Finite model property for guarded fragments, and extending partial isomorphisms

Ian Hodkinson

Joint work with
M Otto

Outline of talk

I aim to explain the context and proof of a recent theorem of M. Otto–I.H. on extending partial isomorphisms of relational structures.

- Definitions
- History
- Applications
- Guarded fragments
- Proof of theorem of Otto–I.H.

Partial isomorphisms & automorphisms

Definition 1 Let L be a relational signature, and A a L-structure.

1. A *partial isomorphism* of A is a partial 1–1 map $p : A \to A$ such that for all n-ary $R \in L$ and all $a_1, \ldots, a_n \in \operatorname{dom} p$,

$$A \models R(a_1, \ldots, a_n) \leftrightarrow R(p(a_1), \ldots, p(a_n)).$$

2. An *automorphism* of A is a bijective partial isomorphism of A.

3. We write $\mathrm{Aut}\ A$ for the set (group) of automorphisms of A.

The problem

Given:

- a finite relational signature L,
- a finite L-structure A,
- some partial isomorphisms $p_1, \ldots, p_n$ of A.

Questions:

1. Can we find a *finite* L-structure $B \supseteq A$ such that $p_1, \ldots, p_n$ extend to automorphisms of B?
2. Can we find such a B with 'nice properties'?

We will see some applications later.

History

0. For sets ($L = \emptyset$), this is easy.

1. J. Truss, 1992 Can extend a single partial isomorphism of a finite graph to an automorphism of a larger finite graph.

2. E. Hrushovski, 1992 Can extend all partial isomorphisms of a finite graph to automorphisms of a (single) larger finite graph.

3. B. Herwig, 1995 For any finite relational signature L and any finite L-structure A, there is a finite L-structure $B \supseteq A$ such that

1. any partial isomorphism of A extends to an automorphism of B,
2. for any $b \in B$, there is $g \in \mathrm{Aut}\ B$ with $g(b) \in A$,
3. if $b_1, \ldots, b_n \in B$, and $B \models R(b_1, \ldots, b_n)$ for some n-ary $R \in L$, then there is some $g \in \mathrm{Aut}\ B$ with $g(b_1), \ldots, g(b_n) \in A$.

Gaifman graph

To explain a later result of Herwig, we need a definition.

Definition 2 Let L be a relational signature, and A an L-structure. The *Gaifman graph* $\mathsf{Gaif}(A)$ of A is the (undirected loop-free) graph defined by:

- its set of nodes is $\mathrm{dom}\, A$,
- (x, y) is an edge iff there are n-ary $R \in L$ and $a_1, \ldots, a_n \in A$ with
  $A \models R(a_1, \ldots, a_n)$,
  $x, y \in \{a_1, \ldots, a_n\}$.

Herwig's 1998 theorem

4. B. Herwig, 1998 For any finite L-structure A, can extend all partial isomorphisms of A to automorphisms of a finite L-structure $B \supseteq A$ such that

1. the 1995 properties hold,
2. if S is an L-structure, $\mathsf{Gaif}(S)$ is a clique, and there is a homomorphism $h : S \to B$, then there is a homomorphism $g : S \to A$.

Corollaries

1. For any $n \geq 3$, can extend all partial isomorphisms of a finite K_n-free graph to automorphisms of a larger finite K_n-free graph.
2. For any class $\mathcal{T}$ of finite tournaments, can extend all partial isomorphisms of a finite digraph omitting all $T \in \mathcal{T}$ to automorphisms of a larger finite digraph omitting all $T \in \mathcal{T}$.

History ctd.

Hrushovski's proof was group-theoretic/combinatorial.

Herwig's papers greatly extended these methods.

5. Herwig–Lascar, 2000 — gave simpler and purely combinatorial proofs of Hrushovski's 1992 and Herwig's 1995 results, connected them to equivalent results in free groups, and extended the results.

A purely combinatorial account of Herwig's 1998 result was missing.

Also, even this result is still not strong enough for some applications.

Applications

1. Small index property.

   A countable structure M has this if any subgroup of $\mathrm{Aut}\ M$ of index $< 2^\omega$ is open in $\mathrm{Aut}\ M$ (in the topology of pointwise convergence).

   Hrushovski's result $\vdash$ the 'random graph' has the small index property.
   Herwig $\vdash$ s.i.p. for universal homogeneous K_n-free graphs, and for Henson digraphs.

2. Finite model theory: hierarchy theorems for fixed-point logics (Grohe, 1996).
   Other work of Grohe too.

3. Finite model property for *guarded fragments* of first-order logic, and classes of 'relativised' algebras in algebraic logic.
   Crs_n, WA, etc.: Andréka, I.H., Németi, 1999.
   Guarded fragment: Grädel, 1999.
   More if we can strengthen Herwig's theorem...

Guarded fragments: a rough guide

'Find out why modal logic is well-behaved (decidable etc), and generalise.'

Guarded fragments (Andréka, van Benthem, Németi, 1998) are 'modal' fragments of first-order logic.

- Any atomic formula is guarded.
- Guarded formulas are closed under booleans.
- If $\varphi(\bar{x}, \bar{y})$ is guarded, and $\gamma(\bar{x}, \bar{y})$ is a guard, then $\exists\bar{y}(\gamma(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{y}))$ is guarded.

In the basic *guarded fragment,* γ must be atomic.

In the *loosely guarded fragment [van Benthem 1997]*, γ can be a conjunction of atomic formulas, if every y in $\bar{y}$ and z in $\bar{x}\bar{y}$ co-occur in a single conjunct.

In the *packed fragment [Marx, 2001],* γ can be a conjunction of atomic and existentially-quantified atomic formulas, if all distinct u, v in $\bar{x}\bar{y}$ co-occur free in a single conjunct.
The guard enforces that $\bar{x}\bar{y}$ is a clique in the Gaifman graph. The *clique-guarded fragment [Grädel, 1999]* does this directly.

Finite model property for guarded fragments

Guarded fragments are well-behaved: decidable in 2EXPTIME, 'back-and-forth' characterisation, some interpolation results, etc.

E. Grädel (1999) showed that the *basic guarded fragment* has the *finite model property:* any guarded sentence with a model has a finite model.

He used Herwig's 1995 theorem. This gives finite $B \supseteq A$ such that whenever $\bar{b} \in B$ satisfies a guard of the basic guarded fragment, there is $g \in \text{Aut } B$ with $g(\bar{b}) \in A$.

But for LGF and PF/CGF, we need more.

Strengthening Herwig's 1998 theorem

6. M. Otto–I.H., 2001 Let L be a relational signature, and A a finite L-structure. There exists a finite L-structure $C \supseteq A$ such that:

1. any partial isomorphism of A extends to an automorphism of C,
2. if $S \subseteq C$ is a clique in $\mathsf{Gaif}(C)$, then there is $g \in \mathsf{Aut}\ C$ with

$$g(S) \stackrel{\text{def}}{=} \{g(x) : x \in S\} \subseteq A.$$

Consequences for finite model property

Whenever $\bar{c} \in C$ satisfies a packed fragment guard, there is $g \in \mathsf{Aut}\ C$ with $g(\bar{c}) \in A$.

Hence can generalise Grädel's FMP proof to the *loosely guarded and packed fragments.*

[Remark: I.H. 2001 proved these fragments have FMP by unpleasant hack of Herwig's proof.]

Proof

Take L, A as stated. By Herwig's 1995 theorem, there is a finite L-structure $B \supseteq A$ such that

1. every partial isomorphism of A extends to an automorphism of B,
2. any $x \in B$ is mapped into A by some $g \in \text{Aut } B$.

If $B = A$, we are done. Assume that $B \supset A$.

Definition 3 A set $U \subseteq B$ is *small* if there is some $g \in \text{Aut } B$ with $g(U) \subseteq A$, and *large* otherwise.

- B is large.
- If U is large then $|U| \geq 2$.
- If U is large and $g \in \text{Aut } B$ then $g(U)$ is large.

Write $\mathcal{U}$ for the set of large subsets of B.

Note: $\mathcal{U} = \{g(U) : U \in \mathcal{U}\}$ for all $g \in \text{Aut } B$.

Domain of C

Definition 4 Let $b \in B$. A map $\chi : \mathcal{U} \to \omega$ is said to be a *b-valuation* if for all $U \in \mathcal{U}$:

- if $b \notin U$ then $\chi(U) = 0$,
- if $b \in U$ then $1 \leq \chi(U) < |U|$. (Note $|U| \geq 2$.)

View χ as a notion $[\![b \in U]\!] = \chi(U)$, for large U.

Value is 0 if $b \notin U$.
Value is positive (many-valued logic!) if $b \in U$.

Definition 5 We let C have domain

$$\{(b, \chi) : b \in B,\ \chi \text{ a } b\text{-valuation}\}.$$

We'll define the L-structure of C in a minute.

Definition 6 Also define the projection $\pi : C \to B$ by $\pi(b, \chi) = b$.

Generic sets

Definition 7 A set $S \subseteq C$ is said to be *generic* if for all distinct $(b, \chi), (c, \psi) \in S$, we have

1. $b \neq c$,
2. $\chi(U) \neq \psi(U)$ for all $U \in \mathcal{U}$ with $b, c \in U$.

A set is generic iff each two-element subset is generic.

Lemma 8 *Let $S \subseteq C$ be generic. Then $\pi(S)$ is small.*

Proof. If $\pi(S) = U \in \mathcal{U}$, then by genericity,

- $\pi \restriction S$ is 1–1, so $|U| = |S|$,
- the map $\theta : S \to \{1, 2, \ldots, |U| - 1\}$ given by

$$\theta(b, \chi) = \chi(U)$$

  is 1–1.

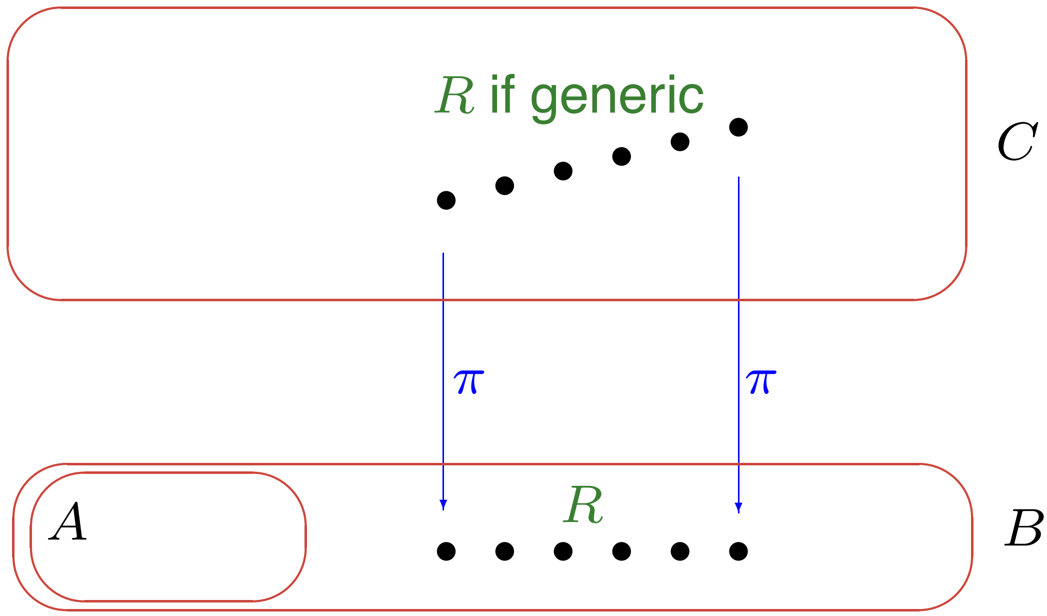
This is impossible. ■

The structure of C

Definition 9 Define C as an L-structure as follows.

If $R \in L$ is n-ary, and $(b_1, \chi_1), \dots, (b_n, \chi_n) \in C$, then let
$C \models R((b_1, \chi_1), \dots, (b_n, \chi_n))$ iff

1. $\{(b_1, \chi_1), \dots, (b_n, \chi_n)\}$ is generic,
2. $B \models R(b_1, \dots, b_n)$.



Embedding A into C

Lemma 10 *A embeds into C.*

Proof. Let $U \in \mathcal{U}$. So there is no $g \in \mathrm{Aut}\ B$ with $g(U) \subseteq A$.

Then $U \not\subseteq A$. So $|U \cap A| < |U|$.

Enumerate $U \cap A$ as $\{a_1^U, \ldots, a_n^U\}$, with $n < |U|$. Do this for all $U \in \mathcal{U}$.

For $a \in A$, define an a-valuation $\chi_a : \mathcal{U} \to \omega$ by

$$\chi_a(U) = \begin{cases} 0, & \text{if } a \notin U, \\ \text{the } i \text{ such that } a = a_i^U, & \text{otherwise.} \end{cases}$$

Now define $\nu : A \to C$ by $\nu(a) = (a, \chi_a)$.

Note that $\nu(A) = \{\nu(a) : a \in A\}$ is generic. So $\nu : A \to C$ is an L-embedding. ■

We can therefore *replace A by $\nu(A) \cong A$,* and prove the theorem for it.

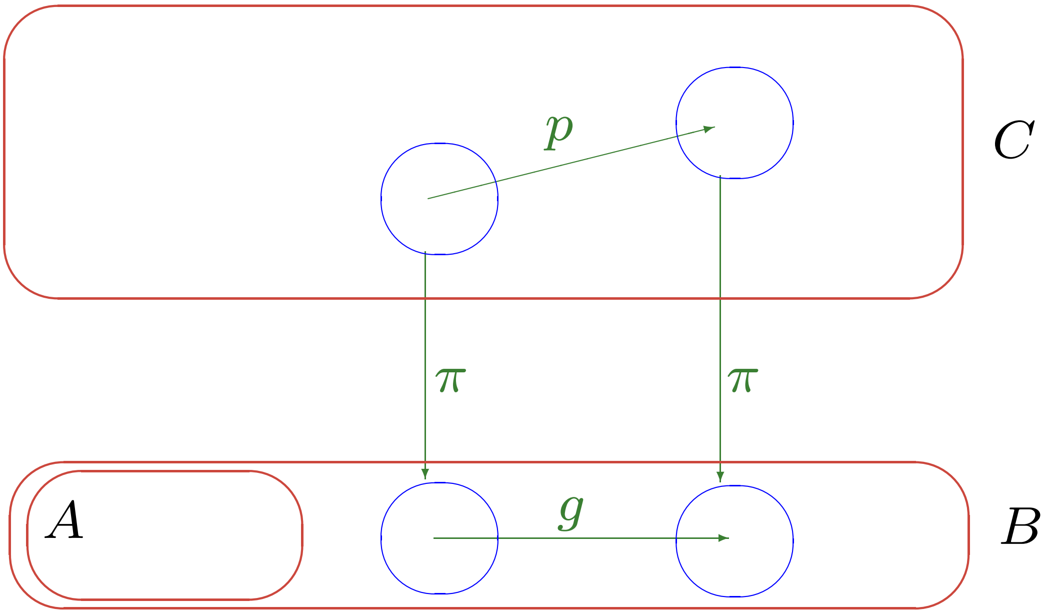
This will be easy after a definition and a lemma.

Compatible maps

Definition 11 Let $p : C \to C$ be a 1–1 partial map, and let $g \in \mathrm{Aut}\ B$. We say that p is *g-compatible* if for all $(b, \chi) \in \mathrm{dom}\, p$ we have

$$p(b, \chi) = (g(b), \chi') \text{ for some } \chi'.$$

(That is, $\pi \circ p \subseteq g \circ \pi$.)



Main lemma

Lemma 12 *Let $p : C \to C$ be a 1–1 partial map with generic domain and range. Let $g \in \mathrm{Aut}\ B$, and suppose that p is g-compatible. Then p extends to some g-compatible $\widehat{p} \in \mathrm{Aut}\ C$.*

Proof. As $\mathrm{dom}\, p$ is generic, can write its elements as (b, χ_b).
Suppose $p(b, \chi_b) = (g(b), \chi'_{g(b)})$, say. We need to define $\widehat{p} : (b, \chi) \mapsto (g(b), \chi')$ for all $(b, \chi) \in C$.

Fix a large set $U \in \mathcal{U}$. Then the set of pairs

$$\{\ \langle \chi_b(U),\ \chi'_{g(b)}(g(U))\rangle\ :\ (b, \chi_b) \in \mathrm{dom}\, p\}$$

is a 1–1 partial map on $|U| = \{0, 1, \ldots, |U| - 1\}$, fixing 0 if defined on it.

Extend it to a permutation θ_U of $|U|$, fixing 0.
Do this for all $U \in \mathcal{U}$.

Define $\widehat{p}(b, \chi) = (g(b), \chi')$, where

$$\chi'(g(U)) = \theta_U(\chi(U)) \text{ for } U \in \mathcal{U}.$$

Then $\widehat{p}$ extends p, and $\widehat{p}$ is g-compatible. Also, $\widehat{p} \in \mathrm{Aut}\ C$ (because $\widehat{p}$ is g-compatible and preserves generic sets). ■

Checking that C is as required

1. Certainly, $C \supseteq \nu(A)$ and C is finite.

2. Let p be a partial isomorphism of $\nu(A)$.
   We need to extend it to $\widehat{p} \in \mathrm{Aut}\ C$.

   Let $p{\downarrow} = \nu^{-1} \circ p \circ \nu$ be the corresponding partial isomorphism of A. By Herwig's theorem, we can extend $p{\downarrow}$ to some $g \in \mathrm{Aut}\ B$.

   Clearly, p is g-compatible.
   And $\mathrm{dom}\, p, \mathrm{rng}\, p$ are generic (as $\subseteq \nu(A)$).

   By lemma 12, p extends to some (g-compatible) $\widehat{p} \in \mathrm{Aut}\ C$.

Mapping cliques back into $\nu(A)$

3. Let $S \subseteq C$ be a clique in $\mathsf{Gaif}(C)$. We want $g \in \mathsf{Aut}\ C$ with $g(S) \subseteq \nu(A)$.

   S is generic.

   So by lemma 8, $\pi(S)$ is small.

   So there is $g \in \mathsf{Aut}\ B$ with $g(\pi(S)) \subseteq A$.

   The map

   $$p : x \mapsto \nu(g(\pi(x))) \in \nu(A) \quad \text{(for } x \in S\text{)}$$

   is 1–1, and has generic domain (S) and range ($\subseteq \nu(A)$), and is g-compatible.

   By lemma 12, p extends to $\widehat{p} \in \mathsf{Aut}\ C$, and $\widehat{p}(S) \subseteq \nu(A)$. ■

Conclusion

- The theorem strengthens Herwig's 1998 results.

- Combined with the combinatorial proof of Herwig's 1995 result by Herwig–Lascar, it gives a purely combinatorial proof of them.

- New and simple proof of finite model property for loosely guarded and packed (and clique-guarded) fragments.
  (Otto has a variant argument to give this, using finite model property of the basic guarded fragment.)