Axiomatising an arbitrary elementary modal logic using hybrid logic

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Thanks to the HyLo organisers for inviting me

introduction

In this talk, *elementary modal logic* means the *modal* logic of an *elementary* class of Kripke frames (defined by a first-order theory).

Question: how to axiomatise?

1. Sahlqvist's theorem
2. Balbiani–Shapirovsky–Shehtman
3. hybrid spin on this: *modal approximants* of hybrid formulas
4. general method for axiomatising any elementary modal logic
5. remarks

in the beginning, there was the canonical model

Lemmon (1966), Makinson, Cresswell . . . and Jónsson–Tarski (1951)

Elegant completeness theorem for the basic (normal) modal logic K:

- axioms: propositional tautologies, $\Box(p \to q) \to (\Box p \to \Box q)$
- rules: modus ponens, substitution, and necessitation $\alpha/\Box\alpha$

Obviously sound: all theorems are valid in every Kripke frame.

Canonical model $\mathcal{M}^c = (\{\text{maximal } K\text{-consistent sets}\}, R^c, V^c)$:
$R^c(\Gamma, \Delta)$ iff $\forall\alpha(\Box\alpha \in \Gamma \Rightarrow \alpha \in \Delta)$ $\qquad V^c(p) = \{\Gamma : p \in \Gamma\}$.

Truth lemma: $\mathcal{M}^c, \Gamma \models \alpha \iff \alpha \in \Gamma$, for all α, Γ.

So every K-consistent formula is satisfied in $\mathcal{M}^c$.

So the K-theorems are precisely the formulas valid in every frame.
K is sound and complete for the class of all Kripke frames.

can the canonical model method do other classes of frames?

The *canonical frame* is $\mathcal{F}^c = (\{\text{maximal consistent sets}\}, R^c)$.

'Consistent' means relative to the ambient logic — K above.

Add axioms, and the canonical frame changes: fewer maximal consistent sets (MCSs), so new properties of R^c on what's left.

Eg: axiom $\Diamond p \to \Box p$ knocks out all Γ with ≥ 2 R^c-successors.
So R^c becomes a *partial function:* $\forall xyz(R^c(x,y) \wedge R^c(x,z) \to y = z)$.
Note: first-order!
And $\Diamond p \to \Box p$ is valid in every frame satisfying this. So:
$K + (\Diamond p \to \Box p)$ is sound and complete for 'partial-function' frames.
So it's an elementary logic.

How far can this massaging of $\mathcal{F}^c$ be taken? What 'shape' of formulas have nice effects on it?
How come axioms enforce *first-order* properties on $\mathcal{F}^c$?

Sahlqvist formulas

A much-loved 'general' canonical frame massaging technique.
Antecedents in Jónsson–Tarski (1951) again.

Definition 1 (Sahlqvist formula)

- any positive modal formula (built from atoms, $\top, \bot$ using $\wedge, \vee, \Box, \Diamond$) is a Sahlqvist formula
- Any negated boxed atom $\neg\Box^n p$ $(n \geq 0)$ is a Sahlqvist formula
- If σ, σ' are Sahlqvist formulas then so are $\sigma \wedge \sigma'$, $\sigma \vee \sigma'$, $\Box\sigma$.

Many common logics are Sahlqvist-axiomatisable.
Eg: $\Diamond p \to \Box p$ — equivalent to $\Box[\neg p] \vee [\Box p]$

Sahlqvist's theorem (1973)

- Every Sahlqvist formula σ has a *local first-order correspondent* $\chi(x)$ (say), in the frame language. χ can be computed from σ.
  For any Kripke frame $\mathcal{F} = (W, R)$ and $w \in W$:
  σ is valid in $\mathcal{F}$ at w iff $\mathcal{F} \models \chi(w)$.
  E.g., for $\Diamond p \to \Box p$, $\chi(x)$ is $\forall yz(R(x, y) \wedge R(x, z) \to y = z)$.
- So σ is valid in $\mathcal{F}$ iff $\mathcal{F} \models \forall x \chi(x)$ — *global correspondent* of σ.
- If a Sahlqvist formula is added to K as an axiom, then the canonical frame $\mathcal{F}^c$ satisfies its global correspondent.

So any Sahlqvist formula axiomatises the modal logic of the class of frames satisfying its global correspondent.
∴ all Sahlqvist-axiomatisable modal logics are elementary.

Generalised by (e.g.,) Goranko, Vakarelov, Kikot (+ others. . .)
But still doesn't cover all elementary logics.

McKinsey–Lemmon logic

This is an interesting logic from the Lemmon notes (1966):

$$KM^\infty = K + \{\, \Diamond\big((\Diamond p_1 \to \Box p_1) \wedge \ldots \wedge (\Diamond p_k \to \Box p_k)\big) : k \geq 1 \}.$$

The class of all frames validating KM^∞ is non-elementary
(Balbiani–Shapirovsky–Shehtman, Goldblatt–IH, 2006).

So *KM^∞ is not Sahlqvist-axiomatisable.*

Still, *KM^∞ is elementary:* Lemmon showed it is the logic of the class $\mathcal{KM}$ of frames satisfying

$$\forall x \exists y (R(x, y) \wedge \underbrace{\forall zt(R(y, z) \wedge R(y, t) \to z = t))}_{\chi(y)}$$

This is $\Diamond$(partial function)!

Balbiani–Shapirovsky–Shehtman

BSS showed how to generalise Sahlqvist's approach to cover KM^∞.

Theorem 2 (BSS, AiML-06) *Let $\sigma(p_1, \ldots, p_n)$ be a Sahlqvist formula with local correspondent $\chi(x)$. Let*

$$\Sigma = K + \{\Diamond\big(\sigma(p_1^1, \ldots, p_n^1) \wedge \ldots \wedge \sigma(p_1^k, \ldots, p_n^k)\big) \quad : \quad k \geq 1\},$$

where the p_j^i are distinct atoms. Then Σ axiomatises the modal logic of the class of frames satisfying $\forall x \exists y (R(x, y) \wedge \chi(y))$.

Proof uses compactness to 'condense' a MCS witnessing $\exists y$.

So all such logics are elementary.

The BSS theorem covers KM^∞ nicely.
Similar logics (e.g., one of Hughes 1990) are also covered.

The full scope of this approach is not clear.

introducing hybrid formulas. . .

Hybrid logic (for this talk):

$$\Phi := i \mid \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Diamond\Phi \mid \Box\Phi \mid \forall i\Phi \mid \exists i\Phi$$

$i, j, \dots$ are the nominals. *Sentence* — no free nominals.

- We only consider (need) *pure formulas* — no propositional atoms.
- We don't include $\downarrow$ (it's expressible) or $@_i$ (not expressible in general. . .).

Semantics as usual.

A hybrid formula φ is *valid* in a frame $\mathcal{F} = (W, R)$ (written $\mathcal{F} \models \varphi$) if $\mathcal{F}, h, w \models \varphi$ for every $w \in W$ and every hybrid valuation $h : \{\text{nominals}\} \to W$.

BSS + hybrid logic: modal approximants of hybrid formulas

Rewrite axioms $\{\Diamond((\Diamond p_1 \to \Box p_1) \wedge \dots \wedge (\Diamond p_k \to \Box p_k)) : k \geq 1)\}$ as

$\{\Diamond((\Box\neg\alpha_1 \vee \Box\alpha_1) \wedge \dots \wedge (\Box\neg\alpha_k \vee \Box\alpha_k)) : k \geq 1,\ \alpha_1, \dots, \alpha_k \text{ formulas}\}$.

In $\mathcal{M}^c$, they condense to a 'limit' $\Diamond \bigwedge\{\Box\neg\alpha \vee \Box\alpha : \text{all formulas } \alpha\}$.

But notice further that the $\pm\alpha$ together pin down ≤ 1 MCS:

$$\boldsymbol{\mathcal{M}^c, \Gamma \models \Box\neg\alpha \vee \Box\alpha \text{ for all } \alpha} \quad \text{iff} \quad \boldsymbol{\mathcal{F}^c, \Gamma \models \exists i \Box i}.$$

The hybrid nominal i is *approximated* by the α:

- *roughly,* in a single axiom (which mentions only finitely many α)
- *exactly,* if all axioms (all α) are taken together

Conclude: the original KM^∞ axioms approximate $\Diamond\exists i\Box i$.
And KM^∞ is the logic of the class of frames validating $\Diamond\exists i\Box i$!

can we recover the KM^∞ axioms from $\Diamond\exists i\Box i$?

Let S be a finite set of modal formulas. For any $X \subseteq S$, we can approximate the *nominal* i by the *modal formula*

- $i_{(S,X)} \quad = \quad \bigwedge\{\alpha : \alpha \in X\} \wedge \bigwedge\{\neg\beta : \beta \in S \setminus X\}$.

Extend approximation to all hybrid formulas:

- $(\varphi \wedge \psi)_{(S,X)} = \varphi_{(S,X)} \wedge \psi_{(S,X)}$, etc
- $(\Box\varphi)_{(S,X)} = \Box(\varphi_{(S,X)})$ and $(\Diamond\varphi)_{(S,X)} = \Diamond(\varphi_{(S,X)})$
- Simulate $\forall i$ by *conjunction*: $(\forall i\varphi)_{(S,X)} = \bigwedge_{Y\subseteq S} \varphi_{(S,Y)}$.
- $(\exists i\varphi)_{(S,X)} = \bigvee_{Y\subseteq S} \varphi_{(S,Y)}$.

We get a *modal approximant* of a hybrid formula φ with respect to S, X.
(Formally, each nominal i gets its own pair (S_i, X_i).)

the KM^∞ axioms 'are' the approximants of $\Diamond\exists i\Box i$

We know $\quad \Diamond((\Box p_1 \vee \Box\neg p_1) \wedge \dots \wedge (\Box p_k \vee \Box\neg p_k)) \quad (k \geq 1)$

axiomatises logic KM^∞ of the class $\mathcal{KM}$ of frames validating

$$\varphi \quad = \quad \Diamond\exists i\Box i.$$

Approximate φ w.r.t. finite set $S = \{p_1, \dots, p_k\}$ of atoms
(X is irrelevant as φ is a sentence):

$$\varphi_S \quad = \quad \overbrace{\Diamond}^{\Diamond}\ \overbrace{\bigvee_{X\subseteq S}}^{\exists i}\ \overbrace{\Box}^{\Box}\ \overbrace{\Big(\bigwedge_{p\in X} p \wedge \bigwedge_{p\in S\setminus X} \neg p\Big)}^{i}.$$

φ_S is equivalent to the kth axiom of KM^∞.
Conclude $\{\varphi_S : S \text{ finite}\}$ axiomatises KM^∞!

The approximants of this hybrid formula φ axiomatise the logic of the class of frames that validate φ!

could there be a general method here?

When do the approximants of a hybrid formula φ axiomatise the logic of the class of frames validating φ?

Can fail: e.g., $\varphi = \forall i\ i$.

To get a positive answer, we need to take a fragment: the *quasipositive fragment* of hybrid logic:

$$\begin{aligned}\Phi := \quad & i \mid \top \mid \bot \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Diamond\Phi \mid \Box\Phi \mid \exists i\Phi \mid \\ & \forall i(\Diamond i \to \Phi) \mid \Diamond j \wedge \forall i(\Box(j \to \Diamond i) \to \Phi) \mid \\ & \Diamond(j \wedge \Diamond j') \wedge \forall i(\Box(j \to \Box(j' \to \Diamond i)) \to \Phi) \mid \cdots\end{aligned}$$

For sentences with unary $\Box, \Diamond$, this is equivalent to *pure positive fragment of* $\mathcal{H}(@, \downarrow)$ — see Areces–Blackburn–Marx (1999).

main theorem, analogy with Sahlqvist

We claim that

Theorem 3 *For any quasipositive hybrid sentence φ, the set of its approximants axiomatises the modal logic of $\{\mathcal{F} : \mathcal{F} \models \varphi\}$.*

axioms for logic	Sahlqvist formula σ	{approximants of φ}
	$\downarrow$	$\uparrow$
definition of frame class	correspondent $\forall x \chi(x)$	quasipositive hybrid sentence φ

soundness...

Lemma 4 *If a quasipositive hybrid sentence φ is valid in a frame $\mathcal{F}$, then all approximants of φ are valid in $\mathcal{F}$.*

— essentially a *monotonicity* principle. Modal formulas are coarser than nominals. So should be OK for any *positive* hybrid φ?

Problem: $\forall$. For some models $\mathcal{M}$ on $\mathcal{F}$, $\bigwedge_{X \subseteq S}$ may include X such that the approximant $i_{(S,X)}$ is true at no world of $\mathcal{M}$.

Solution: *relativise* $\forall$ to exclude such 'inconsistent' X.

$\forall i\ i$ — bad. $\forall i(\Diamond i \to i)$ — OK.

General (quasipositive) form:

$$\underbrace{\Diamond(j \wedge \Diamond j')}_{\text{any length}} \wedge \forall i(\Box(j \to \Box(j' \to \Diamond i)) \to \varphi)$$

But quasipositive is not positive! This causes technical problems.

...and completeness

Lemma 5 *If all approximants of a quasipositive hybrid sentence φ are valid in a* canonical *model $\mathcal{M}^c$ (for any logic), then φ is valid in the canonical frame $\mathcal{F}^c$ of $\mathcal{M}^c$.*

Corollary 6 *φ is valid in the canonical frame of the logic axiomatised by its approximants.*

Can prove by extending proof of Sahlqvist's completeness theorem. Can extend to 'hybrid Sahlqvist formulas', but no real need.

Generalises to sets of quasipositive sentences.

conclusion

Theorem 7 *The approximants of a set Φ of quasipositive sentences axiomatise the logic of the class of frames validating all $\varphi \in \Phi$.*

general method of axiomatising any elementary logic

Theorem 8 *The elementary modal logics are precisely those axiomatised by the approximants of sets of quasipositive sentences.*

Proof. $\Leftarrow$: A set of quasipositive sentences defines an elementary class $\mathcal{K}$ of frames. By theorem 7, its approximants axiomatise the logic of $\mathcal{K}$.

$\Rightarrow$: Take an elementary frame class $\mathcal{K}$ defined by first-order theory T.

step 1 turn T into a 'pseudo-equational' theory U defining a new class $\overline{\mathcal{K}} \supseteq \mathcal{K}$. The modal logics of $\mathcal{K}, \overline{\mathcal{K}}$ are the same.

step 2 turn U into a set Φ of quasipositive hybrid sentences valid in precisely the frames in $\overline{\mathcal{K}}$

step 3 turn Φ into approximants. By theorem 7, they axiomatise the logic of $\overline{\mathcal{K}}$.

step 1: replace T by pseudo-equations

Positive bounded formulas:

$$\Pi := \quad R(x,y) \mid x = y \mid \top \mid \bot \mid \Pi \wedge \Pi \mid \Pi \vee \Pi \mid$$
$$\forall y(R(x,y) \to \Pi) \mid \exists y(R(x,y) \wedge \Pi) \qquad \text{where } x \neq y.$$

Pseudo-equations: $\forall x\, \pi(x)$ where $\pi(x)$ is positive bounded.

Recall $\mathcal{K}$ defined by T. Put $U = \{\varepsilon : \varepsilon$ a pseudo-equation, $T \vdash \varepsilon\}$.

Theorem 9 (Goldblatt 1995) *Mod*(U) *is the closure* $\overline{\mathcal{K}}$ *of* $\mathcal{K}$ *under disjoint unions, bounded morphic images, generated subframes, and ultraroots.*

$\therefore \overline{\mathcal{K}}$ has the same modal logic as $\mathcal{K}$.

So we can and do replace $\mathcal{K}, T$ *by* $\overline{\mathcal{K}}, U$.

Step 1 is not needed if T is already pseudo-equational.

step 2

Proposition 10 *Any pseudo-equation* ε *can be easily translated into a quasipositive hybrid sentence* φ *that is valid in precisely the frames satisfying* ε.

Example:

$$\varepsilon = \forall x \exists y \big(R(x,y) \wedge \forall z (R(y,z) \to R(x,z) \vee R(z,x) \vee y = z)\big)$$

translates to

$$\varphi = \exists x \big[x \wedge \exists y \big(\Diamond y \wedge \forall z \big([\Box(y \to \Diamond z)] \to \Diamond z \vee [\Diamond\Diamond(z \wedge \Diamond x)] \vee \Diamond(y \wedge z)\big)\big)\big].$$

So we *replace the pseudo-equational theory* U *defining* $\overline{\mathcal{K}}$ *by a set* Φ *of quasipositive hybrid sentences valid in precisely the frames in* $\overline{\mathcal{K}}$.

Steps 1 and 2 are not needed if we can define our elementary frame class $\mathcal{K}$ by quasipositive hybrid sentences in the first place.

remarks

Extreme Sahlqvist: axiomatises every elementary modal logic, by approximants. New way to study them. New use for hybrid logic!!

1. 'explains' elementarity of KM^∞ and other non-Sahlqvist logics by Sahlqvist-like means. But BSS do it better?
2. new proof of Fine's theorem (elementary $\Rightarrow$ canonical)
3. proof works for multiple polyadic modalities
4. axioms are r.e. if T is, and can be 'natural' — eg KM^∞
5. some logics need infinitely many quasipositive sentences
6. *open problem* to find *finite* axiomatisation where one exists
7. can we add fixed points (to capture some non-elementary logics)?

references

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