# Axiomatising the modal logic of affine planes

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## introduction

*Projective and affine planes* — incidence systems of points and lines

- axiomatic abstractions of ‘real plane geometry’
- no quantitative information (distances)
- qualitative directional information (collinearity, parallelism)

Heavily studied in *mathematics* as a core part of geometry.

Studied in *modal logic* by Stebletsova, Venema, Balbiani, Goranko, Vakarelov, . . . *Spatial logic* is currently of some interest.

- Venema (1999) proposed 2-sorted treatment, and axiomatised projective planes.
- Balbiani–Goranko (2001) axiomatised ‘weak affine planes’.
- This talk: the true affine planes are *not finitely axiomatisable!*



## outline

- definitions, examples
- existing work (Bruck–Ryser, Venema, Balbiani–Goranko)
- proof of non-finite axiomatisability
- conclusion, problems



## definition — projective plane

A *projective plane* is a triple $(P, L, \mathsf{E})$, where $P, L$ are the sets of points and lines (resp.), $\mathsf{E} \subseteq P \times L$, and

P1. any two distinct points lie on a unique line
$\forall x, y \in P(x \neq y \to \exists!\, l \in L(x \mathrel{\mathsf{E}} l \wedge y \mathrel{\mathsf{E}} l))$

P2. any two distinct lines meet in a unique point
$\forall l, m \in L(l \neq m \to \exists!\, x \in P(x \mathrel{\mathsf{E}} l \wedge x \mathrel{\mathsf{E}} m))$

P3. there exist four points, no three of which are collinear
$\exists x_0 x_1 x_2 x_3 \in P \bigwedge_{i<j<k<4} \neg \exists l \in L(x_i \mathrel{\mathsf{E}} l \wedge x_j \mathrel{\mathsf{E}} l \wedge x_k \mathrel{\mathsf{E}} l)$

## definition — affine plane

An *affine plane* is a triple $(P, L, \mathsf{E}, ||)$, where $P, L$ are the sets of points and lines (resp.), $\mathsf{E} \subseteq P \times L$, $|| \subseteq L \times L$, and

A0. two lines are parallel iff they are equal or disjoint
$\forall l, m \in L(l \mathrel{||} m \quad\leftrightarrow\quad l = m \vee \neg\exists x \in P(x \mathrel{\mathsf{E}} l \wedge x \mathrel{\mathsf{E}} m))$

A1. any two distinct points lie on a unique line
$\forall x, y \in P(x \neq y \to \exists! l \in L(x \mathrel{\mathsf{E}} l \wedge y \mathrel{\mathsf{E}} l))$

A2. there is a unique line through any point parallel to any given line
$\forall x \in P\ \forall l \in L\ \exists! m \in L(x \mathrel{\mathsf{E}} m \wedge m \mathrel{||} l)$

A3. there exist three non-collinear points
$\exists x_0 x_1 x_2 \in P\ \neg\exists l \in L(x_0 \mathrel{\mathsf{E}} l \wedge x_1 \mathrel{\mathsf{E}} l \wedge x_2 \mathrel{\mathsf{E}} l)$
(equivalently: $L \neq \emptyset$, and for any $l \in L$, there is a point not on $l$)



## examples

The real plane $\mathbb{R}^2$ naturally yields an affine plane.
Replace $\mathbb{R}$ by a finite field — you get a finite affine plane.
It can be *completed* to a finite projective plane.

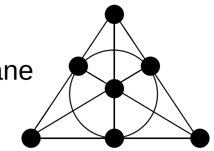
## completion: affine plane $\to$ projective plane

In an affine plane, $||$ is an equivalence relation. Take its equivalence classes to be the points of a new line (the 'line at infinity'). This gives a projective plane — the 'completion' of the affine plane.

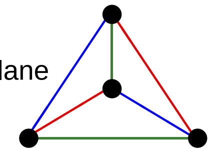
## projective plane $\to$ affine plane

Rip a line and its points out of a projective plane — get affine plane.

projective plane



affine plane





## order

The *order of a projective plane* is (the number of points on a line) $- 1$.
This is well-defined.

The *order of an affine plane* is the number of points on a line.
This is (the number of parallel classes) $- 1$.

*Bruck–Ryser theorem, 1949:* If a finite projective plane has order $n$, and $n \equiv 1$ or $2 \bmod 4$, then $n$ is the sum of two squares.

*Exercise:* For all $n \geq 0$, the number $2 \cdot 3^{2n+1}$ is $\equiv 2 \bmod 4$ and is not the sum of two squares.

So there's no projective (or affine) plane of order $2 \cdot 3^{2n+1}$, for any $n \geq 0$.
No affine plane has $2 \cdot 3^{2n+1} + 1$ parallel classes, for any $n \geq 0$.



## modal logic of projective planes

Yde Venema (1999) proposed treating projective plane $(P, L, \mathsf{E})$ as 2-sorted Kripke frame $(P, L, \mathsf{E}, \exists)$, where $\exists$ is converse of $\mathsf{E}$.

Modal formulas:

$$\begin{array}{rrcl} \text{point formulas} & \pi & ::= & p^0 \mid \neg\pi \mid \pi \wedge \pi \mid [01]\lambda \\ \text{line formulas} & \lambda & ::= & p^1 \mid \neg\lambda \mid \lambda \wedge \lambda \mid [10]\pi \end{array}$$

The boxes $[01], [10]$ have accessibility relations $\mathsf{E}, \exists$ (resp.).

$\langle 01 \rangle, \langle 10 \rangle$ are the corresponding diamonds (the usual abbreviations).

## Venema's axioms and rules

*(Sahlqvist) axioms:*
all propositional tautologies
normality of $[01], [10]$
$[01]$ and $[10]$ are mutually converse
$\langle 01\rangle\top$, $\langle 10\rangle\top$
transitivity of $\langle\cdot\rangle$ and $\langle -\rangle$, where $\langle\cdot\rangle\pi = \langle 01\rangle\langle 10\rangle\pi$, $\langle -\rangle\lambda = \langle 10\rangle\langle 01\rangle\lambda$.
(In a projective plane, these are universal modalities for their sorts.)

*Rules:* modus ponens, generalisation for $[01], [10]$, substitution.

This system is (strongly) sound and complete for projective planes.

Venema also proved that satisfiability for the logic of projective planes is decidable and NEXPTIME-complete.



## what about affine planes?

Regard affine planes as 2-sorted Kripke frames $(P, L, \mathsf{E}, \ni, ||)$.

$$\begin{array}{rrcl} \text{point formulas} & \pi & ::= & p^0 \mid \neg\pi \mid \pi\wedge\pi \mid [01]\lambda \\ \text{line formulas} & \lambda & ::= & p^1 \mid \neg\lambda \mid \lambda\wedge\lambda \mid [10]\pi \mid [||]\lambda \end{array}$$

Balbiani–Goranko (2002) *proposed axioms* for affine planes.
Completeness was left open (though it was proved for a wider class of structures called 'weak affine models').

IH–Hussain (2005): the 2-sorted modal logic of affine planes is *not finitely axiomatisable.*

So B–G's axioms are not complete.



## proof of non-finite axiomatisability

Idea: construct finite affine-like frames $C_n$ $(n < \omega)$ such that:

1. each $C_n$ *is not* the bounded morphic image of any affine plane,
2. their 'limit' $C_\infty$ (e.g., an ultraproduct) *is* a bounded morphic image of an affine plane.

If the logic of affine planes were axiomatisable by a finite set $\Sigma$ of formulas, then

1. each $C_n$ would satisfy at least one of $\{\neg\sigma : \sigma\in\Sigma\}$,
2. $C_\infty$ would validate $\Sigma$.

With suitable construction of 'limit', this is impossible!



## details — configurations (affine-like frames)

Let $\kappa \geq 1$ be a cardinal.

A *$\kappa$-configuration* is a frame $C = (P, L, \mathsf{E}, \ni, ||)$ such that

C0. two lines are parallel iff they are equal or disjoint

C1. For all $x, y \in P$, there are at least $\kappa$ lines $l \in L$ with $x, y \mathrel{\mathsf{E}} l$.

C2. there is a unique line through any point parallel to any given line

C3. $L \neq \emptyset$, and for any $l \in L$, there is a point not on $l$.

C0, C2, C3 are equivalent to affine axioms A0, A2, A3.
C1 is different from A1 ('any two distinct points lie on a unique line').

Any affine plane is a 1-configuration.

## making the $C_n$

*Lemma 1.* For every finite $k \geq 1$, there is a finite $k$-configuration $C_k$ with exactly $c$ parallel classes, where $c = 2 \cdot 3^{2n+1} + 1$ for some $n$.

*Proof.* Pick $c, d, n$ with $k \leq 2^{d-2}$, $4kd^2 \leq 2 \cdot 3^{2n+1} + 1 = c$, $c \leq 2^{d-1}$.
Take any set $P$ with $|P| = 2d$. Put $\mathcal{S} := \{l \subseteq P : |l| = d\}$.
Choose $L \subseteq \mathcal{S}$ with

- $l \in L \Rightarrow P \setminus l \in L$
- $\forall x, y \in P\ \exists_{\geq k} l \in L (x, y \in l)$
  Possible since $|\{l \in \mathcal{S} : x, y \in l\}| \geq \binom{2d-2}{d-2} \geq 2^{d-2} \geq k$.
  Need to pick total of $\leq (2d)^2 k$ sets $l$ (plus their complements).
- $|L|/2 = c$.
  So far, $|L|/2 \leq 4kd^2 \leq c$. Just add more $l \in \mathcal{S}$ to $L$ until $|L|/2 = c$.
  Possible since $|\mathcal{S}| = \binom{2d}{d} \geq 2^d$ so $|\mathcal{S}|/2 \geq 2^{d-1} \geq c$.

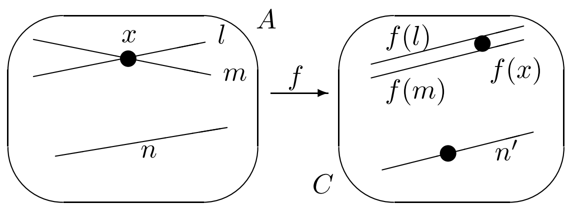
Check $C_k = (P, L, \in, \ni, ||)$ (where $||$ is defined by C0) is as required.



## configurations and affine planes

*Lemma 2.* Let $C$ be a $\kappa$-configuration, $A$ an affine plane, and $f : A \to C$ a surjective homomorphism. Then $A$ and $C$ have the same number of $||$-classes.

*Proof.* Show $f$ induces a bijection on $||$-classes. Take lines $l, m$ of $A$.
$l \mathrel{||} m \Rightarrow f(l) \mathrel{||} f(m)$ — trivial.
Assume $\neg(l \mathrel{||} m)$ but $f(l) \mathrel{||} f(m)$, and get contradiction:




By Bruck–Ryser theorem, no $C_k$ from Lemma 1 is the bounded morphic image of an affine plane.

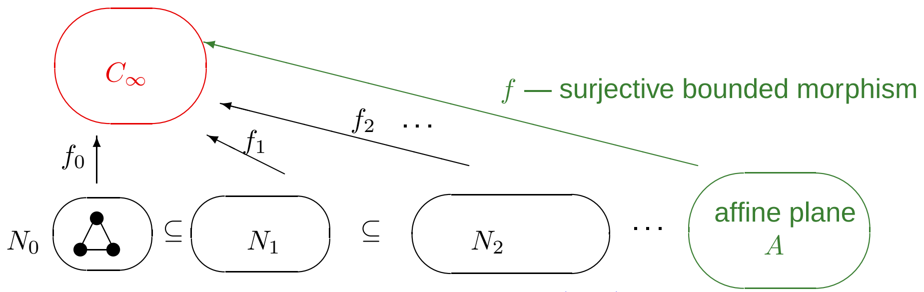


## limit

Let $C_\infty$ be a ‘limit’ (e.g., non-principal ultraproduct) of the $C_k$.
It is a $|C_\infty|$-configuration.

*Lemma 3.* $C_\infty$ is the bounded morphic image of an affine plane.

*Proof.* Build affine plane $A$ and surjective bounded morphism $f : A \to C_\infty$ by a *step-by-step construction* similar to Venema (1999).




chain of networks $N_k$ ($k < |C_\infty|$)



## coherence (soundness) conditions and defects

S1. $|N_k| < |C_\infty|$ for each $k$

S2. each $f_k$ is a homomorphism

S3. $l \mathrel{||} m \iff f_k(l) \mathrel{||} f_k(m)$, for all lines $l, m$ of $N_k$

S4. any two distinct points have at most one line through them

S5. no two distinct parallel lines intersect

S6. there exist three non-collinear points

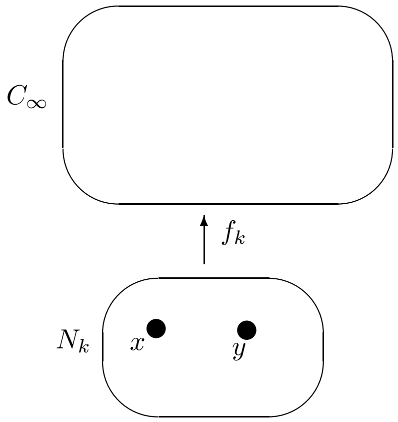
D1. ‘back-defects’ for $f_k$ for each of $\langle 01 \rangle, \langle 10 \rangle, \langle || \rangle$.

D2. two points with no line joining them

D3. parallel-axiom defects

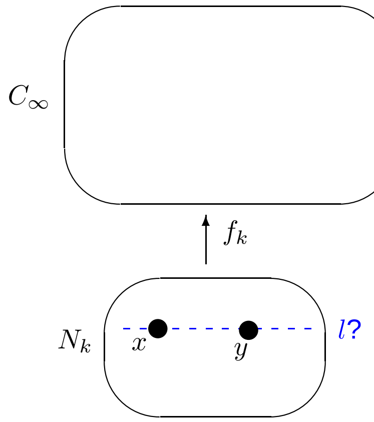
D4. non-parallel lines with no point of intersection

## D2 defects (the main case)



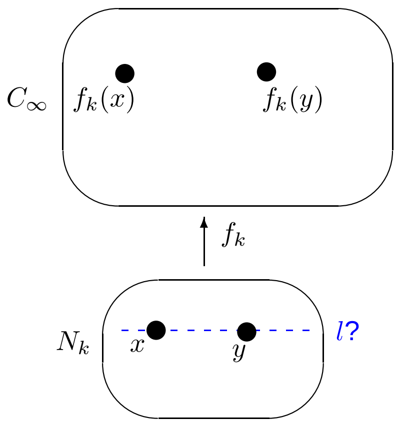



## need to add a line $l$ through $x, y$



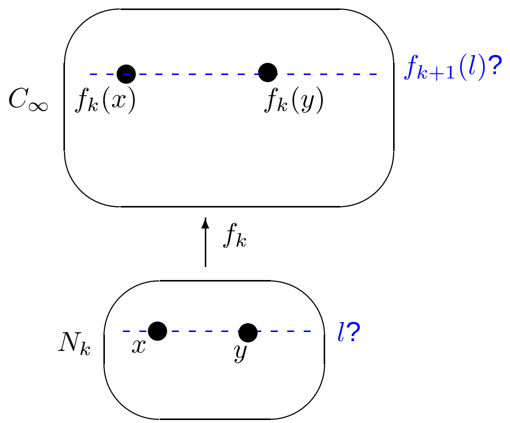



## . . . but $x, y$ already mapped by $f_k$ . . .

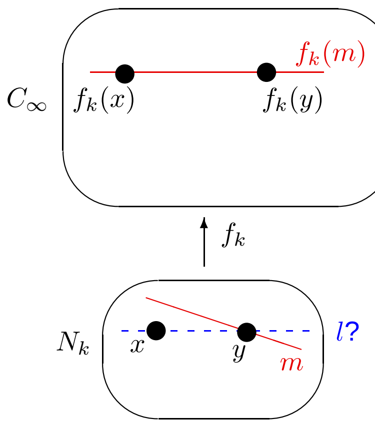





## . . . so need to define the $f_{k+1}$-image of $l$ carefully

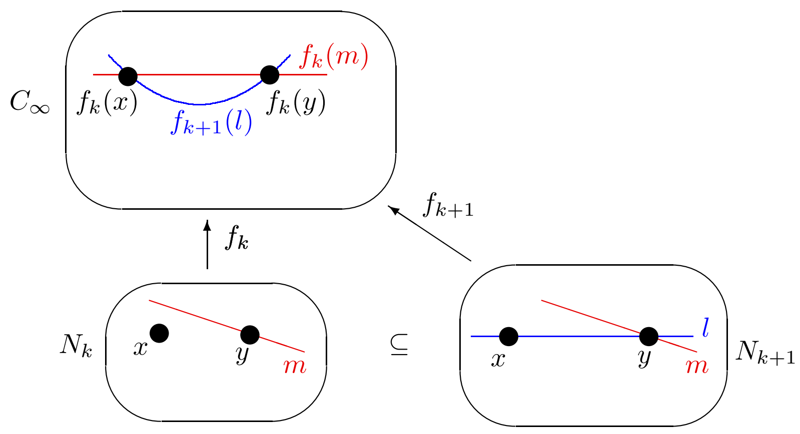


## danger...






## escape






## summary

- Each $C_k$ *is not* the bounded morphic image of an affine plane
- The limit $C_\infty$ *is* the bounded morphic image of an affine plane

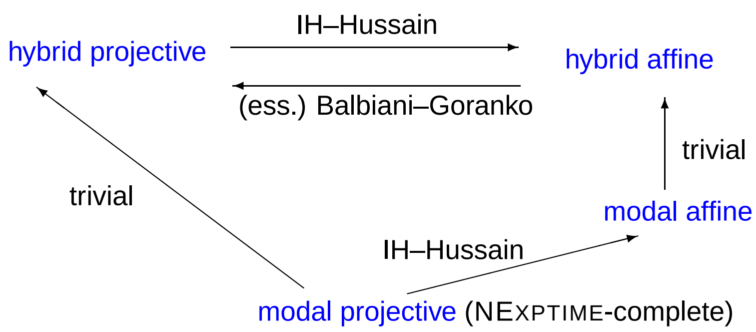
So the modal logic of affine planes is not finitely axiomatisable.

This is a *surprising contrast* with projective planes.

Can view as a *positive result* — explicit axioms may have interesting 'geometrical' content.



## complexity of satisfiability problem



## open questions

1. decidability, complexity of modal logic of affine planes (etc)?
2. find explicit axioms for modal logic of affine planes
3. what if we use hybrid logic?
4. what if we use strict (irreflexive) parallelism?

   Balbiani–Goranko (2002) axiomatised affine planes using non-standard 'irreflexivity' rules.

   decidability and complexity — open



## some references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