each permutation σ of $\{0, 1, \ldots\}$ we have $(\Gamma, f_{\sigma(0)}, f_{\sigma(1)}, \ldots) \cong (\Gamma, f_0, f_1, \ldots)$, so there is $g \in G$ with $g^{-1} f_n g = f_{\sigma(n)}$ for each $n = 0, 1, \ldots$.

[3]See [26, §2.1], [4, p.108], and [20, exercise 7.4.15]. According to [26], the example was found by Cherlin, to refute a different but related conjecture; Hrushovski rediscovered it independently.

[4]Individual 'generic' automorphisms are typically those whose conjugacy class is comeagre. They were considered earlier by (e.g.) Lascar, Rubin, and Truss.

G2. $f_n \in H$ iff n is even, for each $n = 0, 1, \ldots$.

Since the set of even numbers has 2^ω images under permutations of $\{0, 1, \ldots\}$, it follows that H has 2^ω conjugates, so must have large index in G, as claimed.

The f_n are chosen by induction. Given $f_0, \ldots, f_{n-1}$, we use EPPA and another Fraïssé limit to find (within some $G_{\bar{a}}$) a comeagre set of suitable f_n. Comeagre sets arise from the topology on G, which in turn, as we saw, arises from *finite partial isomorphisms* of Γ. Fraïssé's machine ingests a special case of this: *automorphisms of finite substructures* of Γ. *The role of EPPA is to pass between these two.*

Having found a comeagre set of candidates, overlapping both H and its complement, we can then choose f_n in H or not, according to G2. By increasing $\bar{a}$ at each step, we can ensure that the resulting infinite sequence $f_0, f_1, \ldots$ satisfies G1. □

This idea works for the random graph and other Fraïssé limits where EPPA applies, but apparently not $(\mathbb{Q}, <)$ — Truss's method in [29] is still needed there.

4 EPPA

To construct the mutually generic automorphisms just mentioned, we used EPPA: the *extension property for partial automorphisms,* also called the *Hrushovski property.* As the term 'partial automorphism' can seem circular in this context, I will say 'partial isomorphism' instead. Recall that a *partial isomorphism* of a structure is an isomorphism between substructures of it.

DEFINITION 4.1. A class $\mathcal{C}$ of finite structures is said to have EPPA if each $A \in \mathcal{C}$ has an extension $B \in \mathcal{C}$ such that every partial isomorphism of A extends to an automorphism of B.

'Extension' just means that A is a substructure of B. For graphs, A would be an induced subgraph of B.

I believe the first EPPA-style result was due to Truss [30], who showed that any (single) partial isomorphism of a given finite graph extends to an automorphism of some larger finite graph. Hrushovski [23] then showed in a different way that the class of finite graphs has EPPA. This allowed our construction above of the mutually generic automorphisms $f_0, f_1, \ldots$ of the random graph, and it followed that the random graph has the small index property.

Since then, a number of other proofs of Hrushovski's theorem have appeared, both algebraic and combinatorial. A very short one can be found in [18, §4.1]. EPPA has been established for more classes of finite structures, and sometimes with extra benefits (as did [23] implicitly). For example, an early result of Herwig extended it beyond graphs, and gave more information:

THEOREM 4.2 (Herwig, [17]). *Let A be a finite structure in a finite relational signature L. Then there is a finite L-structure $B \supseteq A$, which we will call a* Herwig extension, *such that*

1. *every partial isomorphism of A extends to an automorphism of B,*

2. *every* 'live' *tuple of elements of B (that is, a single element or a tuple $\bar{b}$ such that $B \models R(\bar{b})$ for some relation symbol $R \in L$) is mapped into A by some automorphism of B.*

In addition, [16, 17] proved EPPA for the class of finite triangle-free and K_m-free graphs (and more), and it follows that the corresponding Fraïssé limits have the small index property. Herwig and Lascar [18] linked the subject with free groups. The study of EPPA is now extensive, but as far as I know, it remains open whether the classes of finite partial orders and finite tournaments have EPPA, and whether the corresponding Fraïssé limits have the small index property.

5 Finite model property

Some modal logicians in the 1990s were interested in finding *modal fragments* of first-order logic with the 'nice' properties of modal logic — decidability with good complexity, Craig interpolation, Beth property, finite model property, and so on.

Dov proposed the *finite-variable fragments* of first-order logic, perhaps because modal formulas can be translated to first-order formulas with only two variables — for example, $\Box(p \to \Diamond q)$ translates to

$$\forall y(R(x,y) \to (P(y) \to \exists x(R(y,x) \wedge Q(x)))). \tag{1}$$

For temporal formulas written with Until and Since, some of Dov's favourite connectives, three variables suffice.

For many-dimensional modal logics, Dov's proposal is correct, and the one- and two-variable fragments are quite well behaved, though unfortunately the rest are undecidable.

There were other ideas around. In [3], Andréka, van Benthem and Németi proposed the *bounded* or *guarded fragment,* in which all quantification is relativised to atomic formulas. More formally, assuming a finite relational signature, any atomic formula is guarded; boolean combinations of guarded formulas are guarded; and if $\psi(\bar{x}, \bar{y})$ is guarded, and α is an atomic formula — the *guard* — in which all of $\bar{x}, \bar{y}$ occur, then $\exists \bar{y}(\alpha \wedge \psi)$ is guarded. Hence, $\forall \bar{y}(\alpha \to \psi)$ is guarded. Evidently, (1) is guarded.

The guarded fragment and some of its extensions do have good properties, including the *finite model property:* every satisfiable guarded sentence is true in some finite model [15].

Very strikingly, *EPPA was used to prove this.* To indicate how on earth it could be involved, here is a sketch of the idea, based on [2, 15]. Similar arguments work in algebraic logic [19, 2]. Let σ be a guarded sentence written with n variables, say, and true in some structure M. We find a finite model of σ.

By expanding M by a new relation symbol $P_\varphi(\bar{x})$ for each subformula $\varphi(\bar{x})$ of σ, with $M \models \forall\bar{x}(P_\varphi(\bar{x}) \leftrightarrow \varphi(\bar{x}))$, we can suppose that (†) the quantifier-free type of any tuple $\bar{a}$ in M determines whether or not $M \models \varphi(\bar{a})$, for each subformula φ of σ.

Choose a finite substructure $A \subseteq M$ containing representatives of all quantifier-free types of n-tuples occurring in M. Using theorem 4.2, choose a finite Herwig extension $B \supseteq A$. We show that for each subformula $\varphi(\bar{x})$ of σ,

(∗) $M \models \varphi(\bar{a})$ iff $B \models \varphi(\bar{a})$, for every tuple $\bar{a}$ of elements of A.

Applying this to σ will then show that B is our desired model of σ.

We prove (∗) by induction on φ. It is clear for atomic formulas, and the boolean cases are easy. So consider the case $\varphi(\bar{x}) = \exists\bar{y}(\alpha(\bar{x},\bar{y}) \wedge \psi(\bar{x},\bar{y}))$, assume (∗) for ψ inductively (it also holds trivially for α, which is atomic), and let $\bar{a}$ be a tuple in A.

Suppose that $M \models \varphi(\bar{a})$. Then there is $\bar{b}$ in M with $M \models \alpha(\bar{a},\bar{b}) \wedge \psi(\bar{a},\bar{b})$. Choose a representative $\bar{a}'\bar{b}'$ in A of the quantifier-free type in M of $\bar{a}\bar{b}$. By (†), $M \models \alpha(\bar{a}',\bar{b}') \wedge \psi(\bar{a}',\bar{b}')$ as well. Inductively, $B \models \alpha(\bar{a}',\bar{b}') \wedge \psi(\bar{a}',\bar{b}')$. Now the map $(\bar{a}' \mapsto \bar{a})$ is a partial isomorphism of M, hence also a partial isomorphism of its substructure A. As B is a Herwig extension of A, there is $g \in \operatorname{Aut} B$ with $g(\bar{a}') = \bar{a}$. Let $\bar{c} = g(\bar{b}')$, a tuple in B. As automorphisms preserve all formulas, $B \models \alpha(\bar{a},\bar{c}) \wedge \psi(\bar{a},\bar{c})$. Hence, $B \models \varphi(\bar{a})$ by definition of φ.

Conversely suppose that $B \models \varphi(\bar{a})$. Then there is $\bar{b}$ in B with $B \models \alpha(\bar{a},\bar{b}) \wedge \psi(\bar{a},\bar{b})$. The guard α is an atomic formula in which all of $\bar{x},\bar{y}$ occur. So $\bar{a}\bar{b}$ is a live tuple in B. As B is a Herwig extension of A, there is $g \in \operatorname{Aut} B$ with $g(\bar{a}\bar{b}) = \bar{a}'\bar{b}'$, a tuple in A. Then $B \models \alpha(\bar{a}',\bar{b}') \wedge \psi(\bar{a}',\bar{b}')$ since automorphisms preserve all formulas. Inductively, $M \models \alpha(\bar{a}',\bar{b}') \wedge \psi(\bar{a}',\bar{b}')$, so $M \models \varphi(\bar{a}')$. But the quantifier-free types of $\bar{a},\bar{a}'$ are the same in B (since $g(\bar{a}) = \bar{a}'$) and hence in M since A is a substructure of both. So by (†), $M \models \varphi(\bar{a})$ as required. □

This result extends to stronger guarded fragments such as the loosely guarded, packed, and clique-guarded fragments, into which temporal formulas involving Until and Since can be translated. The proofs needed stronger Herwig-style theorems, such as [22]. So perhaps the finite model property for modal fragments of first-order logic stimulated development of EPPA.

6 Conclusion

Once again, I thank Dov for allowing me to work on the small index property 'in work time' all those years ago. Probably neither of us expected that it would be relevant to Dov's own core interests, but I have tried to show that it was, and in no small way, so his generosity (and forbearance) paid off. Happy 80th birthday, Dov!

References

[1] G. Ahlbrandt and M. Ziegler. Quasi finitely axiomatizable totally categorical theories. *Ann. Pure. Appl. Logic*, 30:63–82, 1986.

[2] H. Andréka, I. Hodkinson, and I. Németi. Finite algebras of relations are representable on finite sets. *J. Symbolic Logic*, 64:243–267, 1999.

[3] H. Andréka, J. van Benthem, and I. Németi. Modal logics and bounded fragments of predicate logic. *J. Philosophical Logic*, 27:217–274, 1998.

[4] P. J. Cameron. *Oligomorphic permutation groups*, volume 152 of *London Math. Soc. Lecture Note Series*. Cambridge University Press, 1990.

[5] J. D. Dixon, P. M. Neumann, and S. Thomas. Subgroups of small index in infinite symmetric groups. *Bull. London Math. Soc.*, 18:580–586, 1986.

[6] D. M. Evans. A note on automorphism groups of countably infinite structures. *Archiv der Mathematik*, 49:479–483, 1987.

[7] D. M. Gabbay. On decidable finitely axiomatisable modal and tense logics without the finite model property, part I. *Israel Journal of Mathematics*, 10:478–495, 1971.

[8] D. M. Gabbay. On decidable finitely axiomatisable modal and tense logics without the finite model property, part II. *Israel Journal of Mathematics*, 10:496–503, 1971.

[9] D. M. Gabbay. Decidability results in non-classical logics, part I. *Ann. Math. Logic*, 8: 237–295, 1975.

[10] D. M. Gabbay. Model theory for tense logics. *Ann Math Logic*, 8:185–235, 1975.

[11] D. M. Gabbay. *Investigations in Modal and Tense Logics with Applications to Problems in Philosophy and Linguistics*. Synthese volume 92. D. Reidel, Dordrecht, 1976.

[12] D. M. Gabbay. Expressive functional completeness in tense logic (preliminary report). In U. Monnich, editor, *Aspects of Philosophical Logic*, pages 91–117. D. Reidel, Dordrecht, 1981.

[13] D. M. Gabbay. An irreflexivity lemma with applications to axiomatizations of conditions on tense frames. In U. Monnich, editor, *Aspects of Philosophical Logic*, pages 67–89. D. Reidel, Dordrecht, 1981.

[14] D. M. Gabbay. Declarative past and imperative future: Executable temporal logic for interactive systems. In B. Banieqbal, H. Barringer, and A. Pnueli, editors, *Proceedings of Colloquium on Temporal Logic in Specification, Altrincham, 1987*, volume 398 of *Lecture Notes in Computer Science*, pages 67–89. Springer-Verlag, 1989.

[15] E. Grädel. On the restraining power of guards. *J. Symbolic Logic*, 64:1719–1742, 1999.

[16] B. Herwig. Extending partial isomorphisms on finite structures. *Combinatorica*, 15: 365–371, 1995.

[17] B. Herwig. Extending partial isomorphisms for the small index property of many ω–categorical structures. *Israel J. Math.*, 107:93–123, 1998.

[18] B. Herwig and D. Lascar. Extending partial isomorphisms and the profinite topology on the free groups. *Trans. Amer. Math. Soc.*, 352:1985–2021, 2000.

[19] R. Hirsch, I. Hodkinson, M. Marx, S. Mikulás, and M. Reynolds. Mosaics and step-by-step. In Orłowska [27], pages 158–167. Appendix to [31].

[20] W. Hodges. *Model theory*, volume 42 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 1993.

[21] W. Hodges, I. Hodkinson, D. Lascar, and S. Shelah. The small index property for ω-stable ω-categorical structures and for the random graph. *J. London Math. Soc.*, 48: 204–218, 1993.

[22] I. Hodkinson and M. Otto. Finite conformal hypergraph covers and Gaifman cliques in finite structures. *Bull. Symbolic Logic*, 9:387–405, 2003.

[23] E. Hrushovski. Extending partial isomorphisms of graphs. *Combinatorica*, 12:411–416, 1992.

[24] J. Hubička, M. Konečný, and J. Nešetřil. All those EPPA classes (strengthenings of the Herwig–Lascar theorem). *Trans. Amer. Math. Soc.*, 375:7601–7667, 2022.

[25] David W. Kueker. Definability, automorphisms, and infinitary languages. In Jon Barwise, editor, *The Syntax and Semantics of Infinitary Languages*, pages 152–165. Springer-Verlag Berlin, Heidelberg, 1968. ISBN 978-3-540-35900-5.

[26] D. Lascar. Autour de la propriété du petit indice. *Proc. London Math. Soc.*, 62:25–53, 1991.

[27] E. Orłowska, editor. *Logic at Work*, volume 24 of *Studies in Fuzziness and Soft Computing*. Physica-Verlag, Heidelberg/New York, 1998. Essays dedicated to the memory of Elena Rasiowa. ISBN 3-7908-1164-5.

[28] S. W. Semmes. Endomorphisms of infinite symmetric groups. *Abstracts Amer. Math. Soc.*, 2:426, 1981.

[29] J. K. Truss. Infinite permutation groups; subgroups of small index. *J. Algebra*, 120: 494–515, 1989.

[30] J. K. Truss. Generic automorphisms of homogeneous structures. *Proc. London Math. Soc.*, 65:121–141, 1992.

[31] Y. Venema and M. Marx. A modal logic of relations. In Orłowska [27]. Report no. IR-396, Faculteit der Wiskunde en Informatica, Vrije Universiteit Amsterdam, 1995.

Received \jreceived