Finite model property for guarded fragments, and extending partial isomorphisms

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Joint work with
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Outline of talk

I aim to explain the context and proof of a recent theorem of M. Otto–I.H. on extending partial isomorphisms of relational structures.

- Definitions
- History
- Applications
- Guarded fragments
- Proof of theorem of Otto–I.H.
Partial isomorphisms & automorphisms

**Definition 1** Let $L$ be a relational signature, and $A$ a $L$-structure.

1. A *partial isomorphism* of $A$ is a partial 1–1 map $p : A \rightarrow A$ such that for all $n$-ary $R \in L$ and all $a_1, \ldots, a_n \in \text{dom } p$,

$$A \models R(a_1, \ldots, a_n) \leftrightarrow R(p(a_1), \ldots, p(a_n)).$$

2. An *automorphism* of $A$ is a bijective partial isomorphism of $A$.

3. We write $\text{Aut } A$ for the set (group) of automorphisms of $A$. 
The problem

Given:

- a finite relational signature $L$,
- a finite $L$-structure $A$,
- some partial isomorphisms $p_1, \ldots, p_n$ of $A$.

Questions:

1. Can we find a finite $L$-structure $B \supseteq A$ such that $p_1, \ldots, p_n$ extend to automorphisms of $B$?

2. Can we find such a $B$ with ‘nice properties’?

We will see some applications later.
History

0. For sets \((L = \emptyset)\), this is easy.

1. **J. Truss, 1992** Can extend a single partial isomorphism of a finite graph to an automorphism of a larger finite graph.

2. **E. Hrushovski, 1992** Can extend all partial isomorphisms of a finite graph to automorphisms of a (single) larger finite graph.

3. **B. Herwig, 1995** For any finite relational signature \(L\) and any finite \(L\)-structure \(A\), there is a finite \(L\)-structure \(B \supseteq A\) such that

1. any partial isomorphism of \(A\) extends to an automorphism of \(B\),

2. for any \(b \in B\), there is \(g \in \text{Aut } B\) with \(g(b) \in A\),

3. if \(b_1, \ldots, b_n \in B\), and \(B \models R(b_1, \ldots, b_n)\) for some \(n\)-ary \(R \in L\), then there is some \(g \in \text{Aut } B\) with \(g(b_1), \ldots, g(b_n) \in A\).
Gaifman graph

To explain a later result of Herwig, we need a definition.

**Definition 2** Let $L$ be a relational signature, and $A$ an $L$-structure. The *Gaifman graph* $\text{Gaif}(A)$ of $A$ is the (undirected loop-free) graph defined by:

- its set of nodes is $\text{dom} \ A$,
- $(x, y)$ is an edge iff there are $n$-ary $R \in L$ and $a_1, \ldots, a_n \in A$ with
  \[
  A \models R(a_1, \ldots, a_n),
  \]
  \[
  x, y \in \{a_1, \ldots, a_n\}.
  \]
4. **B. Herwig, 1998** For any finite $L$-structure $A$, can extend all partial isomorphisms of $A$ to automorphisms of a finite $L$-structure $B \supseteq A$ such that

1. the 1995 properties hold,

2. if $S$ is an $L$-structure, $Gaif(S)$ is a clique, and there is a homomorphism $h : S \to B$, then there is a homomorphism $g : S \to A$.

**Corollaries**

1. For any $n \geq 3$, can extend all partial isomorphisms of a finite $K_n$-free graph to automorphisms of a larger finite $K_n$-free graph.

2. For any class $\mathcal{T}$ of finite tournaments, can extend all partial isomorphisms of a finite digraph omitting all $T \in \mathcal{T}$ to automorphisms of a larger finite digraph omitting all $T \in \mathcal{T}$.
History ctd.

Hrushovski’s proof was group-theoretic/combinatorial.

Herwig’s papers greatly extended these methods.

5. Herwig–Lascar, 2000 — gave simpler and purely combinatorial proofs of Hrushovski’s 1992 and Herwig’s 1995 results, connected them to equivalent results in free groups, and extended the results.

A purely combinatorial account of Herwig’s 1998 result was missing.

Also, even this result is still not strong enough for some applications.
Applications

1. **Small index property.**

A countable structure $M$ has this if any subgroup of $\text{Aut } M$ of index $< 2^\omega$ is open in $\text{Aut } M$ (in the topology of pointwise convergence).

Hrushovski’s result $\vdash$ the ‘random graph’ has the small index property.

Herwig $\vdash$ s.i.p. for universal homogeneous $K_n$-free graphs, and for Henson digraphs.

2. **Finite model theory:** hierarchy theorems for fixed-point logics (Grohe, 1996).

Other work of Grohe too.

3. **Finite model property** for **guarded fragments** of first-order logic, and classes of ‘relativised’ algebras in algebraic logic.

Crs$_n$, WA, etc.: Andréka, I.H., Németi, 1999.


More if we can strengthen Herwig’s theorem...
Guarded fragments: a rough guide

‘Find out why modal logic is well-behaved (decidable etc), and generalise.’

Guarded fragments (Andréka, van Benthem, Németi, 1998) are ‘modal’ fragments of first-order logic.

- Any atomic formula is guarded.
- Guarded formulas are closed under booleans.
- If \( \varphi(x, y) \) is guarded, and \( \gamma(x, y) \) is a guard,
  then \( \exists y (\gamma(x, y) \land \varphi(x, y)) \) is guarded.

In the basic guarded fragment, \( \gamma \) must be atomic.

In the loosely guarded fragment [van Benthem 1997], \( \gamma \) can be a conjunction of atomic formulas, if every \( y \) in \( y \) and \( z \) in \( xy \) co-occur in a single conjunct.

In the packed fragment [Marx, 2001], \( \gamma \) can be a conjunction of atomic and existentially-quantified atomic formulas, if all distinct \( u, v \) in \( xy \) co-occur free in a single conjunct.

The guard enforces that \( xy \) is a clique in the Gaifman graph. The clique-guarded fragment [Grädel, 1999] does this directly.
Finite model property for guarded fragments

Guarded fragments are well-behaved: decidable in 2EXPTIME, ‘back-and-forth’ characterisation, some interpolation results, etc.

E. Grädel (1999) showed that the basic guarded fragment has the finite model property: any guarded sentence with a model has a finite model.

He used Herwig’s 1995 theorem. This gives finite $B \supseteq A$ such that whenever $\bar{b} \in B$ satisfies a guard of the basic guarded fragment, there is $g \in \text{Aut } B$ with $g(\bar{b}) \in A$.

But for LGF and PF/CGF, we need more.
Strengthening Herwig’s 1998 theorem

6. M. Otto–I.H., 2001 Let $L$ be a relational signature, and $A$ a finite $L$-structure. There exists a finite $L$-structure $C \supseteq A$ such that:
   1. any partial isomorphism of $A$ extends to an automorphism of $C$,
   2. if $S \subseteq C$ is a clique in $\text{Gaif}(C)$, then there is $g \in \text{Aut } C$ with
      \[ g(S) \overset{\text{def}}{=} \{g(x) : x \in S\} \subseteq A. \]

Consequences for finite model property

Whenever $\bar{c} \in C$ satisfies a packed fragment guard, there is $g \in \text{Aut } C$ with $g(\bar{c}) \in A$.

Hence can generalise Grädel’s FMP proof to the \textit{loosely guarded and packed fragments}.

[Remark: I.H. 2001 proved these fragments have FMP by unpleasant hack of Herwig’s proof.]
Proof

Take \( L, A \) as stated. By Herwig’s 1995 theorem, there is a finite \( L \)-structure \( B \supseteq A \) such that

1. every partial isomorphism of \( A \) extends to an automorphism of \( B \),
2. any \( x \in B \) is mapped into \( A \) by some \( g \in \text{Aut } B \).

If \( B = A \), we are done. Assume that \( B \supset A \).

**Definition 3** A set \( U \subseteq B \) is *small* if there is some \( g \in \text{Aut } B \) with \( g(U) \subseteq A \), and *large* otherwise.

- \( B \) is large.
- If \( U \) is large then \( |U| \geq 2 \).
- If \( U \) is large and \( g \in \text{Aut } B \) then \( g(U) \) is large.

Write \( \mathcal{U} \) for the set of large subsets of \( B \).

Note: \( \mathcal{U} = \{g(U) : U \in \mathcal{U}\} \) for all \( g \in \text{Aut } B \).
**Domain of $C$**

**Definition 4** Let $b \in B$. A map $\chi : U \rightarrow \omega$ is said to be a $b$-valuation if for all $U \in \mathcal{U}$:

- if $b \notin U$ then $\chi(U) = 0$,
- if $b \in U$ then $1 \leq \chi(U) < |U|$. (Note $|U| \geq 2$.)

View $\chi$ as a notion $[b \in U] = \chi(U)$, for large $U$.

Value is 0 if $b \notin U$.
Value is positive (many-valued logic!) if $b \in U$.

**Definition 5** We let $C$ have domain

$$\{(b, \chi) : b \in B, \ \chi \text{ a } b\text{-valuation}\}.$$ 

We’ll define the $L$-structure of $C$ in a minute.

**Definition 6** Also define the projection $\pi : C \rightarrow B$ by $\pi(b, \chi) = b$. 

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**Generic sets**

**Definition 7** A set $S \subseteq C$ is said to be *generic* if for all distinct $(b, \chi), (c, \psi) \in S$, we have

1. $b \neq c,$
2. $\chi(U) \neq \psi(U)$ for all $U \in \mathcal{U}$ with $b, c \in U$.

A set is generic iff each two-element subset is generic.

**Lemma 8** Let $S \subseteq C$ be generic. Then $\pi(S)$ is small.

**Proof.** If $\pi(S) = U \in \mathcal{U}$, then by genericity,

- $\pi \upharpoonright S$ is 1–1, so $|U| = |S|$, 
- the map $\theta : S \to \{1, 2, \ldots, |U| - 1\}$ given by

$$\theta(b, \chi) = \chi(U)$$

is 1–1.

This is impossible. \hfill \blacksquare
The structure of $C$

**Definition 9** Define $C$ as an $L$-structure as follows.

If $R \in L$ is $n$-ary, and $(b_1, \chi_1), \ldots, (b_n, \chi_n) \in C$, then let

$$C \models R((b_1, \chi_1), \ldots, (b_n, \chi_n)) \text{ iff }$$

1. $\{(b_1, \chi_1), \ldots, (b_n, \chi_n)\}$ is generic,

2. $B \models R(b_1, \ldots, b_n)$. 

![Diagram showing the structure of $C$ with $R$ if generic, $A$, and $B$.](diagram.png)
Embedding $A$ into $C$

**Lemma 10** $A$ embeds into $C$.

**Proof.** Let $U \in \mathcal{U}$. So there is no $g \in \text{Aut } B$ with $g(U) \subseteq A$.

Then $U \not\subseteq A$. So $|U \cap A| < |U|$.

Enumerate $U \cap A$ as $\{a_1^U, \ldots, a_n^U\}$, with $n < |U|$. Do this for all $U \in \mathcal{U}$.

For $a \in A$, define an $a$-valuation $\chi_a : \mathcal{U} \to \omega$ by

$$\chi_a(U) = \begin{cases} 0, & \text{if } a \notin U; \\ \text{the } i \text{ such that } a = a_i^U, & \text{otherwise.} \end{cases}$$

Now define $\nu : A \to C$ by $\nu(a) = (a, \chi_a)$.

Note that $\nu(A) = \{\nu(a) : a \in A\}$ is generic. So $\nu : A \to C$ is an $L$-embedding.

We can therefore replace $A$ by $\nu(A) \cong A$, and prove the theorem for it.

This will be easy after a definition and a lemma.
**Compatible maps**

**Definition 11** Let \( p : C \rightarrow C \) be a 1–1 partial map, and let \( g \in \text{Aut } B \). We say that \( p \) is \( g\text{-compatible} \) if for all \((b, \chi) \in \text{dom } p\) we have

\[
p(b, \chi) = (g(b), \chi') \quad \text{for some } \chi'.
\]

(That is, \( \pi \circ p \subseteq g \circ \pi \).)
Main lemma

Lemma 12 Let $p : C \rightarrow C$ be a 1–1 partial map with generic domain and range. Let $g \in \text{Aut } B$, and suppose that $p$ is $g$-compatible. Then $p$ extends to some $g$-compatible $\hat{p} \in \text{Aut } C$.

Proof. As $\text{dom } p$ is generic, can write its elements as $(b, \chi_b)$.
Suppose $p(b, \chi_b) = (g(b), \chi_{g(b)})$, say. We need to define $\hat{p} : (b, \chi) \mapsto (g(b), \chi')$ for all $(b, \chi) \in C$.

Fix a large set $U \in \mathcal{U}$. Then the set of pairs

\[
\{ \langle \chi_b(U), \chi'_{g(b)}(g(U)) \rangle : (b, \chi_b) \in \text{dom } p \}
\]

is a 1–1 partial map on $|U| = \{0, 1, \ldots, |U| - 1\}$, fixing 0 if defined on it.

Extend it to a permutation $\theta_U$ of $|U|$, fixing 0.
Do this for all $U \in \mathcal{U}$.

Define $\hat{p}(b, \chi) = (g(b), \chi')$, where

$$\chi'(g(U)) = \theta_U(\chi(U)) \text{ for } U \in \mathcal{U}.$$ 

Then $\hat{p}$ extends $p$, and $\hat{p}$ is $g$-compatible. Also, $\hat{p} \in \text{Aut } C$ (because $\hat{p}$ is $g$-compatible and preserves generic sets).
Checking that $C$ is as required

1. Certainly, $C \supseteq \nu(A)$ and $C$ is finite.

2. Let $p$ be a partial isomorphism of $\nu(A)$. We need to extend it to $\hat{p} \in \text{Aut } C$.

   Let $p\downarrow = \nu^{-1} \circ \text{op} \circ \nu$ be the corresponding partial isomorphism of $A$. By Herwig’s theorem, we can extend $p\downarrow$ to some $g \in \text{Aut } B$.

   Clearly, $p$ is $g$-compatible.
   And $\text{dom } p, \text{rng } p$ are generic (as $\subseteq \nu(A)$).

   By lemma 12, $p$ extends to some ($g$-compatible) $\hat{p} \in \text{Aut } C$. 
Mapping cliques back into $\nu(A)$

3. Let $S \subseteq C$ be a clique in $\text{Gaif}(C')$. We want $g \in \text{Aut } C$ with $g(S) \subseteq \nu(A)$.

$S$ is generic.

*So by lemma 8, $\pi(S)$ is small.*

So there is $g \in \text{Aut } B$ with $g(\pi(S)) \subseteq A$.

The map

$$p : x \mapsto \nu(g(\pi(x))) \in \nu(A) \quad (\text{for } x \in S)$$

is 1–1, and has generic domain $(S)$ and range ($\subseteq \nu(A)$), and is $g$-compatible.

By lemma 12, $p$ extends to $\hat{p} \in \text{Aut } C$, and $\hat{p}(S) \subseteq \nu(A)$.  

\[\blacksquare\]
Conclusion

- The theorem strengthens Herwig’s 1998 results.

- Combined with the combinatorial proof of Herwig’s 1995 result by Herwig–Lascar, it gives a purely combinatorial proof of them.

- New and simple proof of finite model property for loosely guarded and packed (and clique-guarded) fragments. (Otto has a variant argument to give this, using finite model property of the basic guarded fragment.)