Loosely guarded fragment of first-order logic has the finite model property

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Abstract

We show that the loosely guarded and packed fragments of first-order logic have the finite model property. We use a construction of Herwig. We point out some consequences in temporal predicate logic and algebraic logic.

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1 Introduction

Perhaps because beginning students of modal logic are often told that modal logic is more expressive than first-order logic and indeed has some second-order expressive power, or perhaps because they are hoping for something new, it can come as a surprise to them that every modal formula has a 'standard translation' into first-order logic. For example, $\Diamond(p \to \Box q)$ is translated to

$$\exists y(R(x,y) \wedge (P(y) \to \forall z(R(y,z) \to Q(z)))). \tag{1}$$

The translation mimics the Kripke semantics for modal logic. Not every first-order formula (with one free variable in the appropriate signature) is the translation of a modal formula; so the formulas that are form a proper *fragment* of first-order logic, and one that inherits the nice properties of modal logic, such as decidability with reasonable complexity, interpolation, and the finite model property. The situation is similar for various multimodal, temporal, and dynamic logics — each corresponds by standard translation to a well-behaved modal-style fragment of classical logic.

Finding 'modal fragments' of first-order logic is an old problem in modal correspondence theory. One way to take it is to try to identify the first-order formulas that are equivalent to translations of modal formulas. Van Benthem [22, 25] proved that a first-order formula is equivalent to the translation of a modal one iff it is preserved under bisimulation. However, we cannot effectively identify these formulas, since it is undecidable whether a first-order formula is bisimulation-invariant [23, remark 4.19]. In certain restricted situations, this difficulty disappears. For example, the expressive completeness results of Kamp [16] show that over Dedekind-complete linear time, *every* first-order formula is

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equivalent to the translation of a temporal formula written with the binary modalities Until and Since. In such cases, the whole of first-order logic can be viewed as a modal fragment. But the situation here is rather special, and the lowish computational complexity of temporal logic (PSPACE-complete over a wide range of flows of time including the natural numbers) is not matched by first-order logic (non-elementary over the natural numbers). These problems are to be expected if we do not explicitly bound the complexity of proving the equivalence of a first-order formula to a modal one.

Let us focus instead on first-order formulas that are not merely equivalent to but actually are the translations of modal formulas of various kinds. The aim here is to define syntactically fragments of first-order logic containing the standard translations of various modal logics, and sharing their nice properties 'for the same reasons'. Gabbay [4] suggested that the true modal fragments of first-order logic in this sense were the finite-variable fragments, since the standard translation of modal formulas can always be done with a bounded number of variables (by re-using them — in (1), we could have used x instead of z). This proposal is natural from the first-order viewpoint, general in that it is not confined to special situations, and for many-dimensional modal logic it is provably correct (cf. [4, lemma 3, p. 115]); but it suffered the objection that finite-variable fragments do not share the nice properties that (one-dimensional) modal logic has.

A different kind of (hopefully) modal fragment of first-order logic, the *guarded fragment,* was put forward by Andréka, van Benthem, and Németi [2]. Their idea was to look at quantification patterns instead. Only relativised quantification (along the accessibility relation of the Kripke frame) is allowed in modal formulas; so in the guarded fragment, all quantification must be relativised to some atomic formula. Thus, if $\varphi(x,y,z)$ is a formula of the guarded fragment, then so are $\exists yz(R(x,y,z) \wedge \varphi(x,y,z))$ and $\forall yz(R(x,y,z) \to \varphi(x,y,z))$, because x and the quantified y,z are 'guarded' by the relation symbol R. The plain $\exists yz\varphi(x,y,z)$ would not be acceptable.

The guarded fragment does have the hoped-for nice properties. Decidability, and other results such as a Łoś–Tarski theorem, were proved in [2, 23, 24]. Complexity results are established in [5, 18]: deciding validity for sentences of the guarded fragment, with or without equality, is complete for double-exponential time, but n-variable fragments of the guarded fragment (for finite $n \geq 2$) are EXPTIME-complete, and some 2-variable guarded fragments are even in PSPACE. It was proved in [5] that the guarded fragment has the finite model property — any guarded sentence with a model has a finite model. (For further discussion of surrounding issues, see, e.g., [3, 19, 26] as well as the citations already given.) Because of these results and others, the guarded fragment and various extensions of it (e.g., by fixed-point operators) have become rather popular. But the guarded fragment also was objected to on the ground that the standard translations of some quite respectable modal-style formulas, such as temporal formulas involving Since and Until, fall outside the fragment. (The translation of $U(p,q)$ is

$$\exists y(x<y \wedge P(y) \wedge \forall z(x<z \wedge z<y \to Q(z))) \tag{2}$$

— this is not in the guarded fragment because $\forall z(x<z \wedge z<y \to Q(z))$ is not.) However, the $\forall z$ is clearly guarded to some extent in (2): z doesn't occur with x,y in a single atomic formula, but each pair of variables from x,y,z do (x and y become guarded in this way higher up the formula, by $x<y$). So van Benthem [24] proposed the *loosely guarded fragment,* which he also calls the *pairwise guarded fragment.* This fragment is our main topic here.

Roughly speaking, in the loosely guarded fragment, quantified variables must be pairwise guarded by atomic formulas. For example, if $\varphi(x,y,z)$ is a formula of the loosely guarded fragment then so is $\exists yz(R(x,y) \wedge R(y,z) \wedge S(x,z) \wedge \varphi(x,y,z))$. (See definition 3.1 for details.) The loosely guarded fragment does contain (2). It is much more powerful than the guarded fragment, but still has many nice properties, such as decidability and reasonable complexity [2, 5, 18, 24]: identical complexity results to those already cited for the guarded fragment hold for the loosely guarded fragment.

Our results In the current paper (corollary 3.4), we prove that the loosely guarded fragment has the finite model property. In theorem 3.3, we do the same for the 'packed fragment', in which the guards themselves may be existentially quantified — this fragment was defined by Marx in order to characterise the loosely guarded fragment in terms of back-and-forth systems of partial isomorphisms defined on packed sets [19]. Our proofs use a slight modification, rather along the lines of [9], of part of a model-theoretic construction of Herwig [8]. The construction is effective and yields the decidability of these fragments, but we have not tried to fine-tune it to obtain smallest possible models or the complexity results already cited.

Our results add weight to the idea that these fragments are useful, though of course they do not exclude the possibility that larger 'modal' fragments exist. As a bonus, we derive some corollaries for predicate temporal logic, algebraic logic, and arrow logic, concerning decidability and the finite base/model property. Some of them were already known.

In a recent technical report [10], a tableau decision prodecure for the 'clique guarded fragment' is given. This fragment is related to the packed fragment referred to above. It is claimed that the finite model property for this fragment and the loosely guarded fragment follows from the proof, though the current version (May 19, 2000) states that there is still a gap in one of the lemmas needed for this corollary.

Outline of paper In section 2 we explain the modified Herwig construction, and then in section 3 we use it to derive the finite model property for the packed fragment and (as a corollary) the loosely guarded fragment. Some consequences in temporal, algebraic, and arrow logic are given in section 4.

NOTATION 1.1 We write $\bar{a} \in A$ to denote that $\bar{a}$ is an n-tuple of elements of the set A, for some finite n. For a tuple $\bar{a} = (a_1, \ldots, a_n)$, we let $rng(\bar{a})$ denote the set $\{a_1, \ldots, a_n\}$. We write $|\bar{a}|$ for the length of the tuple $\bar{a}$ (so $\bar{a}$ is an $|\bar{a}|$-tuple). If $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_m)$ are tuples, we write $\bar{a}\bar{b}$ for their concatenation $(a_1, \ldots, a_n, b_1, \ldots, b_m)$. We often regard an element as a 1-tuple, so that $a\bar{b}$ denotes the tuple $(a, b_1, \ldots, b_m)$, for example.

All maps are treated formally (as sets of ordered pairs, so we can write $f \subseteq g$, $f \cup g$, etc), and are written on the left of their arguments — $f(x)$ rather than xf, so that $fg(x)$ and $f \circ g(x)$ denote $f(g(x))$. For partial maps p, q, composition $p \circ q$ (or just pq) and inverse p^{-1} (for one-one p) are defined in the obvious way; note that $\circ$ is associative. We write the domain and range of a map f as $dom\, f, rng\, f$ respectively. If f is a map defined on $rng(\bar{a})$, where $\bar{a} = (a_1, \ldots, a_n)$, we let $f(\bar{a})$ denote the tuple $(f(a_1), \ldots, f(a_n))$, and if $A \subseteq dom\, f$, we write $f(A)$ for $\{f(a) : a \in A\}$. (The potential ambiguity if $\bar{a}$ or A is a member of $dom\, f$ is never a problem here.)

We usually use the same notation for a structure M as for its domain. Unless otherwise stated, when we write a formula as $\varphi(\bar{x})$ it is implicit that $\bar{x}$ is a tuple of distinct variables containing at least the free variables of φ.

2 Herwig's construction

Herwig's theorems [7, 8] give a way of extending a finite structure to a larger one, still finite and inheriting some properties of the original structure, in such a way that all partial isomorphisms of the smaller structure extend to automorphisms of the larger one. Earlier results in this direction include [21], and in particular, those of Hrushovski [15]; the currently-used construction originated in [15] and the techniques used by Herwig are based on it. The construction consists of a 'type-realising' step and a second, 'amalgamation' step. In the Herwig–Lascar paper [9], the amalgamation step was absorbed into the more general context of free groups — this paper established striking equivalences between partial isomorphism extension results and known theorems in free groups, and proved a very strong extension theorem.

This work was motivated by pure model-theoretic considerations. Using it to prove the finite model property originated in joint work with several people [13, 1], and in Grädel [5] the finite model property for the guarded fragment was proved this way. For applications in other areas, see, e.g., [6].

Here, we will only need (a modification of) the type-realising part of the construction. Our approach is based on both [8, 9] but our notation is closer to that of the latter paper. The main new features are theorem 2.2(2) and definition 2.7. We need the following definition; the first four items are standard in model theory.

DEFINITION 2.1 Let L be a finite relational signature (i.e., with no function or constant symbols), and M,N be L-structures.

1. We write $M \subseteq N$, and say that M is a *substructure* of N and that N is an *extension* of M, if $dom(M) \subseteq dom(N)$, and for all n-ary $R \in L$ and all n-tuples $\bar{a} \in M$, we have $M \models R(\bar{a})$ iff $N \models R(\bar{a})$. An *expansion* of M is a structure in a larger signature got by adding interpretations of the new symbols; no new elements are added to the domain. A *reduct* of M to a smaller signature is got by forgetting the interpretations of the surplus symbols; no domain elements are removed.

2. A partial map $p : M \to N$ is said to be a *partial isomorphism* if it is one-one, and for all $R \in L$, of arity n, say, and for all n-tuples $\bar{a} \in dom(p)$, we have $M \models R(\bar{a})$ iff $N \models R(p(\bar{a}))$.

3. An *automorphism* of M is a bijective partial isomorphism from M to M. The set of all automorphisms of M is written $Aut\,M$; it is a group under composition of maps.

4. A *homomorphism* from M to N is a total map $f : M \to N$ such that for all n-ary $R \in L$ and n-tuples $\bar{a} \in M$, if $M \models R(\bar{a})$ then $N \models R(f(\bar{a}))$.

5. For $n > 0$, an n-tuple $\bar{a} \in M$ is said to be *live* (in M) if $n = 1$ or there is an n-ary relation symbol $R \in L$ with $M \models R(\bar{a})$.

6. A subset A of M is said to be *packed* (in M) if whenever $a,b \in A$ are distinct, there is $\bar{c} \in A$ which is live in M and with $a,b \in rng(\bar{c})$. A tuple $\bar{a} \in M$ is said to be packed if $rng(\bar{a})$ is packed.

The dependence on M in (6) is really only a dependence on the signature of M; we will use that a set packed in M is packed in any expansion of M. Every subset of M of size at most 1 is vacuously packed. Note that for non-binary signatures, with relation symbols of higher arity than two, not every subset of a packed set need be packed.

We aim to prove the following, which (we repeat) is a modification of work in [8, 9].

THEOREM 2.2 *Let L be a finite relational signature and K a finite L-structure. Let $r < \omega$. Then there is a finite L-structure $H = H_r(K)$, a substructure $\overline{K} \subseteq H$, and a surjective homomorphism $\pi : \overline{K} \to K$, with the following properties:*

1. *For every partial isomorphism p of K, there is $\widehat{p} \in Aut\,H$ such that if $a \in \overline{K}$ and $\pi(a) \in dom\,p$ then $\widehat{p}(a) \in \overline{K}$ and $p\pi(a) = \pi\widehat{p}(a)$. That is, $p \circ \pi \subseteq \pi \circ \widehat{p}$.*

2. *If $A \subseteq H$ and A is packed in H then there is $g \in Aut\,H$ with $g(A) \subseteq \overline{K}$.*

3. *For every prenex universal L-formula $\theta(\bar{x}) = \forall\bar{y}\varphi(\bar{x},\bar{y})$, where φ is quantifier-free and $\bar{x}\bar{y}$ is a non-repeating tuple of variables of length at most r, and every $\bar{a} \in \overline{K}$ such that $\pi\restriction rng(\bar{a})$ is one-one and $rng(\bar{a}) \subseteq A$ for some packed $A \subseteq H$, we have $\overline{K} \models \theta(\bar{a}) \Rightarrow K \models \theta(\pi(\bar{a}))$.*

This will be proved in section 2.3.

PROBLEM 2.3 (MARX) *For L,K as above, is there a finite L-structure $H \supseteq K$ such that any partial isomorphism of K extends to an automorphism of H, and any packed subset of H can be mapped by an automorphism of H into K?*

Theorem 2.2 is an approximation to this, as is one of the main results in [8], which (roughly speaking) constructs H such that any partial isomorphism of K extends to an automorphism of H, any live tuple of H can be mapped by an automorphism of H into K, and any packed subset of H can be mapped by a homomorphism into K. A positive solution to problem 2.3 would allow a simpler proof of our results.

2.1 The construction informally

It may help to outline the proof of theorem 2.2 to be given in sections 2.2 and 2.3. Those readers not interested may of course skip this section. Those who are are hereby warned that while we will make every effort not to mislead, because of lack of space we will not be able to discuss every detail of the proof, nor even to be completely accurate about the details we do discuss. Nothing said here should be taken as contradicting the formal definitions given in the proof later on.

2.1.1 The structure $K^{(r)}$ and its parts

To prove the theorem, we will construct an auxiliary structure $K^{(r)} \supseteq K$.

- $K^{(r)}$ will consist of disjoint blocks or *components* $K = K_0, K_1, \dots, K_r$.
- Part of each component is (roughly speaking) a copy of K. $\overline{K}^{(r)}$ will denote the the substructure of $K^{(r)}$ whose domain is the union of these copies of K.
- We let $\widehat{K}^{(r)}$ denote the substructure of $K^{(r)}$ with domain $K^{(r)} \setminus K$; this structure will be the $H_r(K)$ of theorem 2.2.
- The part of $\overline{K}^{(r)}$ inside $\widehat{K}^{(r)}$ will be the substructure $\overline{K}$ of theorem 2.2. The fact that $\overline{K}$ consists of 'copies' of K yields part 3 of theorem 2.2 (π projects each copy back onto K).

See figure 1.

2.1.2 Extending partial isomorphisms of K to $K^{(r)}$

$K^{(r)}$ is *stretched* in that no component K_i *for* $i > 0$ contains more than one element of any live tuple of elements of $K^{(r)}$, though this element can crop up several times in the tuple. (The term 'stretched' is from [9]. Note that K_0 has a special status.) The 'copy' of K in a component K_i ($i > 0$) is therefore not an exact copy, because it contains no live tuple of length > 1. We call it a 'copy' because if a_1, $\dots, a_n$ are elements of K and its copies, the corresponding elements of the original K being $b_1, \dots,$ b_n, then for any n-ary $R \in L$, $R(a_1, \dots, a_n)$ will hold in $K^{(r)}$ iff $R(b_1, \dots, b_n)$ holds in K and $|\{a_1, \dots,$ $a_n\} \cap K_i| \leq 1$ for each $i > 0$.

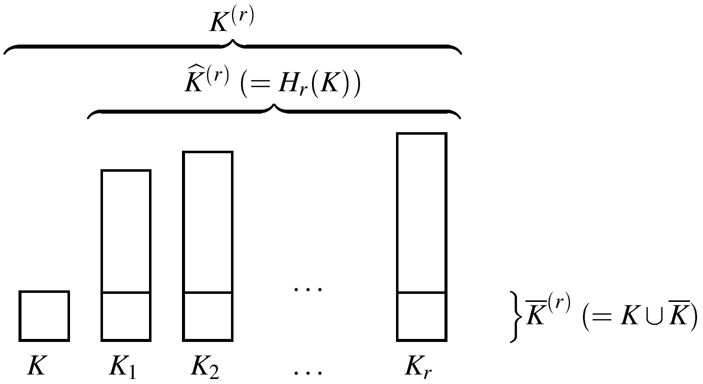

Figure 1: The structure $K^{(r)}$, components $K, K_1, \ldots, K_r$ and substructures $\widehat{K}^{(r)}, \overline{K}^{(r)}$.

Stretchedness of $K^{(r)}$ means that the only 'definable' relations that hold within a single component are essentially unary. This makes it quite easy to extend any partial isomorphism p of K to the new components of $K^{(r)}$, and even so as to form a permutation of each component extending the map induced by p on the 'copy' of K in the component. To do it, we will build the components $K_1, \ldots, K_r$ by induction, ensuring that each new component has just the right number of elements of any given isomorphism type over the existing components.

Let us discuss in more detail what this means. Suppose inductively that we have built $K^{(r)}$, for some $r \geq 0$, and extended p to a partial isomorphism ψ of $K^{(r)}$ such that $\psi \restriction K_i$ is a permutation of K_i for $1 \leq i \leq r$. $K^{(r+1)}$ will be some stretched extension of $K^{(r)}$, including a new component K_{r+1}. Here we discuss what properties it needs to have.

Let ψ' be any extension of ψ such that $\psi' \restriction K_{r+1}$ is a permutation of K_{r+1}. Consider what is required for ψ' to be a partial isomorphism of $K^{(r+1)}$. By stretchedness, it is sufficient (and necessary) that for any $a \in K_{r+1}$, the tuples $a\bar{b}$ and $\psi'(a)\psi(\bar{b})$ satisfy exactly the same atomic formulas (in $K^{(r+1)}$ of course), for any tuple $\bar{b} \in dom\,\psi$. Here and below, 'atomic' formulas exclude equalities.

The set t of atomic formulas $\alpha(x, \bar{b})$, for $\bar{b} \in dom\,\psi$, satisfied by a is called the *type* of a (over $dom\,\psi$), and $\bar{b}$ is a tuple of *parameters* of t. Abstractly, a type over a set $D \subseteq K^{(r)}$ is a set of atomic formulas with parameters from D. Now ψ 'translates' t to the type $\psi(t) = \{\alpha(x, \psi(\bar{b})) : \alpha(x, \bar{b}) \in t\}$. So what we require for ψ' to be a partial isomorphism of $K^{(r+1)}$ is that for any $a \in K_{r+1}$ with type t over $dom\,\psi$, $\psi'(a)$ has type $\psi(t)$ over $rng\,\psi$.

For any $D \subseteq K^{(r)}$ and type t over D, let E_t^D denote the set of elements of K_{r+1} with type t over D. K_{r+1} is both the disjoint union of the $E_t^{dom\,\psi}$ and the disjoint union of the $E_{\psi(t)}^{rng\,\psi}$, as t ranges over types over $dom\,\psi$. Then ψ' is a partial isomorphism of $K^{(r+1)}$ iff $\psi' \restriction E_t^{dom\,\psi}$ is a bijection from $E_t^{dom\,\psi}$ to $E_{\psi(t)}^{rng\,\psi}$ for each type t over $dom\,\psi$. So to have any hope of extending ψ to such a ψ', we need to construct K_{r+1} so that *for any type t over $dom\,\psi$, $E_t^{dom\,\psi}$ and $E_{\psi(t)}^{rng\,\psi}$ have the same size.* But given this, since the 'copy' q of p induced on the copy of K in K_{r+1} is necessarily a partial one-one map from $E_t^{dom\,\psi}$ to $E_{\psi(t)}^{rng\,\psi}$, we may extend q to a bijection : $E_t^{dom\,\psi} \to E_{\psi(t)}^{rng\,\psi}$. This can be done for all t, yielding a partial isomorphism of $K^{(r+1)}$ extending both ψ and the copy of p and acting as a permutation on K_{r+1}. In fact, this argument works for any q such that $\psi \cup q$ is a partial isomorphism of $K^{(r+1)}$.

So we want to construct K_{r+1} so that $E_t^{dom\,\psi}$ and $E_{\psi(t)}^{rng\,\psi}$ have the same cardinality, *for every t and for the extension ψ of each partial isomorphism p of K.* The type over $dom\,\psi$ of a new element introduced into K_{r+1} is determined once and for all at the time we add it, for each ψ, and its types over different $dom\,\psi$s interact. For example, if we add elements satisfyng precisely the formulas in the type t, the sets $E_t^{dom\,\psi}$ increase in size for all ψ defined on the parameters of t. So it seems we have a difficult problem here. The ingenious method used to solve it comes from [15, 7, 8, 9], and constitutes the most crucial idea in the proof. We shift attention from an element *having* a given type, to its *realising* a type.

An element realises type t if it satisfies all the formulas in t. Clearly, it may satisfy *more* formulas than these. Write R_t for the set of elements of K_{r+1} that realise t. Unlike E_t^D, R_t depends only on t, not on any set D. Because of this simpler situation, we can easily arrange that the number $|R_t|$ of elements of K_{r+1} realising a given type t depends only on $|t|$, the number of formulas in t. This can be done as follows. We construct K_{r+1} starting with a fresh copy of K. The copies of elements of K have the same types over $K^{(r)}$ as the originals. We then add further elements in stages, realising first the largest possible types over $K^{(r)}$, then the next largest, and so on, down to the empty type. Each stage is used to 'pad out' the numbers of elements realising the types of the currently-considered size to be the same. In this way we can control the interaction between types: elements added at earlier stages may realise types considered later, but not vice versa, because the sizes of types are decreasing. So our work in the current stage will not be destroyed in later stages. The totality of elements added in this way (after deleting any that violate stretchedness) constitute the new component K_{r+1}, and these elements plus the old ones in $K^{(r)}$ constitute the extension $K^{(r+1)}$.

It can now be shown by induction that $|E_t^{dom\,\psi}| = |E_{\psi(t)}^{rng\,\psi}|$ for all ψ, t as above, using the obvious fact that $|t| = |\psi(t)|$. Assume inductively that $|E_u^{dom\,\psi}| = |E_{\psi(u)}^{rng\,\psi}|$ for all types u over $dom\,\psi$ with $u \supset t$. The elements of $E_t^{dom\,\psi}$ are those elements of K_{r+1} that (a) realise t, and (b) are not in $E_u^{dom\,\psi}$ for any $u \supset t$. Similarly, $E_{\psi(t)}^{rng\,\psi}$ is the set of elements of K_{r+1} that (a) realise $\psi(t)$ and (b) are not in $E_{\psi(u)}^{rng\,\psi}$ for any $u \supset t$. By construction, the number of (a)-elements is the same on both sides, and by the inductive hypothesis the same goes for the number of elements excluded by (b). So we obtain $|E_t^{dom\,\psi}| = |E_{\psi(t)}^{rng\,\psi}|$. See lemma 2.15 for details.

2.1.3 Admissible types

We can get this far, proving part 1 of theorem 2.2, without caring *which* types over $K^{(r)}$ the elements of K_{r+1} have, but only *how many* elements have a given type. To impose extra properties on the structures $K^{(r)}$ — in particular, to obtain part 2 of the theorem, that any packed subset of $\widehat{K}^{(r)}$ is mapped by an automorphism of $\widehat{K}^{(r)}$ into the union '$\overline{K}^{(r)}$' of the copies of K — we need to control which types are realised in K_{r+1}. This is in fact quite easy to do: K_{r+1} is built in stages and we have great freedom at each stage to choose which types to realise, so long as we then equalise the numbers of realisations as already described.

Much of the evolution of the construction through [15, 7, 8, 9] can be viewed (at least by the author) as refining the selection of types to realise in K_{r+1}. We will call the types chosen to be realised the *admissible* types. To extend partial isomorphisms as already described, it is sufficient that the chosen notion of admissibility satisfies the following axioms (cf. lemmas 2.8–2.11):

1. the type over $K^{(r)}$ of any element of K is admissible [so we can start K_{r+1} with a copy of K],
2. no tuple of parameters in an admissible type violates stretchedness of $K^{(r+1)}$,
3. any restriction of an admissible type to a smaller parameter set is admissible,
4. if ψ is a partial isomorphism of $K^{(r)}$ inducing a permutation of each non-zero component, and t is an admissible type over $dom\,\psi$, then $\psi(t)$ is also admissible.

In the construction of K_{r+1}, we now arrange at each stage that all types of new elements over $K^{(r)}$ are admissible and that the number of elements realising a given admissible type t depends only on $|t|$. The axioms (especially the last two) allow us to replicate the argument of section 2.1.2 and obtain theorem 2.2(1). But they do not appear to be sufficient to prove theorem 2.2(2): we need further restrictions on which types are realised in K_{r+1}.

Roughly, we will define a type t over $K^{(r)}$ to be admissible if there is a partial isomorphism ψ of $K^{(r)}$ inducing a permutation of each non-zero component, defined on t, and such that $\psi(t)$ is the type of some element of K (i.e., K_0). The above axioms all easily follow from this. Furthermore, assuming that all elements of K_{r+1} have admissible types over $K^{(r)}$ in this sense, we can show that any packed subset of $K^{(r)}$ is mapped into $\overline{K}^{(r)}$ by a partial isomorphism ψ of $K^{(r)}$ that induces (by restriction) an automorphism of $\widehat{K}^{(r)}$. This clearly implies theorem 2.2(2).

The way we show this constitutes the second most crucial idea of the proof. Again, we assume the property inductively for $K^{(r)}$ and try to prove it for $K^{(r+1)}$. So let $A \subseteq K^{(r+1)}$ be a packed set. If $A \subseteq K^{(r)}$, the result is easily proved using the inductive hypothesis. So assume that $A \not\subseteq K^{(r)}$. As A is packed and $K^{(r+1)}$ is stretched, $A \cap K_{r+1}$ consists of a single element, say a. By packedness, the definition of admissibility, and the argument of section 2.1.2, we can assume without loss of generality that $a \in \overline{K}^{(r+1)}$. Let $B \subseteq K^{(r)}$ be obtained by replacing a by the corresponding element, say b, of the original K ($= K_0$). Then B is also packed, so inductively we may take a partial isomorphism ϕ of $K^{(r)}$ that induces by restriction a permutation of each K_i ($i > 0$), and satisfies $\phi(B) \subseteq \overline{K}^{(r)}$. Let $p = \phi \restriction K$, and let q be the map induced by p on the copy of K in the new component K_{r+1}. By construction of K_{r+1}, $\phi \cup q$ is a partial isomorphism of $K^{(r+1)}$, so by the argument of section 2.1.2, it extends to a partial isomorphism ϕ' of $K^{(r+1)}$ that induces a permutation on K_{r+1}. Now a corresponds to b, so $\phi'(a) = q(a)$ corresponds to $p(b)$. Since $p(b) \in K$, we have $\phi'(a) \in \overline{K}^{(r+1)}$. So $\phi'(A) \subseteq \overline{K}^{(r+1)}$, as required.

2.2 The construction formally

We adopt these conventions for the proof:

CONVENTION 2.4

1. All structures mentioned are finite L-structures.
2. By *atomic formula* we will mean an atomic L-formula other than an equality. We write $\alpha(\bar{x})$ for such a formula (see notation 1.1 here).

We prove theorem 2.2 by establishing an auxiliary proposition, which also gives more information about the structure of $H_r(K)$. Figure 1 may help in picturing it.

PROPOSITION 2.5 *Let L be a finite relational signature and K a finite L-structure. For each $r < \omega$, there is a finite L-structure $K^{(r)} \supseteq K$, whose domain is the disjoint union of 'components' $K = K_0$, $K_1, \ldots, K_r$, and one-one maps $\nu_i : K \to K_i$ for each $i \leq r$ (where ν_0 is the identity map on K), with properties 1– 4 below. We use the following notation:*

- *We write $\widehat{K}^{(r)}$ for the substructure of $K^{(r)}$ with domain $K^{(r)} \setminus K$.*
- *We write $\overline{K}^{(r)}$ for the substructure of $K^{(r)}$ with domain $\bigcup_{i \leq r} \nu_i(K)$.*
- *A set A of elements of $K^{(r)}$ is said to be* stretched *(in $K^{(r)}$) if $|A \cap K_i| \leq 1$ for every i with $1 \leq i \leq r$. A tuple $\bar{a}$ is stretched if $rng(\bar{a})$ is stretched. (This notion occurs in about the same form in [9].)*

1. *Any live tuple of $K^{(r)}$ is stretched.*
2. *For each $i \leq r$, each atomic formula $\alpha(x,\bar{y})$, and each $a \in K$, $\bar{b} \in \bigcup_{j<i} K_j$, we have*
$$K^{(r)} \models \alpha(a,\bar{b}) \leftrightarrow \alpha(\nu_i(a),\bar{b}).$$
3. *For every partial isomorphism p of K and each i with $1 \leq i \leq r$, there is a permutation p_i of K_i such that:*
   - *$p_i \circ \nu_i$ extends $\nu_i \circ p$. That is, $p_i(\nu_i(a)) = \nu_i(p(a))$ for all $a \in dom(p)$.*
   - *The map $p \cup \bigcup_{1\leq i\leq r} p_i$ is a partial isomorphism of $K^{(r)}$.*
4. *If $A \subseteq K^{(r)}$ is packed (in $K^{(r)}$) then there is a partial isomorphism ψ of $K^{(r)}$ such that $\psi \restriction K_i$ is a permutation of K_i for all i with $1 \leq i \leq r$, and satisfying $\psi(A) \subseteq \overline{K}^{(r)}$.*

PROOF The proof will occupy most of this section. For the duration, we fix L, K as in the formulation of the proposition. The proof is by induction on r. For $r = 0$, we may let $K^{(0)} = K$; for condition 4 we take ψ to be the identity map on K. Assume inductively that $K^{(r)}, \nu_0, \ldots, \nu_r$ have been constructed, satisfying the conditions of the proposition. We will obtain $K^{(r+1)}$ from $K^{(r)}$ by adding a new component K_{r+1} disjoint from $K^{(r)}$. To do this, we will use *types.*

2.2.1 Types

DEFINITION 2.6

1. A *type over $K^{(r)}$*, or for short, *type,* is a set t of atomic formulas with one free variable, always x, and parameters in $K^{(r)}$: i.e., formulas of the form $\alpha(x,\bar{c})$, where $\alpha(x,\bar{y})$ is atomic (not an equality — see convention 2.4) in which x occurs free, and $\bar{c} \in K^{(r)}$.
2. For a type t, $|t|$ denotes the cardinality of (i.e., number of formulas in) t, and *base* t denotes the set of all elements of K that genuinely occur in formulas in t — so *base* $t = \{a \in K : \alpha(x,a,\bar{b}) \in t$ for some $\bar{b} \in K^{(r)}$ and atomic $\alpha(x,y,\bar{z})$ in which x, y occur free$\}$.
3. If t is a type and $D \subseteq K$, we write $t \restriction D$ for $\{\alpha(x,\bar{a}) \in t : \bar{a} \in \widehat{K}^{(r)} \cup D\}$.
4. Let $M \supseteq K^{(r)}$ and $a \in M$.
   (a) If t is a type, a is said to *realise* t if $M \models \alpha(a,\bar{c})$ for all $\alpha(x,\bar{c}) \in t$.
   (b) If $D \subseteq K$, we write $tp(a/D)$ for the set
   $$\{\alpha(x,\bar{c}) : \alpha(x,\bar{y}) \text{ atomic with } x \text{ among its free variables, } \bar{c} \in \widehat{K}^{(r)} \cup D, M \models \alpha(a,\bar{c})\}.$$

The definition of *base* reflects our interest in parameters from K; use of parameters from $\widehat{K}^{(r)}$ in types is unrestricted and the reader should always bear in mind that formulas in a type can have parameters that are not in the base of the type. It is critical to note that a realising t does not imply that $tp(a/base\ t) = t$, but only $tp(a/base\ t) \supseteq t$. Also, $base\,(tp(a/D)) \subseteq D$, but we may not have equality. These and other simple facts about bases, such as $tp(a/D) = tp(a/base\,(tp(a/D)))$ for any $a \in K$, $D \subseteq K$, will be used without mention later on.

Strictly, the definitions in (4) depend on M, so we should say that a realises t in M, and write $tp_M(a/D)$. But formulas in types are atomic, so a realises t in M iff a realises t in M', for any $M' \supseteq M$, and similarly for $tp(a/D)$. Consequently, the dependence never makes a difference in practice, and therefore we refrain from overloading the notation.

2.2.2 Admissible types

There is a lot of freedom in the construction to choose the kinds of type we wish to realise. We will choose the 'admissible' types.

DEFINITION 2.7

1. We write $\Psi^{(r)}$ for the set of all partial isomorphisms ψ of $K^{(r)}$ such that $\psi \upharpoonright K_i$ is a permutation of K_i for all $1 \leq i \leq r$.
2. If $\psi \in \Psi^{(r)}$ and t is a type with $base\, t \subseteq dom\, \psi$, we write $\psi(t)$ for the type $\{\alpha(x, \psi(\bar{a})) : \alpha(x, \bar{a}) \in t\}$. A *conjugate* of t is any type of the form $\psi(t)$ for $\psi \in \Psi^{(r)}$ with $base\, t \subseteq dom\, \psi$.
3. A type t is said to be *admissible* if it is a conjugate of $tp(a/D)$ for some $a \in K$ and $D \subseteq K$.

So if t is admissible then $t = \psi(tp(a/D))$ for some ψ, a, D as above; so $base\,(tp(a/D)) \subseteq dom\, \psi$, but D may be larger than $base\,(tp(a/D))$ and we may not have $D \subseteq dom\, \psi$. The only properties of admissible types needed to extend partial isomorphisms are encapsulated in the following simple lemmas.

LEMMA 2.8 *Any conjugate of an admissible type is admissible.*

PROOF Since $\Psi^{(r)}$ contains the identity map on $K^{(r)}$ and is closed under inverse and composition, conjugacy is an equivalence relation on types. The lemma follows immediately from this. ❑

LEMMA 2.9 *If $a \in K$ then $tp(a/K)$ is admissible.*

PROOF Trivial. ❑

LEMMA 2.10 *If t is an admissible type, $\alpha(x, \bar{y})$ is an atomic formula in which all variables of $x\bar{y}$ occur, and $\alpha(x, \bar{c}) \in t$, then $\bar{c}$ is stretched in $K^{(r)}$.*

PROOF Let $t, \alpha, \bar{c}$ be as stated. As t is admissible, there are $a \in K$, $D \subseteq K$, and $\psi \in \Psi^{(r)}$ with $base\,(tp(a/D)) \subseteq dom\, \psi$ and $\psi(tp(a/D)) = t$. So $\alpha(x, \psi^{-1}(\bar{c})) \in tp(a/D)$ and $K^{(r)} \models \alpha(a, \psi^{-1}(\bar{c}))$. So $rng(a\psi^{-1}(\bar{c})) = rng(\bar{b})$ for some live tuple $\bar{b} \in K^{(r)}$. Inductively, by proposition 2.5(1) for $K^{(r)}$, $\bar{b}$, and hence $a\psi^{-1}(\bar{c})$, $\psi^{-1}(\bar{c})$, and $\bar{c}$, are stretched in $K^{(r)}$. ❑

LEMMA 2.11 *Let t be any admissible type. Then $t \upharpoonright D$ is admissible for every $D \subseteq K$.*

PROOF We have $t = \psi(tp(a/E))$ for some $a \in K$, $E \subseteq K$ and $\psi \in \Psi^{(r)}$ with $base\,(tp(a/E)) \subseteq dom\, \psi$. Let $D' = \psi^{-1}(D \cap rng\, \psi)$. Then $base\, t \subseteq rng\, \psi$, so clearly, $t \upharpoonright D = t \upharpoonright (D \cap rng\, \psi) = t \upharpoonright \psi(D') = \psi(tp(a/E) \upharpoonright D') = \psi(tp(a/E \cap D'))$. This is of the required form, and so $t \upharpoonright D$ is admissible. ❑

2.2.3 Building the structure $K^{(r+1)}$

We are going to start the construction of $K^{(r+1)}$ now, by building its new component K_{r+1}. This component will itself be built by an inductive construction: we build a chain of sets $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_l = K_{r+1}$, and a chain of structures $K^{(r)} \subseteq K_0^{(r+1)} \subseteq K_1^{(r+1)} \subseteq \cdots \subseteq K_l^{(r+1)} = K^{(r+1)}$, where l is the largest cardinality of any admissible type. See figure 2.

We will also define a bijection $\nu_{r+1} : K \to S_0$. The structures $K_j^{(r+1)}$ (for each $j = 0, 1, \ldots, l$) will have the following properties:

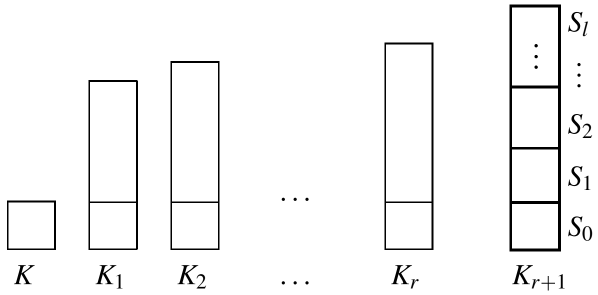

Figure 2: The new component K_{r+1} of $K^{(r+1)}$

(i) The domain of $K_j^{(r+1)}$ is the disjoint union of $K^{(r)}$ and S_j. The components of $K_j^{(r+1)}$ are $K, K_1, \ldots, K_r$, and S_j. (So in $K_j^{(r+1)}$, the notion of stretched tuple makes sense.)

(ii) Any live tuple in $K_j^{(r+1)}$ is stretched.

(iii) For any atomic formula $\alpha(x, \bar{y})$, $a \in K$, and $\bar{b} \in K^{(r)}$, we have $K_j^{(r+1)} \models \alpha(a, \bar{b}) \leftrightarrow \alpha(\nu_{r+1}(a), \bar{b})$.

(iv) For any admissible type t with $|t| > l - j$, the number of elements of S_j that realise t in $K_j^{(r+1)}$ depends only on $|t|$.

(v) $tp(a/K)$ is admissible for every $a \in S_j$.

Below, we will refer to these as 'property (i)', etc.

S_0 is obtained simply by taking a new copy of K, disjoint from $K^{(r)}$. Formally, we may let $S_0 = K \times \{K^{(r)}\}$. Thus, $K_0^{(r+1)}$ has domain $K^{(r)} \cup S_0$, for $i \leq r$, its ith component is defined to be the same as that of $K^{(r)}$, and its $(r+1)$th component is defined to be S_0. We let $\nu_{r+1} : K \to S_0$ be given by $\nu_{r+1}(a) = (a, K^{(r)})$, for $a \in K$.

We define $K_0^{(r+1)}$ as an L-structure as follows. Let $\mu : K_0^{(r+1)} \to K^{(r)}$ be the map that is the identity on $K^{(r)}$ and is ν_{r+1}^{-1} on S_0. Then if $R \in L$ is n-ary and $\bar{a} \in K_0^{(r+1)}$ is an n-tuple, we define

$$K_0^{(r+1)} \models R(\bar{a}) \quad \text{iff} \quad \bar{a} \text{ is stretched in } K_0^{(r+1)} \text{ and } K^{(r)} \models R(\mu(\bar{a})). \tag{3}$$

Note that μ is a homomorphism.

LEMMA 2.12 *$K_0^{(r+1)} \supseteq K^{(r)}$, and $K_0^{(r+1)}$ satisfies properties (i)–(v) above.*

PROOF By the definition of $K_0^{(r+1)}$ ((3) above), $K_0^{(r+1)} \supseteq K^{(r)}$. Clearly, $K_0^{(r+1)}$ has components $K, K_1, \ldots, K_r, S_0$ and all live tuples in it are stretched.

We check property (iii). Take $\alpha(x, \bar{y}), a, \bar{b}$ as in property (iii); we may assume that all variables in $x\bar{y}$ occur in α. Then $K_j^{(r+1)} \models \alpha(a, \bar{b})$ iff $a\bar{b}$ is stretched and $K_j^{(r)} \models \alpha(a, \bar{b})$ [by (3), noting as in lemma 2.10 that the change from R to α does not affect stretchedness], iff $\nu_{r+1}(a)\bar{b}$ is stretched and $K_j^{(r)} \models \alpha(\mu(\nu_{r+1}(a)), \bar{b})$ [by definition of 'stretched' and because $\mu(\nu_{r+1}(a)) = a$], iff $K_j^{(r+1)} \models \alpha(\nu_{r+1}(a), \bar{b})$ [by (3)], as required.

Property (iv) holds vacuously since there is no admissible type t with $|t| > l$. To check property (v), let $a \in S_0$ and let $t = tp(a/K)$. By property (iii), $tp(\mu(a)/K) = t$. By lemma 2.9, t is admissible. ❑

Assume now that $j < l$ and that $K_j^{(r+1)}$ is defined. We will construct $K_{j+1}^{(r+1)}$ along the lines of section 2.1.2, which the reader may wish to review. Remember that types have parameters in $K^{(r)}$.

First, a definition. If $M \supseteq K^{(r)}$ and t is an admissible type, we write $M + t$ for the L-structure extending M obtained by adding to M a single new point b, say, and defining $M + t \models \alpha(b, \bar{c})$ iff $\alpha(x, \bar{c}) \in t$, for all atomic formulas $\alpha(x, \bar{y})$ and all $\bar{c} \in M$. Of course, as the parameters of t are in $K^{(r)}$, $M + t \models \alpha(b, \bar{c})$ implies $\bar{c} \in K^{(r)}$. $M + t$ is unique up to isomorphism. For types $t_1, \ldots, t_n$, we define $M + t_1 + \cdots + t_n$ to be $(\cdots((M + t_1) + t_2) + \cdots) + t_n$.

Let the admissible types of cardinality $l - j$ be $t_0, \ldots, t_{s-1}$, say, enumerated without repetitions. Let

$$\begin{aligned} m_i &= |\{a \in S_j : a \text{ realises } t_i \text{ in } K_j^{(r+1)}\}|, \qquad \text{for each } i < s, \\ m &= \max\{m_i : i < s\}, \\ K_{j+1}^{(r+1)} &= K_j^{(r+1)} + \underbrace{t_0 + t_0 + \cdots + t_0}_{m - m_0 \text{ times}} + \cdots + \underbrace{t_{s-1} + t_{s-1} + \cdots + t_{s-1}}_{m - m_{s-1} \text{ times}}, \end{aligned}$$

and define S_{j+1} to consist of S_j plus all the new points of $K_{j+1}^{(r+1)} \setminus K_j^{(r+1)}$. So $S_{j+1} = K_{j+1}^{(r+1)} \setminus K^{(r)}$. Define the components of $K_{j+1}^{(r+1)}$ to be $K_0, \ldots, K_r, S_{j+1}$.

LEMMA 2.13 *$K_{j+1}^{(r+1)} \supseteq K_j^{(r+1)}$, and $K_{j+1}^{(r+1)}$ satisfies properties (i)–(v).*

PROOF Clearly, $K_{j+1}^{(r+1)} \supseteq K_j^{(r+1)}$. Property (i) holds for $K_{j+1}^{(r+1)}$ by construction. For property (ii), assuming inductively that the property holds for $K_j^{(r+1)}$, we need only show that if $\bar{a}$ is live in $K_{j+1}^{(r+1)}$ and $rng(\bar{a}) \not\subseteq K_j^{(r+1)}$ then $\bar{a}$ is stretched. We assume for notational simplicity that $\bar{a} = b\bar{c}$ where $b \in S_{j+1} \setminus S_j$ and $\bar{c} \in K_{j+1}^{(r+1)}$, and that $K_{j+1}^{(r+1)} \models \alpha(b, \bar{c})$ for some atomic $\alpha(x, \bar{y})$ in which all variables in $x\bar{y}$ occur free. But then by definition of $K_{j+1}^{(r+1)}$, $\alpha(x, \bar{c}) \in t_i$ for some $i < s$. As t_i is admissible, lemma 2.10 yields that $\bar{c}$ is stretched in $K^{(r)}$. As $b \in S_{j+1}$, $b\bar{c}$ is clearly stretched in $K_{j+1}^{(r+1)}$.

Property (iii) is immediate, because it holds for the substructure $K_0^{(r+1)}$ of $K_{j+1}^{(r+1)}$.

We now check property (iv). First, note that if $i, i' < s$ and $i \neq i'$ then $|t_i| = |t_{i'}|$, $t_i \neq t_{i'}$, so that $t_{i'} \not\subseteq t_i$. Because by construction the elements of $S_{j+1} \setminus S_j$ realise exactly the formulas in some t_i, and no more, each of them realises exactly one type t_i (for some $i < s$). So, any point of S_{j+1} realising t_i is in fact in $K_j^{(r+1)} + t_i + t_i + \cdots + t_i$ (in the obvious sense). There are m_i such points in S_j, and we added $m - m_i$ more to S_{j+1}. So the number of realisations of t_i in S_{j+1}, for any $i < s$, is exactly m.

We must check that we have not destroyed the inductive hypothesis (for larger types than the t_i) by adding the new points. For any admissible type u with $|u| > l - j$, we have $u \not\subseteq t_i$ for any $i < s$. Because the points of $S_{j+1} \setminus S_j$ satisfy only the formulas of some unique t_i, none of them realise u. So all realisations of u in S_{j+1} are in fact in S_j. By property (iv) for $K_j^{(r+1)}$, the number of them depends only on $|u|$. Thus, property (iv) holds for all admissible types of size $> l - (j+1)$.

Property (v) is easily seen. Inductively, it holds for all $a \in S_j$. Let $a \in S_{j+1} \setminus S_j$, and suppose that a realises type t_i for (unique) $i < s$. Then $tp(a/K) = t_i$, which is admissible. ❑

We now define $K^{(r+1)} = K_l^{(r+1)}$ and $K_{r+1} = S_l$. As $K^{(r+1)} \supseteq K^{(r)}$, the inductive hypothesis and properties (i)–(iii) above imply that the first two conditions of proposition 2.5 hold and that the components of $K^{(r+1)}$ are as demanded.

2.2.4 Defining permutations of K_{r+1}

Here, we show how to take any extension of a map ψ in $\Psi^{(r)}$ to a partial isomorphism of $K^{(r+1)}$, and extend it to an element of $\Psi^{(r+1)}$. (Here, as for r, $\Psi^{(r+1)}$ is the set of all partial isomorphisms ψ' of $K^{(r+1)}$ such that $\psi' \restriction K_i$ is a permutation of K_i for each i with $1 \le i \le r+1$.) The crucial observation needed to do it is property (iv) of the construction of K_{r+1} — that in $K^{(r+1)}$, the number of realisations of any admissible type depends only on the cardinality of the type. This extension result will be used in section 2.2.5 to establish conditions 3 and 4 of proposition 2.5.

Fix $\psi \in \Psi^{(r)}$, and define:

$$\begin{array}{rrcl} & D & = & K \cap dom\,\psi, \\ & D' & = & \psi(D), \\ & \mathcal{A} & = & \{t : t \text{ an admissible type with } base\ t \subseteq D\}, \\ \text{and for } t \in \mathcal{A}, & E_t & = & \{a \in K_{r+1} : tp(a/D) = t\}, \\ \text{and} & E'_t & = & \{a \in K_{r+1} : tp(a/D') = \psi(t)\}. \end{array}$$

E_t is the set of elements of K_{r+1} whose type over D is Exactly t. The E_t ($t \in \mathcal{A}$) are pairwise disjoint, as are the E'_t. By property (v) of the construction of $K^{(r+1)}$, and lemma 2.11, $K_{r+1} = \bigcup_{t\in\mathcal{A}} E_t = \bigcup_{t\in\mathcal{A}} E'_t$.

LEMMA 2.14 *Let q be a 1–1 partial map $: K_{r+1} \to K_{r+1}$. Then in the notation above, the following are equivalent:*

1. *$\psi \cup q$ is a partial isomorphism of $K^{(r+1)}$.*

2. *Whenever $\alpha(x,\bar{y})$ is atomic, x occurs free in α, $a \in dom\,q$, and $\bar{c} \in dom(\psi)$, we have $K^{(r+1)} \models \alpha(a,\bar{c}) \leftrightarrow \alpha(q(a),\psi(\bar{c}))$.*

3. *$\psi(tp(a/D)) = tp(q(a)/D')$ for any $a \in dom\,q$.*

4. *For any $t \in \mathcal{A}$, $q \restriction E_t$ is a 1–1 partial map from E_t to E'_t.*

PROOF As ψ is a partial isomorphism of $K^{(r)}$, $K^{(r)} \subseteq K^{(r+1)}$, and all live tuples in $K^{(r+1)}$ are stretched, it is clear that $\psi \cup q$ is a partial isomorphism of $K^{(r+1)}$ iff condition 2 holds.

Now let $\alpha, a, \bar{c}$ be as in (2). Then $K^{(r+1)} \models \alpha(a,\bar{c})$ iff $\alpha(x,\bar{c}) \in tp(a/D)$, since $\bar{c} \in dom(\psi) = D \cup \widehat{K}^{(r)}$. Similarly, $K^{(r+1)} \models \alpha(q(a),\psi(\bar{c}))$ iff $\alpha(x,\psi(\bar{c})) \in tp(q(a)/D')$. We conclude that (2) holds iff $\alpha(x,\bar{c}) \in tp(a/D) \iff \alpha(x,\psi(\bar{c})) \in tp(q(a)/D')$ for all $a, \alpha, \bar{c}$ as above. But this just says that $\psi(tp(a/D)) = tp(q(a)/D')$ for all $a \in dom\,q$. Hence, (2) and (3) are equivalent.

For (3) $\Rightarrow$ (4), if $t \in \mathcal{A}$ and $a \in E_t \cap dom\,q$ then $tp(a/D) = t$, and assuming (3), we obtain $tp(q(a)/D') = \psi(t)$, so $q(a) \in E'_t$. Thus, (4) holds. For the converse, assume (4) and let $a \in dom\,q$. As $K_{r+1} = \bigcup_{t\in\mathcal{A}} E_t$, $a \in E_t$ for some (unique) $t \in \mathcal{A}$, so that $tp(a/D) = t$. By (4), $q(a) \in E'_t$ so $tp(q(a)/D') = \psi(t)$, giving (3). ❑

The following originated in [7] and is the heart of the matter.

LEMMA 2.15 *Let $t \in \mathcal{A}$. Then $|E_t| = |E'_t|$.*

PROOF For $t \in \mathcal{A}$ let

$$\begin{array}{rcl} R_t & = & \{a \in K_{r+1} : a \text{ realises } t \text{ in } K^{(r+1)}\}, \\ R'_t & = & \{a \in K_{r+1} : a \text{ realises } \psi(t) \text{ in } K^{(r+1)}\}. \end{array}$$

If $a \in K_{r+1}$, then $a \in R_t$ iff $tp(a/D) \supseteq t$, and because $tp(a/D)$ is by property (v) and lemma 2.11 admissible, this is iff $a \in E_u$ for some $u \in \mathcal{A}$ with $u \supseteq t$. Similarly, $a \in R'_t$ iff $a \in E'_u$ for some $u \in \mathcal{A}$ with $u \supseteq t$. We obtain

$$R_t = \bigcup_{t \subseteq u \in \mathcal{A}} E_u \qquad \text{and} \qquad R'_t = \bigcup_{t \subseteq u \in \mathcal{A}} E'_u, \qquad \text{for all } t \in \mathcal{A}.$$

As $\mathcal{A}$ is finite, $(\mathcal{A}, \supset)$ is well-founded; we now proceed by induction on it. Let $t \in \mathcal{A}$, and assume inductively that $|E_u| = |E'_u|$ for any $u \in \mathcal{A}$ with $u \supset t$. We show that $|E_t| = |E'_t|$. As the E_u for varying $u \in \mathcal{A}$ are pairwise disjoint, and similarly for the E'_u, we have

$$E_t = R_t \setminus \bigcup_{t \subset u \in \mathcal{A}} E_u, \qquad \text{and} \qquad E'_t = R'_t \setminus \bigcup_{t \subset u \in \mathcal{A}} E'_u.$$

Obviously, $|t| = |\psi(t)|$, and $\psi(t)$ is admissible (by lemma 2.8). So by construction of $K^{(r+1)}$, we have $|R_t| = |R'_t|$. With the inductive hypothesis, this gives:

$$|E_t| \;=\; |R_t| - \sum_{t \subset u \in \mathcal{A}} |E_u| \;=\; |R'_t| - \sum_{t \subset u \in \mathcal{A}} |E'_u| \;=\; |E'_t|.$$

This completes the induction and the proof. ❑

We now obtain:

PROPOSITION 2.16 *Let q be a 1–1 partial map $: K_{r+1} \to K_{r+1}$ such that $\psi \cup q$ is a partial isomorphism of $K^{(r+1)}$. Then $\psi \cup q$ extends to a partial isomorphism in $\Psi^{(r+1)}$.*

PROOF We know that the E_t are pairwise disjoint, as are the E'_t. For any type $t \in \mathcal{A}$, lemma 2.15 shows that $|E_t| = |E'_t|$, and by lemma 2.14, $q \restriction E_t$ is a partial 1–1 map $: E_t \to E'_t$. So for each t, we may extend $q \restriction E_t$ to a bijection $: E_t \to E'_t$. We let q^+ be the union of all these extensions. Since $K_{r+1} = \bigcup\{E_t : t \in \mathcal{A}\} = \bigcup\{E'_t : t \in \mathcal{A}\}$, q^+ is a well-defined permutation of K_{r+1}. By lemma 2.14, $\psi \cup q^+$ is a partial isomorphism of $K^{(r+1)}$, and so $\psi \cup q^+ \in \Psi^{(r+1)}$. ❑

This has a corollary:

COROLLARY 2.17 *Any $\psi \in \Psi^{(r)}$ extends to $\psi' \in \Psi^{(r+1)}$ such that $\nu_{r+1}\psi'(a) = \psi'\nu_{r+1}(a)$ for all $a \in K \cap dom\,\psi$ (that is, $\nu_{r+1} \circ \psi' \subseteq \psi' \circ \nu_{r+1}$).*

PROOF Let ψ be as in the formulation of the corollary. Define a partial one-one map $q : K_{r+1} \to K_{r+1}$ by

$$q = \nu_{r+1}\psi\nu_{r+1}^{-1}.$$

By proposition 2.16, we need only show that $\psi \cup q$ is a partial isomorphism of $K^{(r+1)}$. For this, it suffices by lemma 2.14 to prove that for any atomic $\alpha(x, \bar{y})$ in which x is free, and any $a \in dom\,q$ and $\bar{c} \in dom\,\psi$,

$$K^{(r+1)} \models \alpha(a, \bar{c}) \leftrightarrow \alpha(q(a), \psi(\bar{c})).$$

Well, take $a' \in K$ with $\nu_{r+1}(a') = a$. Note that $a' \in dom\,\psi$. Then $K^{(r+1)} \models \alpha(a, \bar{c})$ iff $K^{(r+1)} \models \alpha(a', \bar{c})$ (by property (iii)), iff $K^{(r+1)} \models \alpha(\psi(a'), \psi(\bar{c}))$ (as ψ is a partial isomorphism of $K^{(r+1)}$), iff $K^{(r+1)} \models \alpha(\nu_{r+1}\psi(a'), \psi(\bar{c}))$ (by property (iii) again), iff $K^{(r+1)} \models \alpha(q(a), \psi(\bar{c}))$ (by definition of q). ❑

2.2.5 Proving conditions 3 and 4 of proposition 2.5

We now obtain condition 3 of proposition 2.5 for $K^{(r+1)}$ by taking ψ in corollary 2.17 to be $p \cup \bigcup_{1 \le i \le r} p_i$, for any partial isomorphism p of K, where $p_1, \ldots, p_r$ are given inductively. Letting ψ' be as in the corollary, we may let $p_{r+1} = \psi' \restriction K_{r+1}$.

We also obtain the last condition (condition 4) of the proposition, in the manner sketched in section 2.1.3. Let $A \subseteq K^{(r+1)}$ be packed. We must find $\psi \in \Psi^{(r+1)}$ with $A \subseteq dom\,\psi$ and $\psi(A) \subseteq \overline{K}^{(r+1)}$.

If $A \subseteq K^{(r)}$, then by condition 4 for $K^{(r)}$, there is $\psi \in \Psi^{(r)}$ with $A \subseteq dom\,\psi$ and $\psi(A) \subseteq \overline{K}^{(r)}$; by corollary 2.17, ψ extends to $\psi' \in \Psi^{(r+1)}$, and clearly, $\psi'(A) \subseteq \overline{K}^{(r+1)}$. We are done.

So assume that $A \cap K_{r+1} \neq \emptyset$. As A is packed and live tuples in $K^{(r+1)}$ are stretched, we have $A \cap K_{r+1} = \{a\}$ for some a. Let

$$t = tp(a/K), \text{ and } D = base\, t.$$

By property (v), t is admissible, so there are $b \in K$, $E \subseteq K$, and $\psi \in \Psi^{(r)}$ with $D \subseteq dom\,\psi$ and $tp(b/E) = \psi(t)$. By restricting ψ and E, we may assume that $K \cap dom\,\psi = D$ and $E = base(tp(b/E)) = \psi(D)$. By property (iii), we have $tp(\nu_{r+1}(b)/E) = \psi(t)$, too. Let q be the partial map : $K_{r+1} \to K_{r+1}$ defined only on a and with $q(a) = \nu_{r+1}(b)$. Since $K \cap dom\,\psi = D$, lemma 2.14(3) applies, telling that $\psi \cup q$ is a partial isomorphism of $K^{(r+1)}$. By proposition 2.16, $\psi \cup q$ extends to $\psi' \in \Psi^{(r+1)}$ (which may not commute with ν_{r+1}).

Now $A \cap K \subseteq D$. To see this, let $d \in A \cap K$. As A is packed, there is atomic $\alpha(x,y,\bar{z})$ in which x, y occur, and $\bar{b} \in A \cap K^{(r)}$, with $K^{(r+1)} \models \alpha(a,d,\bar{b})$. Then $\alpha(x,d,\bar{b}) \in t$ and $d \in base\, t = D$, as required. Hence, $A \setminus \{a\} \subseteq dom\,\psi$, and we may define

$$B = \psi(A \setminus \{a\}) \cup \{b\} \subseteq K^{(r)}.$$

Then by property (iii), the map $\mu\psi' \restriction A : A \to B$ preserves all L-relations, so B is packed. So by proposition 2.5(4) for $K^{(r)}$, there is $\phi \in \Psi^{(r)}$ with $B \subseteq dom\,\phi$ and $\phi(B) \subseteq \overline{K}^{(r)}$. By corollary 2.17, ϕ extends to $\phi' \in \Psi^{(r+1)}$ such that $\nu_{r+1} \circ \phi' \subseteq \phi' \circ \nu_{r+1}$.

Now we have $\phi'\psi' \in \Psi^{(r+1)}$, $A \subseteq dom\,(\phi'\psi')$, and $\phi'\psi'(A) \subseteq \overline{K}^{(r+1)}$. The first statement is clear. For the others, we consider elements of A in turn. First consider a. We have $\phi'\psi'(a) = \phi'\nu_{r+1}(b)$. Because $b \in dom\,\phi$, we have $\nu_{r+1}\phi'(b) = \phi'\nu_{r+1}(b)$ by choice of ϕ'. Hence, $\phi'\psi'(a) = \nu_{r+1}\phi'(b) \in \overline{K}^{(r+1)}$. For $c \in A \setminus \{a\}$, we have $\phi'\psi'(c) \in \phi'(B) \subseteq \overline{K}^{(r)} \subseteq \overline{K}^{(r+1)}$. This completes the proof of condition 4 of proposition 2.5 for $K^{(r+1)}$.

We have now finished the induction on r and proved proposition 2.5. ❑

2.3 Proof of theorem 2.2

PROOF Let K, L, r be as in the formulation of the theorem. We assume $r > 0$, set $H = H_r(K) \stackrel{\text{def}}{=} \widehat{K}^{(r)}$ as in proposition 2.5, let $\overline{K}$ be the substructure of H with domain $\overline{K}^{(r)} \cap \widehat{K}^{(r)}$, and let $\pi \stackrel{\text{def}}{=} \bigcup_{1 \le i \le r} \nu_i^{-1} : \overline{K} \to K$. Then π is obviously surjective. To check that it is a homomorphism, we will show a little more: that

$$(\dagger) \qquad \text{if } \bar{a} \in \overline{K} \text{ is stretched then } K^{(r)} \models \alpha(\bar{a}) \leftrightarrow \alpha(\pi(\bar{a})) \text{ for all atomic } \alpha(\bar{x}).$$

We may assume that $\bar{a} = (a_0, \ldots, a_{n-1})$ with $a_j \in K_{i_j}$ for each $j < n$, where $0 < i_0 < i_1 < \cdots < i_{n-1} \le r$. Then for each $j < n$, we have $\nu_{i_j}\pi(a_j) = a_j$ and $a_0, \ldots, a_{j-1}, \pi(a_{j+1}), \ldots, \pi(a_{n-1}) \in \bigcup_{l < i_j} K_l$, so by proposition 2.5(2) we obtain

$$\begin{aligned}
K^{(r)} &\models \alpha(a_0,\ldots,a_{n-1}) \leftrightarrow \alpha(a_0,\ldots,a_{n-2},\pi(a_{n-1})),\\
K^{(r)} &\models \alpha(a_0,\ldots,a_{n-2},\pi(a_{n-1})) \leftrightarrow \alpha(a_0,\ldots,a_{n-3},\pi(a_{n-2}),\pi(a_{n-1})),\\
&\vdots\\
K^{(r)} &\models \alpha(a_0,\pi(a_1),\ldots,,\pi(a_{n-1})) \leftrightarrow \alpha(\pi(a_0),\ldots,\pi(a_{n-1})),
\end{aligned}$$

proving (†). Now, if $R \in L$ and $\overline{K} \models R(\bar{a})$ then $K^{(r)} \models R(\bar{a})$; by proposition 2.5(1), $\bar{a}$ must be stretched, so by (†), $K^{(r)} \models R(\pi(\bar{a}))$. As $K \subseteq K^{(r)}$, we get $K \models R(\pi(\bar{a}))$, and hence π is a homomorphism.

We check the other requirements of the theorem.

1. If p is a partial isomorphism of K, let $\widehat{p} = p_1 \cup \ldots \cup p_r$ as in proposition 2.5(3). Then $\widehat{p} \in Aut\,H$. Let $a \in \overline{K}$ with $\pi(a) \in dom\,p$. We require $p\pi(a) = \pi\widehat{p}(a)$. Say, $a \in K_i$ (some i, $1 \leq i \leq r$). Then by the proposition again, $\nu_i p\pi(a) = \widehat{p}\nu_i(\pi(a)) = \widehat{p}(a)$. Since ν_i is one-one, $p\pi(a) = \nu_i^{-1}\widehat{p}(a) = \pi\widehat{p}(a)$.

2. This is immediate from proposition 2.5(4).

3. Take a universal formula $\theta(\bar{x}) = \forall\bar{y}\varphi(\bar{x},\bar{y})$, where φ is quantifier-free, $\bar{x}\bar{y}$ is non-repeating, $|\bar{x}| = n$, $|\bar{y}| = m$, say, and $n+m \leq r$. Take $\bar{a} = (a_0,\ldots,a_{n-1}) \in \overline{K}$ with $rng(\bar{a})$ contained in a packed subset of $\widehat{K}^{(r)}$ and such that $\pi\restriction rng(\bar{a})$ is one-one, and with $\overline{K} \models \theta(\bar{a})$. Write $\bar{b}$ for $\pi(\bar{a})$, so $b_i = \pi(a_i)$ for $i < n$. We show $K \models \theta(\bar{b})$.

   Let $\bar{c} = (c_0,\ldots,c_{m-1}) \in K$ be any m-tuple; we require $K \models \varphi(\bar{b},\bar{c})$. Choose $1 \leq l_0,\ldots,l_{m-1} \leq r$ satisfying, for all $i, j < m$ and $k < n$:

   - if $c_i = b_k$ then $a_k \in K_{l_i}$,
   - if $c_i \notin rng(\bar{b})$ then $K_{l_i} \cap rng(\bar{a}) = \emptyset$,
   - if $c_i = c_j$ then $l_i = l_j$,
   - if $c_i \neq c_j$ and $c_i, c_j \notin rng(\bar{b})$ then $l_i \neq l_j$.

   This is possible since $\pi\restriction rng(\bar{a})$ is one-one and $n+m \leq r$. Let $d_i = \nu_{l_i}(c_i)$ for $i < m$, and $\bar{d} = (d_0, \ldots, d_{m-1})$.

   Claim. The map $\pi\restriction rng(\bar{a}\bar{d})$ preserves all quantifier-free formulas.

   Proof of claim. Because π is one-one on $rng(\bar{a})$, by choice of the l_i it is one-one on $rng(\bar{a}\bar{d})$ too, so preserves all equalities and inequalities. Further, because $\bar{a}$ is contained in a packed subset of $\widehat{K}^{(r)}$, it is stretched in $K^{(r)}$; by choice of the l_i, it follows that $\bar{a}\bar{d}$ is also stretched. By (†) above, $\pi\restriction rng(\bar{a}\bar{d})$ preserves all atomic relations in both directions. This proves the claim.

   But $\overline{K} \models \theta(\bar{a})$, so $\overline{K} \models \varphi(\bar{a},\bar{d})$. Since φ is quantifier-free, and by definition $\pi(\bar{a}\bar{d}) = \bar{b}\bar{c}$, the claim gives $K \models \varphi(\bar{b},\bar{c})$, as required.

This completes the proof of theorem 2.2. ❑

3 Loosely guarded and packed fragments

Here we apply the combinatorics of section 2, together with some simple, tedious syntactic manipulations, to prove the finite model property for the loosely guarded and packed fragments. Convention 2.4 is no longer in force, so that 'atomic' formulas may be equalities; but notations $\varphi(\bar{x})$ still indicate by default that $\bar{x}$ is non-repeating and contains the free variables of φ.

3.1 The loosely guarded and packed fragments

We begin by recalling the definition of the loosely guarded fragment.

DEFINITION 3.1 (VAN BENTHEM, [24, p. 9]) *Let L be a signature without function symbols. The* loosely guarded fragment $\mathsf{LGF}(L)$ *over L consists of (just) the following kinds of L-formula:*

- *Any atomic L-formula is in $\mathsf{LGF}(L)$.*
- *$\mathsf{LGF}(L)$ is closed under Boolean combinations.*
- *If*
  - *γ (the 'guard') is a conjunction of atomic L-formulas,*[1]
  - *$\varphi \in \mathsf{LGF}(L)$,*
  - *every free variable of φ is free in γ,*
  - *$\bar{y}$ is a tuple of free variables of γ,*
  - *if x is a free variable of γ, y is a variable from $\bar{y}$, and $x \neq y$,*[2] *then there is a conjunct of γ in which x, y both occur,*

  then $\exists\bar{y}(\gamma \wedge \varphi) \in \mathsf{LGF}(L)$.

Note that γ may have more free variables than φ, and more than two variables may be 'guarded' by a single conjunct of γ. The reader may check that the standard translation of $U(p,q)$ (formula (2) of section 1) is loosely guarded.

We will also prove the finite model property for the 'packed fragment'.

DEFINITION 3.2 (MARX, [19]) *Let L be a signature without function symbols. An L-formula γ is said to be a* packing guard *if γ is a conjunction of atomic and existentially-quantified atomic formulas such that for any distinct free variables x, y of γ, there is a conjunct of γ in which x, y both occur free.*

The packed fragment $\mathsf{PF}(L)$ *consists of (just) the following formulas:*

- *Any atomic L-formula is in $\mathsf{PF}(L)$.*
- *$\mathsf{PF}(L)$ is closed under boolean combinations.*
- *If the L-formula γ is a packing guard, $\varphi \in \mathsf{PF}(L)$, every free variable of φ is free in γ, and $\bar{y}$ is a tuple of free variables of γ, then $\exists\bar{y}(\gamma \wedge \varphi) \in \mathsf{PF}(L)$.*[3]

$\mathsf{LGF}(L)$ is not a subfragment of $\mathsf{PF}(L)$ because guards γ of $\mathsf{PF}(L)$ must bind *every* pair of free variables of φ. For example, the standard translation of $U(p,q)$ (formula (2) of section 1) is loosely guarded but not packed. However, as we will see in lemma 3.9, every $\mathsf{LGF}(L)$-sentence is equivalent to a sentence of $\mathsf{PF}(L)$. An example, due to Marx, of a packed sentence which is not equivalent to a loosely guarded sentence is $\exists xyz(\exists wR(x,y,w) \wedge \exists wR(x,z,w) \wedge \exists wR(z,y,w) \wedge \neg R(x,y,z))$.

Note that if γ is a quantifier-free packing guard with free variables precisely $x_0, \ldots, x_{n-1}$, M is an L-structure, $a_0, \ldots, a_{n-1} \in M$, and $M \models \gamma(a_0, \ldots, a_{n-1})$, then $\{a_0, \ldots, a_{n-1}\}$ is packed in M.

[1] The original definition [24] did not allow equalities in guards, but their presence does not affect the expressive power or the finite model property so we include them.

[2] [5, definition 2.2] omits the restriction $x \neq y$; this does not reduce the expressive power if equalities are allowed as conjuncts of γ.

[3] [19] requires that γ and φ have exactly the same free variables. We do not need this restriction, but adding it does not reduce the expressive power since we may add conjuncts $x = x$ to φ until its free variables are the same as γ's. Also, [19] does not allow existentially-quantified equalities in guards, and not all variables in $\bar{y}$ need be free in γ; it is plain that these differences do not change the expressive power either.

Packed subformulas It is convenient to introduce the notion of *packed subformula* of a formula $\varphi \in \mathsf{PF}(L)$. This is done by induction on the construction of φ. If φ is atomic then φ is a packed subformula of itself. The packed subformulas of a boolean combination φ of formulas in $\mathsf{PF}(L)$ are φ and the packed subformulas of the combinants. And if $\varphi \in \mathsf{PF}(L)$ and γ is a packing guard, then the packed subformulas of $\exists\bar{y}(\gamma \wedge \varphi)$ are itself and those of φ.

3.2 Finite model property

In section 3.3 we will prove:

THEOREM 3.3 *For any relational signature L, the packed fragment* $\mathsf{PF}(L)$ *has the finite model property.*

This easily gives the following, which will be proved in section 3.4:

COROLLARY 3.4 *For any relational signature L, the loosely guarded fragment* $\mathsf{LGF}(L)$ *has the finite model property.*

REMARK 3.5 It is not hard to see that we can add constants to L and keep the finite model property for $\mathsf{PF}(L)$ and $\mathsf{LGF}(L)$. (Note that guards need not guard constants, so, e.g., $\forall x(x = x \to R(c,x))$ is a loosely guarded sentence.) However, the loosely guarded fragment in signatures with function symbols is undecidable (this follows from results in [20]) and therefore does not have the finite model property.

3.3 Proof of the finite model property for the packed fragment

To prove theorem 3.3, it clearly suffices to show that for any *finite* relational signature L, any sentence of $\mathsf{PF}(L)$ with a model has a finite model. Fix such an L. The proof proceeds in two stages: first, for sentences with only quantifier-free packing guards, and then for arbitrary sentences. The first stage is done in two lemmas: first, assuming that the sentence has an irreflexive model, and then the general case.

DEFINITION 3.6 An L-structure M is said to be *irreflexive* if whenever $R \in L$, $a_1, \ldots, a_n \in M$, and $M \models R(a_1, \ldots, a_n)$, then $a_1, \ldots, a_n$ are distinct.

LEMMA 3.7 *Suppose that* $\sigma \in \mathsf{PF}(L)$*, all guards in* σ *are quantifier-free, and* σ *has an irreflexive model* M*. Then* σ *has a finite model.*

PROOF Assume the hypotheses. Let σ be written with variables $v_0, \ldots, v_{r-1}$ only, where $0 < r < \omega$, and write $\bar{v}$ for $(v_0, \ldots, v_{r-1})$. For any formula φ written with variables $\bar{v}$, and any r-tuple $\bar{a} \in M$, we will write $M \models \varphi(\bar{a})$ to mean that φ is true in M when v_i is assigned to a_i for each $i < r$.

Introduce a new r-ary relation symbol R_φ for each packed subformula φ of σ, and define an expansion M^+ of M by interpreting each R_φ equivalently in M to φ:

$$M^+ \models R_\varphi(a_0, \ldots, a_{r-1}) \iff M \models \varphi(a_0, \ldots, a_{r-1}),$$

for any $a_0, \ldots, a_{r-1} \in M$. Of course, if some v_i is not free in φ then we can change a_i without changing the truth of φ or R_φ; and if φ is a sentence then R_φ will hold for all r-tuples in M, or for none. M^+ is in general not irreflexive. Take a finite substructure $K \subseteq M^+$ containing an isomorphic copy of every substructure $A \subseteq M^+$ with $|A| \leq 2r$. Let $H = H_r(K)$, $\overline{K}$, and π be as in theorem 2.2. $K, \overline{K}, H$ are finite structures in the expanded language. We will show that $H \models \sigma$.

Claim. For every packed subformula φ of σ and any r-tuple $\bar{a} \in H$ that is packed in the L-reduct $H \restriction L$ of H, we have $H \models R_\varphi(\bar{a}) \leftrightarrow \varphi(\bar{a})$.

Proof of claim. First note that because $\bar{a}$ is packed in $H \restriction L$ and hence in H, by theorem 2.2(2) there is $g \in Aut\, H$ with $g(\bar{a}) \in \overline{K}$. Since automorphisms preserve all first-order formulas, we may assume that $\bar{a} \in \overline{K}$. Next, note that because M is irreflexive and π is a homomorphism, if $\bar{a} \in \overline{K}$ is packed in $H \restriction L$ then $\pi \restriction rng(\bar{a})$ is one-one. So by theorem 2.2(3), $\pi \restriction rng(\bar{a})$ preserves forwards all 'small' prenex universal formulas, and all quantifier-free formulas.

We prove the claim by induction on φ. For atomic φ it is clear, since $K \models \varphi(\pi(\bar{a})) \leftrightarrow R_\varphi(\pi(\bar{a}))$ and $\pi \restriction rng(\bar{a})$ preserves quantifier-free formulas. For conjunction, inductively assume the claim for θ, φ, and let $\bar{a} \in \overline{K}$ be packed in $H \restriction L$. Then inductively, $H \models (\theta \wedge \varphi)(\bar{a})$ iff $H \models R_\theta(\bar{a}) \wedge R_\varphi(\bar{a})$. Since $\pi \restriction rng(\bar{a})$ preserves quantifier-free formulas, this is iff $M^+ \models R_\theta(\pi(\bar{a})) \wedge R_\varphi(\pi(\bar{a}))$. By definition of M^+ this is iff $M \models (\theta \wedge \varphi)(\pi(\bar{a}))$, iff $M^+ \models R_{\theta\wedge\varphi}(\pi(\bar{a}))$. Again using preservation, this is iff $H \models R_{\theta\wedge\varphi}(\bar{a})$, as required. Negation is handled similarly.

Now consider the case $\theta(\bar{x}) = \exists\bar{y}(\gamma(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{y}))$, where $\varphi \in \mathsf{PF}(L)$, γ is a quantifier-free packing guard, $\bar{x}, \bar{x}\bar{y}$ are non-repeating tuples of variables from $\bar{v}$ enumerating the free variables of θ, γ respectively, and $\bar{y}$ is non-empty. Let $\bar{z}$ enumerate the variables of $\bar{v}$ that are not in $\bar{x}\bar{y}$, if any, and let y be any variable in $\bar{y}$. We will use the following notation: if $\bar{w} = (v_{i_0}, \ldots, v_{i_{k-1}})$ is a tuple of variables from $\bar{v} = (v_0, \ldots, v_{r-1})$, and $\bar{a} = (a_0, \ldots, a_{r-1})$ is any r-tuple, we write $\bar{a}_{\bar{w}}$ for the tuple $(a_{i_0}, \ldots, a_{i_{k-1}})$, corresponding to $\bar{a}$ as $\bar{w}$ does to $\bar{v}$. Similarly, we write $\bar{a}_{v_i}$ for a_i.

Let $\bar{a} \in \overline{K}$ be packed in $H \restriction L$, and inductively assume the claim for φ.

$\Rightarrow$ First assume that $H \models R_\theta(\bar{a})$; we show $H \models \theta(\bar{a})$. Write $\bar{b}$ for $\pi(\bar{a})$. As π is a homomorphism, $M^+ \models R_\theta(\bar{b})$. By definition of R_θ in M^+, $M \models \theta(\bar{b})$, so there is an r-tuple $\bar{c} \in M$ with $\bar{c}_{\bar{x}} = \bar{b}_{\bar{x}}$ and $M \models (\gamma \wedge \varphi)(\bar{c})$, and thus, $M^+ \models \gamma(\bar{c}) \wedge R_\varphi(\bar{c})$. Since the variables of $\bar{z}$ are not free in $\gamma \wedge \varphi$, we may assume that each element of $\bar{c}_{\bar{z}}$ is equal to $\bar{c}_y$.

As $|rng(\bar{b}\bar{c})| \leq 2r$, there is $\bar{b}'\bar{c}' \in K$ isomorphic to $\bar{b}\bar{c}$. As γ, R_φ are quantifier-free, $K \models \gamma(\bar{c}') \wedge R_\varphi(\bar{c}')$. Let $\chi(\bar{x})$ be the prenex existential formula

$$\exists\bar{y}\bar{z}\Big(\gamma \wedge \Big(\bigwedge_{z \text{ in } \bar{z}} z = y\Big) \wedge R_\varphi(v_0, \ldots, v_{r-1})\Big).$$

Then clearly, $K \models \chi(\bar{b}'_{\bar{x}})$.

Let $p : \bar{b} \mapsto \bar{b}'$, a partial isomorphism of K, and take $\widehat{p} \in Aut\, H$ satisfying the provisions of theorem 2.2(1). Letting $\bar{a}' = \widehat{p}(\bar{a})$, it is clear that $\bar{a}'$ is packed in $H \restriction L$ and so $\pi \restriction rng\, \bar{a}'$ is one-one, and $\pi(\bar{a}') = \bar{b}'$. Since $|\bar{x}\bar{y}\bar{z}| \leq r$, by theorem 2.2(3) $\pi \restriction rng\, \bar{a}'_{\bar{x}}$ preserves $\neg\chi$ forwards, and hence $\overline{K} \models \chi(\bar{a}'_{\bar{x}})$. As $\overline{K} \subseteq H$ and χ is existential, $H \models \chi(\bar{a}'_{\bar{x}})$. As $\widehat{p} \in Aut\, H$ and $\widehat{p}(\bar{a}) = \bar{a}'$, $H \models \chi(\bar{a}_{\bar{x}})$. So there is an r-tuple $\bar{d} \in H$ with $\bar{d}_{\bar{x}} = \bar{a}_{\bar{x}}$, each element of $\bar{d}_{\bar{z}}$ equal to $\bar{d}_y$, and with $H \models \gamma(\bar{d}) \wedge R_\varphi(\bar{d})$. By the form of γ and because $rng(\bar{d}) = rng(\bar{d}_{\bar{x}\bar{y}})$, $\bar{d}$ is packed in $H \restriction L$,[4] so inductively, $H \models \gamma(\bar{d}) \wedge \varphi(\bar{d})$. Thus, $H \models \theta(\bar{a})$ by definition of θ.

$\Leftarrow$ Conversely, suppose that $H \models \theta(\bar{a})$; we require $H \models R_\theta(\bar{a})$. There is an r-tuple $\bar{b} \in H$ with $\bar{b}_{\bar{x}} = \bar{a}_{\bar{x}}$ and $H \models (\gamma \wedge \varphi)(\bar{b})$. As before, we may assume that each element of $\bar{b}_{\bar{z}}$ is equal to $\bar{b}_y$. By the form of γ, $\bar{b}$ is packed in $H \restriction L$ (see footnote 4), and hence in H. So by theorem 2.2(2) there is

[4] This fails in the loosely guarded fragment, which is why we use the packed fragment with quantifier-free packing guards.

$g \in Aut\, H$ with $g(\bar{b}) \in \overline{K}$. Write $\bar{c}$ for $g(\bar{b})$. Inductively, $H \models \gamma(\bar{b}) \wedge R_\varphi(\bar{b})$, so $H \models \gamma(\bar{c}) \wedge R_\varphi(\bar{c})$. Writing $\bar{d} = \pi(\bar{c})$, because π is a homomorphism we have $M^+ \models \gamma(\bar{d}) \wedge R_\varphi(\bar{d})$. By definition of relations in M^+, we have $M \models \varphi(\bar{d})$. Hence, certainly, $M \models \theta(\bar{d})$, so $M^+ \models R_\theta(\bar{d})$ by definition of M^+.

There are now two cases. If θ is a sentence, then every r-tuple is related by R_θ in M^+. In particular, $K \models R_\theta(\pi(\bar{a}))$. By theorem 2.2(3), $\overline{K} \models R_\theta(\bar{a})$, and so $H \models R_\theta(\bar{a})$ as required.

Assume otherwise, so $\bar{x}$ is non-empty. Choose x in $\bar{x}$; for any r-tuple $\bar{w}$, write $\bar{w}'$ for the r-tuple given by $\bar{w}'_{\bar{x}} = \bar{w}_{\bar{x}}$, and $\bar{w}'_u = \bar{w}_x$ for every u in $\bar{y}\bar{z}$. Note that

$$(*) \qquad \overline{K} \models R_\theta(\bar{m}) \leftrightarrow R_\theta(\bar{m}') \text{ for any } r\text{-tuple } \bar{m} \in \overline{K} \text{ that is packed in } H \restriction L.$$

$(*)$ follows because $\bar{m}$ is packed so $\pi \restriction rng(\bar{m})$ preserves the formula $R_\theta(\bar{v}) \leftrightarrow R_\theta(\bar{v}')$, and $K \models \forall \bar{v}(R_\theta(\bar{v}) \leftrightarrow R_\theta(\bar{v}'))$ because $\bar{v}, \bar{v}'$ agree on the free variables of θ.

We know $M^+ \models R_\theta(\bar{d})$, and as $\bar{c}$ is certainly packed in $H \restriction L$, we obtain $\overline{K} \models R_\theta(\bar{c})$. By $(*)$, $\overline{K} \models R_\theta(\bar{c}')$. As $g \in Aut\, H$ and (clearly) $g(\bar{a}') = \bar{c}'$, we have $\overline{K} \models R_\theta(\bar{a}')$. By $(*)$ again, $\overline{K} \models R_\theta(\bar{a})$, so $H \models R_\theta(\bar{a})$ as required. This proves the claim.

Take any packed r-tuple $\bar{a} \in \overline{K}$ (for example, a tuple of equal elements). Since $M \models \sigma$, we have $M \models R_\sigma(\pi(\bar{a}))$. We now obtain $H \models R_\sigma(\bar{a})$ by theorem 2.2(3), and thus, by the claim, $H \models \sigma$. We have found a finite model of σ, completing the proof of the lemma. ❑

LEMMA 3.8 *Let σ be a sentence of $\mathsf{PF}(L)$ with only quantifier-free guards, and suppose that σ has a model, say M. Then σ has a finite model.*

PROOF We make M into an irreflexive structure, adjust σ accordingly, and apply the preceding lemma. This will show that σ has a finite model.

For each n-ary $R \in L$ and each equivalence relation ε on n, with k equivalence classes, say, introduce a new k-ary relation symbol R_ε, and define an expansion $M^\sharp$ of M interpreting the new symbols as follows. Let $0 = e_0 < e_1 < \cdots e_{k-1} < n$ be representatives of the ε-classes, each being minimal in its ε-class. We call the e_i the canonical representatives of ε. For distinct elements $b_0, \ldots, b_{k-1} \in M$, define $a_0, \ldots, a_{n-1} \in M$ by $a_i = b_j$ iff $i \,\varepsilon\, e_j$, for each $i < n$, $j < k$, and define

$$M^\sharp \models R_\varepsilon(b_0, \ldots, b_{k-1}) \text{ iff } M \models R(a_0, \ldots, a_{n-1}).$$

For example, if $n = 5$ and the ε-classes are $\{0,2\}, \{1,3\}, \{4\}$, then $e_0 = 0$, $e_1 = 1$, $e_2 = 4$, and if $a, b, c \in M$ are distinct, we define $M^\sharp \models R_\varepsilon(a,b,c)$ iff $M \models R(a,b,a,b,c)$.

Write $M^{\neq}$ for the reduct of $M^\sharp$ to the new relation symbols R_ε (for all n-ary $R \in L$ and equivalence relations ε on n). Then $M^{\neq}$ is irreflexive.

Write $\mathcal{E}_n$ for the set of all equivalence relations on n. For each atomic L-formula of the form $R(\bar{x})$, for n-ary $R \in L$ and any n-tuple $\bar{x} = (x_0, \ldots, x_{n-1})$ of variables, perhaps with repetitions, and for $\varepsilon \in \mathcal{E}_n$ with canonical representatives $e_0, \ldots, e_{k-1}$, define $R(\bar{x})^\varepsilon$ to be the following formula, with the same free variables $\bar{x}$:

$$\Big(\bigwedge_{i \varepsilon j} x_i = x_j\Big) \wedge \bigwedge \Big\{ R_\varepsilon(x_{i_0}, \ldots, x_{i_{k-1}}) \;:\; i_0, \ldots, i_{k-1} < n,\;\; i_l \,\varepsilon\, e_l \text{ for each } l < k \Big\}.$$

Now define $R(\bar{x})^{\neq} = \bigvee_{\varepsilon \in \mathcal{E}_n} R(\bar{x})^\varepsilon$. Observe that for any $h : \{x_0, \ldots, x_{n-1}\} \to M$, if $h(x_i) = a_i$ $(i < n)$, $\bar{a} = (a_0, \ldots, a_{n-1})$, and $\eta \in \mathcal{E}_n$ is defined by $i \,\eta\, j$ iff $a_i = a_j$, then $M^\sharp \models R(\bar{a}) \leftrightarrow R(\bar{x})^\eta(\bar{a})$. (That is,

$R(\bar{x}) \leftrightarrow R(\bar{x})^{\eta}$ is true in $M^{\sharp}$ under assignment h.) Also, for any $\varepsilon \in \mathcal{E}_n$, if $M^{\neq} \models R(\bar{x})^{\varepsilon}(\bar{a})$ then because $M^{\neq}$ is irreflexive, $\varepsilon = \eta$. Hence, $M^{\sharp} \models \forall\bar{x}(R(\bar{x}) \leftrightarrow R(\bar{x})^{\neq})$.

For any formula $\varphi \in \mathsf{PF}(L)$ with quantifier-free guards, let $\varphi^{\neq}$ be obtained from φ by replacing every atomic subformula $R(\bar{x})$ of φ (for $R \in L$) by $R(\bar{x})^{\neq}$ — this includes both packed subformulas and formulas occurring in guards.

Claim. $\varphi^{\neq}$ is logically equivalent to a packed formula with quantifier-free guards.
Proof of claim. This can be seen by induction on φ. We sketch the argument. The case of atomic φ is easy, as are the boolean cases. For formulas $\exists\bar{y}(\gamma(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{y}))$, we observe that for each $R \in L$ and any $\bar{x}$, every pair of free variables of $R(\bar{x})$ also occur free in some conjunct of each disjunct $R(\bar{x})^{\varepsilon}$ of $R(\bar{x})^{\neq}$. Hence, by distributing the conjunctions from γ over the disjunctions in the $R(\bar{x})^{\neq}$, we see that $\gamma^{\neq}(\bar{x},\bar{y})$ can be put in disjunctive normal form $\bigvee_i \gamma_i(\bar{x},\bar{y})$ such that each pair of distinct variables of $\bar{x}\bar{y}$ occurs in a conjunct of each γ_i. So $\exists\bar{y}(\gamma(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{y}))^{\neq}$ is equivalent to $\bigvee_i \exists\bar{y}(\gamma_i(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{y})^{\neq})$, which inductively is equivalent to a packed formula with quantifier-free guards. This proves the claim.

We can now rapidly conclude the proof. Since for all R, $R(\bar{x})$ and $R(\bar{x})^{\neq}$ are equivalent in $M^{\sharp}$, and $M \models \sigma$, we have $M^{\neq} \models \sigma^{\neq}$. As $M^{\neq}$ is irreflexive, by the claim and lemma 3.7 there is a finite model $N \models \sigma^{\neq}$. Now make N an L-structure $N^{\sharp}$ by interpreting each $R(\bar{x})$ as $R(\bar{x})^{\neq}$. More formally, for n-ary $R \in L$, take an n-tuple $\bar{y}$ of distinct variables and let $N^{\sharp} \models R(\bar{a})$ iff $N \models R(\bar{y})^{\neq}(\bar{a})$, for any n-tuple $\bar{a} \in N$. It is easily checked that if $\bar{x}$ is any n-tuple of variables, then $N^{\sharp} \models \forall\bar{x}(R(\bar{x}) \leftrightarrow R(\bar{x})^{\neq})$. So clearly, $N^{\sharp} \models \sigma$. ❑

We can now prove theorem 3.3.

PROOF Let σ be a sentence of $\mathsf{PF}(L)$, and let M be a model of σ. For each existential conjunct $\beta(\bar{z}) = \exists\bar{v}\alpha(\bar{z},\bar{v})$ of each guard in σ, where $\alpha(\bar{z},\bar{v})$ is atomic with free variables just those in $\bar{z}\bar{v}$, introduce a new $|\bar{z}|$-ary relation symbol $R_{\beta(\bar{z})}$ and interpret it in M in the same way as $\beta(\bar{z})$: i.e., let $M \models R_{\beta(\bar{z})}(\bar{a})$ iff $M \models \beta(\bar{a})$, for all $\bar{a} \in M$ of length $|\bar{z}|$. We continue to write M for this definitional expansion. Notice that $M \models \chi_{\beta(\bar{z})}$, where

$$\chi_{\beta(\bar{z})} = \forall\bar{z}(R_{\beta(\bar{z})}(\bar{z}) \to \exists\bar{v}(\alpha(\bar{z},\bar{v}) \wedge \top)) \wedge \forall\bar{z}\bar{v}(\alpha(\bar{z},\bar{v}) \to R_{\beta(\bar{z})}(\bar{z})).$$

Of course, $\chi_{\beta(\bar{z})}$ is logically equivalent to $\forall\bar{z}(R_{\beta(\bar{z})}(\bar{z}) \leftrightarrow \beta(\bar{z}))$, but we write it in the above form to obtain a sentence of the packed fragment with quantifier-free guards. Let σ' be the result of replacing each existential conjunct $\beta(\bar{z})$ of each guard of each subformula of σ by $R_{\beta(\bar{z})}(\bar{z})$. Let σ'' be the conjunction of σ' and all the sentences $\chi_{\beta(\bar{z})}$, for each existential conjunct $\beta(\bar{z})$ of each guard in σ. Then σ'' is a sentence of the packed fragment with only quantifier-free guards, and $M \models \sigma''$.

By lemma 3.8, there is a finite model $N \models \sigma''$. For each β as above, $N \models \chi_{\beta(\bar{z})}$, so $R_{\beta(\bar{z})}(\bar{z})$ is equivalent in N to $\beta(\bar{z})$. So it is clear that $N \models \sigma$. ❑

3.4 Finite model property for the loosely guarded fragment

We can easily derive this from the finite model property for the packed fragment.

LEMMA 3.9 *Let L be a relational signature, let the L-formula δ be a packing guard (definition 3.2) with free variables precisely $\bar{v}$, and let $\varphi(\bar{v}) \in \mathsf{LGF}(L)$. Then $\delta \wedge \varphi$ is logically equivalent to $\delta \wedge \varphi'$ for some formula $\varphi'(\bar{v}) \in \mathsf{PF}(L)$.*

PROOF By induction on φ. If φ is atomic, we let $\varphi' = \varphi$. Assuming the result for φ, ψ, we let $(\neg\varphi)' = \neg\varphi'$ and $(\varphi \wedge \psi)' = \varphi' \wedge \psi'$, as usual. Then $\delta \wedge \neg\varphi$ is equivalent to $\delta \wedge \neg(\delta \wedge \varphi)$, which is equivalent to $\delta \wedge \neg(\delta \wedge \varphi')$ and to $\delta \wedge \neg\varphi' = \delta \wedge (\neg\varphi)'$; the case of $\wedge$ is similar.

Finally let $\varphi(\bar{x}) = \exists\bar{y}(\gamma(\bar{x},\bar{y}) \wedge \psi(\bar{x},\bar{y}))$ be a loosely guarded L-formula, where the free variables of γ are precisely $\bar{x}\bar{y}$, and $rng(\bar{x}) \subseteq rng(\bar{v})$. Assume the result for ψ, and consider $\delta(\bar{v}) \wedge \varphi(\bar{x})$. For each pair of distinct variables x, x' from $\bar{x}$, pick a conjunct $\chi_{x,x'}(x,x',\bar{z})$ of δ in which they both occur free, and let $\chi^{\circ}_{x,x'}(x,x') = \exists\bar{z}\chi_{x,x'}$. Let $\delta^{\circ}(\bar{x}) = \bigwedge_{x,x'}\chi^{\circ}_{x,x'}$. Then $\delta \vdash \delta^{\circ}$, and $\delta \wedge \varphi$ is equivalent to $\delta \wedge \exists\bar{y}(\gamma(\bar{x},\bar{y}) \wedge \delta^{\circ}(\bar{x}) \wedge \psi(\bar{x},\bar{y}))$. Now $\gamma \wedge \delta^{\circ}$ is a packing guard with free variables $\bar{x}\bar{y}$, so inductively, $\gamma \wedge \delta^{\circ} \wedge \psi(\bar{x},\bar{y})$ is equivalent to $\gamma \wedge \delta^{\circ} \wedge \psi'$ for some $\psi'(\bar{x},\bar{y}) \in \mathsf{PF}(L)$. Then $\delta \wedge \varphi$ is equivalent to $\delta \wedge \exists\bar{y}(\gamma \wedge \delta^{\circ} \wedge \psi')$; the second conjunct here is in $\mathsf{PF}(L)$ and its free variables are all from $\bar{v}$. ❑

PROOF OF COROLLARY 3.4 It is immediate from the lemma that any sentence of $\mathsf{LGF}(L)$ is logically equivalent to a sentence of $\mathsf{PF}(L)$. The corollary now follows from theorem 3.3. ❑

4 Applications

We end by outlining some applications of our results.

4.1 Decidable fragments of predicate temporal logic

In [14], certain decidable fragments of predicate temporal logic with Until and Since (the 'monodic' fragments) were introduced. The idea is to restrict the predicate part of the logic to a known decidable fragment of first-order logic (such as the loosely guarded fragment) and restrict temporal operations to formulas with at most one free variable. Monodic fragments are decidable over a wide range of linear flows of time. Moreover, if it is decidable whether a sentence of the chosen first-order fragment (roughly speaking) has a finite model, the corresponding monodic temporal logic with finite domains is also decidable.

Since the loosely guarded fragment is decidable and has the finite model property, it is decidable whether a sentence of the loosely guarded fragment has a finite model, and thus the loosely guarded monodic fragment of predicate temporal logic and finite domains is decidable.

4.2 Finite base property in algebraic logic

The 'finite algebra on finite base property' for weakly associative algebras [17] follows easily from the finite model property for the loosely guarded fragment. Let $\mathcal{A} = (A, +, -, 0, 1, 1', \breve{}, ;)$ be a finite weakly associative algebra. Regard each $a \in A$ as a binary relation symbol. Then a *relativised representation* of $\mathcal{A}$ is a model of the following theory:

$$\begin{array}{rl}
\forall xy[1'(x,y) \leftrightarrow x = y] & \\
\forall xy[r(x,y) \leftrightarrow s(x,y) \vee t(x,y)] & \text{for each } r,s,t \in A \text{ with } r = s+t \\
\forall xy[1(x,y) \to (r(x,y) \leftrightarrow \neg s(x,y))] & \text{for each } r,s \in A \text{ with } r = -s \\
\forall xy[r(x,y) \leftrightarrow s(y,x)] & \text{for each } r,s \in A \text{ with } r = \breve{s} \\
\forall xy[1(x,y) \to (r(x,y) \leftrightarrow \exists z(s(x,z) \wedge t(z,y)))] & \text{for each } r,s,t \in A \text{ with } r = s\,;t \\
\exists xy\; r(x,y) & \text{for each } r \in A \text{ with } r \neq 0.
\end{array}$$

Every weakly associative algebra has a relativised representation [17].

It is easily seen that the conjunction of the above theory can be written as a loosely guarded sentence. Thus, by corollary 3.4, any finite weakly associative algebra has a finite relativised representation. One may also show in much the same way that WA has the finite base property: any universal sentence true in every finite weakly associative algebra is true in all weakly associative algebras. (These results were proved in [1].)

Various related results can also be derived. For example, similar arguments will show that for finite $n \geq 3$, any finite relation algebra in RA_n has a finite n-square relativised representation, and that if $n \geq 4$, any subalgebra of the relation algebra reduct of a finite n-dimensional cylindric algebra has a finite n-flat relativised representation. For definitions of these terms, see [12, 11]; these results solve open problems stated there.

4.3 Finite model property for arrow logic in relativised interpretation

Arrow logic is the logical counterpart of relation algebra, and fits into the paradigm of *dynamic logic*: see [23] for more information. Formulas of arrow logic can be given relativised semantics, corresponding to weakly associative algebras. It is immediate from the finite base property for WA (above) that any formula of arrow logic with a relativised model has a finite relativised model.

References

[1] H Andréka, I Hodkinson, and I Németi, *Finite algebras of relations are representable on finite sets*, J. Symbolic Logic **64** (1999), 243–267.

[2] H Andréka, J van Benthem, and I Németi, *Modal logics and bounded fragments of predicate logic*, J. Philosophical Logic **27** (1998), 217–274.

[3] C. Areces, C. Monz, H. de Nivelle, and M. de Rijke, *The guarded fragment: Ins and outs*, JFAK (J. Gerbrandy, M. Marx, M. de Rijke, and Y. Venema, eds.), Vossiuspers, AUP, Amsterdam, 1999, Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday.

[4] D M Gabbay, *Expressive functional completeness in tense logic (preliminary report)*, Aspects of Philosophical Logic (U Monnich, ed.), Reidel, Dordrecht, 1981, pp. 91–117.

[5] E Grädel, *On the restraining power of guards*, J. Symbolic Logic **64** (1999), 1719–1742.

[6] M Grohe, *Arity hierarchies*, Ann. Pure Appl. Logic **82** (1996), 103–163.

[7] B. Herwig, *Extending partial isomorphisms on finite structures*, Combinatorica **15** (1995), 365–371.

[8] ______, *Extending partial isomorphisms for the small index property of many ω–categorical structures*, Israel J. Math. **107** (1998), 93–124.

[9] B. Herwig and D. Lascar, *Extending partial isomorphisms and the profinite topology on the free groups*, Trans. Amer. Math. Soc. **352** (2000), 1985–2021.

[10] C Hirsch and S Tobies, *A tableau algorithm for the clique guarded fragment*, Tech. Report LTCS-Report 00-03, Aachen University of Technology, 2000. See `http://www-lti.informatik.rwth-aachen.de/Forschung/Reports.html`.

[11] R Hirsch and I Hodkinson, *Relation algebras from cylindric algebras, I*, Ann. Pure Appl. Logic, provisionally accepted.

[12] ______, *Relation algebras with n-dimensional relational bases*, Ann. Pure Appl. Logic **101** (2000), 227–274.

[13] R Hirsch, I Hodkinson, M Marx, Sz Mikulás, and M Reynolds, *Mosaics and step-by-step*, Logic at Work (E. Orłowska, ed.), Studies in Fuzziness and Soft Computing, vol. 24, Physica-Verlag, Heidelberg/New York, 1998, Appendix to 'A Modal Logic of Relations' by Y Venema and M Marx, pp. 158–167.

[14] I Hodkinson, F Wolter, and M Zakharyaschev, *Decidable fragments of first-order temporal logics*, Ann. Pure Appl. Logic, to appear.

[15] E Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica **12** (1992), 411–416.

[16] J A W Kamp, *Tense logic and the theory of linear order*, Ph.D. thesis, University of California, Los Angeles, 1968.

[17] R Maddux, *Some varieties containing relation algebras*, Trans. Amer. Math. Soc. **272** (1982), 501–526.

[18] M Marx, *Complexity of modal logics of relations*, preprint, 1997. Available at `http://www.wins.uva.nl/research/illc/Publications/ResearchReports/ML-1997-02.text.ps.gz`

[19] ______, *Tolerance logic*, Tech. Report IR–469, Vrije Universiteit, Amsterdam, 1999, to appear in J. Logic, Language and Information. Available at `http://www.wins.uva.nl/~marx/pub/tolerance.ps.gz`

[20] Sz Mikulás and M Marx, *Undecidable relativizations of algebras of relations*, J. Symbolic Logic **64** (1999), 747–760.

[21] J K Truss, *Generic automorphisms of homogeneous structures*, Proc. London Math. Soc. **64** (1992), 121–141.

[22] J van Benthem, *Modal correspondence theory*, Ph.D. thesis, Mathematical Institute, University of Amsterdam, 1976.

[23] ______, *Exploring logical dynamics*, Studies in Logic, Language and Information, CSLI Publications & FoLLI, Stanford, 1996.

[24] ______, *Dynamic bits and pieces*, Tech. Report LP-97-01, ILLC, University of Amsterdam, 1997. Available at `http://preprint.beta.uva.nl/server/bp_search.show_doc?p_srvr_id=3&p_doc_id=468`

[25] ______, *Modal logic and classical logic*, Bibliopolis, Naples, 1985.

[26] M Vardi, *Why is modal logic so robustly decidable?*, Descriptive complexity and finite models, Amer. Math. Soc., Providence, RI, 1996, pp. 149–183.