# A construction of many uncountable rings using SFP domains

Ian Hodkinson[1]
Saharon Shelah[2]

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**Abstract**

The paper in in two parts. In Part I we describe a construction of a certain kind of subdirect product of a family of rings. We endow the index set of the family with the partial order structure of an SFP domain, as introduced by Plotkin, and provide a commuting system of homomorphisms between those rings whose indices are related in the ordering. We then take the subdirect product consisting of those elements of the direct product having finite support in the sense of this domain structure. We examine the properties of rings obtainable in this way.

In Part II we prove an 'anti-structure theorem' by exhibiting $2^{\aleph_1}$ pairwise non-embeddable rings of cardinality $\aleph_1$ with various higher-order properties. The construction uses Aronszajn trees.

## Introduction

This paper presents a blend of ideas from ring theory, set-theoretic combinatorics and computer science. It is divided into two parts: part I will perhaps be of more interest to algebraists, and part II to logicians.

In part I we develop a method of constructing a subdirect product of certain families of rings. To do this we impose a partial order structure on the index set of the family. We will take this poset structure to be that of an SFP domain, a notion introduced in [P] and well known to domain theorists in computer science. We will analyse the behaviour of the ideals of the resulting subdirect product and show that *inter alia* they carry information about the underlying poset structure on the index set. We can then exert control over the subdirect product by purely partial order-theoretic means.

We exploit this in part II. Using a variant of the construction of Aronszajn trees in set theory we will construct, using ZFC only, $2^{\aleph_1}$ SFP domains such that, assuming that all component rings are countable, any subdirect products obtained with them will be pairwise non-embeddable rings. We can impose conditions on the component rings themselves to obtain

[1] Department of Computing, Imperial College, London SW7 2BZ, England
[2] Department of Mathematics, Hebrew University, Jerusalem 91904, Israel. PAPER 297

stronger results.

A typical product is:

*THEOREM: Let S be a countable Boolean ring. There are* $2^{\aleph_1}$ $L_{\infty\omega}$*-equivalent pairwise non-embeddable Boolean rings* $R_i$ ($i<2^{\aleph_1}$) *of cardinality* $\aleph_1$ *extending S. Each* $R_i$ *is existentially closed and rigid, and each of its maximal ideals has a countable set of generators.*

This suggests that there are too many such rings to classify fully. It is thus an *anti-structure theorem* in the spirit of, for example, the result of [Sh] that if T is a first order non-superstable complete first order theory of cardinality $\kappa$ then there are $2^\lambda$ pairwise non-elementarily embeddable models of T of cardinality $\lambda$ for all regular $\lambda>\kappa$.

The technique tends to produce rings with many orthogonal central idempotents, so is most at home when constructing Boolean or von Neumann regular rings.

The work in this paper simplifies the construction of the doctoral thesis [Hk] of the first author, which also uses the continuum hypothesis. The argument there is more complicated and less general because SFP domains are not used. The motivation for [Hk] came from the paper of Ziegler [Z]. If I is a left ideal of a ring R, we say that I is *densely decomposable* if whenever A is a left ideal properly extending I then there are left ideals X, Y $\subseteq$ A properly extending I but with X$\cap$Y = I (see Section 3 of Part I). If R is countable, commutative and von Neumann regular then a proper ideal I of R is densely decomposable iff the ring R/I is atomless (has no principal maximal ideals), iff the injective hull of the left R-module R/I has no indecomposable direct summand. If R is additionally assumed to be countable and atomless then R has $2^{\aleph_0}$ (i.e. $2^{|R|}$) maximal ideals; this was generalised to arbitrary countable rings in [Z] (7.1(1), 7.2, 8.3). Our initial objective was to show that the result fails for $|R| = \aleph_1$. This is established by the theorem quoted above. Every maximal ideal of each $R_i$ of the theorem is countably generated, so they are at most $2^{\aleph_0}$ in number - this can be less than $2^{\aleph_1} = 2^{|R_i|}$. The construction in [BK] gives an atomless Boolean ring of cardinality $\aleph_1$ also illustrating this, but Jensen's $\diamond$ (diamond) is used. On the other hand, unlike the construction in [BK], an atomless Boolean algebra constructed by the methods we give will generally have an uncountable set of pairwise incomparable elements.

It would be interesting to prove an intrinsic characterisation theorem for rings arising by our construction, analogous to that for varieties and reduced products. Possibly the work of Smyth [Sm] would be relevant.

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# Part I: SFP systems

This part of the paper contains the results of a more algebraic nature. We will define the notion of an SFP system of rings, and study some of the properties of its limit.

Let us describe the approach in rather more detail than above. Let $(P,\leqslant)$ be a poset such that for every $p \in P$ we have a ring $R_p$. Suppose further that for every $p\leqslant q$ in P we have a ring homomorphism $\nu_{pq} : R_p \to R_q$. We require that the $\nu_{pq}$ ($p\leqslant q$ in P) form a commuting system in the usual sense.

Assume that P has a least element $\perp$. Then the presence of the maps $\nu$ allow us to embed the ring $R_\perp$ diagonally into the direct product $\Pi(R_p : p \in P)$, via $r \mapsto (\nu_{\perp p}(r) : p \in P)$ for $r \in R_\perp$. We would like to generalise this as follows. Let $N \subseteq P$ be finite. Can we embed the finite direct product $\Pi(R_n : n \in N)$ diagonally into the full direct product?

So let $r \in \Pi(R_n : n \in N)$. We need to define its image $r'$ in $\Pi(R_p : p \in P)$. By analogy with the case $N = \{\perp\}$, for each $p \in P$ we would like to define $r'(p)$ to be $\nu_{np}(r(n))$, where n is an appropriate element of N, depending on p. To force a unique choice of n we will assume that N satisfies: for all $p \in P$ there is a unique maximal element of $\{n \in N : n \leqslant p\}$. This would hold if for example N is linearly ordered. We write this maximal element as $\mathbf{p / N}$. We can now define $r'$ to be $(\nu_{p/N,p}(r(p/N)) : p \in P)$. Then N is in effect a *finite support* of $r'$ in $\Pi(R_p : p \in P)$.

So we consider the set $R^*$ of all elements of $\Pi(R_p : p \in P)$ having a finite support in this sense. We require that $R^*$ be a subring of $\Pi(R_p : p \in P)$. To obtain closure under + and - we will need any two finite supports to be contained in a third, and to avoid redundancy of any $R_p$ we will formally require that (*) *any finite subset of P extends to a finite support $N \subseteq P$.* For example, if P is linearly ordered with a least element this is trivially true. So we could take P to be $(\mathbb{Q}\cup\{-\infty\},<)$, each $R_p$ to be $\{0,1\}$ and all $\nu_{pq}$ to be the identity map. In this case $R^*$ turns out to be the countable atomless Boolean ring (see Example 3.4). However, the condition (*) holds in much more general cases and is closely related to the SFP domains of Plotkin [P]. Any such P extends canonically to an SFP domain by adding where necessary a least upper bound h for each directed subset D of P. These extra points h turn out to be very useful: $\langle R_d, \nu_{dd'} : d\leqslant d' \text{ in } D\rangle$ forms a direct system and it is technically convenient to define $R_h$ to be its direct limit, and extend $\nu$ accordingly. Hence we will work with SFP domains throughout.

It is easy to show that if the 'component rings' $R_p$ ($p \in P$) have various properties then so does $R^*$. Examples of properties preserved in this way are: commutative; Boolean; von Neumann regular; existentially closed commutative. The cardinality of $R^*$ is also related to $|P|$ and the $|R_p|$. We also show that the $L_{\infty\omega}$-theory of $R^*$ is determined by the $L_{\infty\omega}$-theory of P together with the $R_p$ and the maps $\nu_{pq}$.

So far the construction could be undertaken for any model-theoretic structure. We consider rings because we can fruitfully study their ideals. (Generalisations to structures such as lattices are probably possible here.) An important class of ideals arises as follows. If I is a (left) ideal of $R_S$ for some $s \in P$ then $I^* = \{r \in R^* : r(s) \in I\}$ is a left ideal of $R^*$. Ideals of this form are called *full ideals:* they are in a sense 'locally determined'. We can recover I and s from $I^*$, so the full ideals are closely related to the poset structure of P. They are a kind of basis for the set of all ideals of $R^*$. Using the extra elements h of P we can show that any maximal, prime or irreducible ideal of $R^*$ must be full, and every ideal of $R^*$ is the intersection of the full ideals that contain it.

The layout of this part of paper is as follows. In Section 1 we discuss SFP domains and formally lay out the subdirect product construction. In Section 2 we discuss ideals of $R^*$ and use the results in the next section to enforce that $R^*$ has a property generalising 'atomlessness' in Boolean algebras. Finally, in Section 4 we discuss $L_{\infty\omega}$-equivalence.

## 1 SFP systems

In this section we give most of the definitions that we will need, plus some examples and useful lemmas for illustration.

*Algebraic dcpos*

Recall that a partially ordered set, or **poset**, is a (usually non-empty) set equipped with a reflexive transitive binary relation, written here as '$\leqslant$'. A poset $(D,\leqslant)$ is **directed** if D is non-empty and whenever $d_1, d_2 \in D$, then there is $d_3 \in D$ with $d_3 \geqslant d_1, d_2$.

A non-empty poset P is said to be **directed complete** (a 'dcpo') if whenever $D \subseteq P$ is directed then D has a least upper bound in P. We write this bound as **lub(D)**, or more explicitly $\mathbf{lub}_P(\mathbf{D})$; it is necessarily unique.

An element p of a dcpo P is said to be **finite** if whenever $D \subseteq P$ is directed and $p \leqslant \text{lub}(D)$ then $p \leqslant d$ for some $d \in D$. We write $\mathbf{P^0}$ for the set of finite elements of P; $P^0$ is called the **base** of P. P is said to be **algebraic** if for all $p \in P$, the set $\mathbf{p\downarrow} = \{q \in P : q \leqslant p\}$ is such that $p\downarrow \cap P^0$ is directed and $\text{lub}(p\downarrow \cap P^0) = p$. That is, p is the lub of the set of finite elements beneath it, and we can usually replace p by this set. It follows that in this case P is determined by its base (see below). Algebraic dcpos P with countable base are usually called **domains** in the computer science literature.

Examples of algebraic dcpos are all finite (non-empty) posets and all successor ordinals. If X is a non-empty set then $\wp(X)$, ordered by inclusion, is an algebraic dcpo, and the finite elements are just the finite subsets of X - hence the name. The half-open real interval (0,1] has no finite elements and shows that a dcpo need not be algebraic, as does the following dcpo:

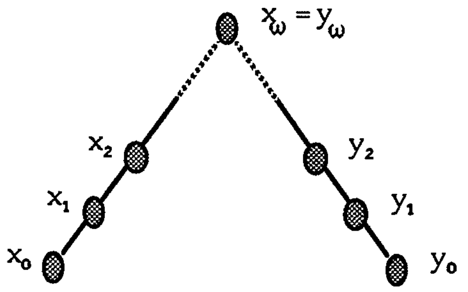


**Figure 1.1**

*Ideals*

Let P be any poset. An **ideal** of P is a subset I of P that is closed downwards (i.e. if $x \leqslant y \in I$ then $x \in I$) and directed. Clearly if $p \in P$ then $p\downarrow$ is an ideal; ideals of this form are said to be **principal**. It is well known that if P is an arbitrary non-empty poset, the set of ideals of P, ordered by inclusion, forms an algebraic dcpo whose finite elements are just the principal ideals; these are in order-isomorphism with P. Hence any P can be 'completed' to an algebraic dcpo by taking this **'ideal completion'**. Moreover, any algebraic dcpo P is isomorphic to the ideal completion of its base $P^0$. We will often identify a poset P with the set of finite elements of its ideal completion. A similar ideal completion can be undertaken for preorders also.

*Locally directed sets*

Now let P be a poset. A subset N of P is said to be **locally directed in P** (written $N \trianglelefteq P$) if for all $p \in P$, $p\downarrow \cap N$ is directed. Equivalently, $N \trianglelefteq P$ iff $N \cap I$ is directed for all ideals I of P. For example, if P is an algebraic dcpo then $P^0 \trianglelefteq P$. If P contains a least element $\perp$, then any linearly ordered subset N of P with $\perp \in N$ is locally directed in P. $N \subseteq \wp X$ is locally directed in $(\wp X, \subseteq)$ iff N is closed under finite (including empty) unions. Since $P \trianglelefteq P$ for any P, locally directed does not imply directed. The converse also fails, as if $\perp$ is the least element of P then $N \trianglelefteq P \Rightarrow \perp \in N$.

It is easily seen that $\trianglelefteq$ is a reflexive and transitive relation on posets, and that if $N \trianglelefteq P$ and $N \subseteq Q \subseteq P$ then $N \trianglelefteq Q$.

Now assume that P is a dcpo. If $N \trianglelefteq P$ and $p \in P$, we write **p / N** for $\mathrm{lub}_P(p\downarrow \cap N)$; this exists since P is a dcpo, and indeed if N is finite, or more generally a dcpo such that $\mathrm{lub}_P(D) = \mathrm{lub}_N(D)$ for all directed $D \subseteq N$, then $p/N \in N$. We can view $p/N$ as N's best approximation to p. We have $p/N \leqslant p$ for all p; further, P is algebraic iff $P^0 \trianglelefteq P$ and $p/P^0 = p$ for all $p \in P$. If $N \trianglelefteq P$ we can define an equivalence relation $\sim_N$ on P by $x \sim_N y$ iff $x/N = y/N$. We will see in Section 2 that the equivalence classes are related to the well known 'patch' topology on P.

*SFP domains*

We can now define the strain of poset of interest to us here. A poset P is said to be **nice** if any finite subset $X \subseteq P$ can be extended to a finite locally directed subset of P. An **SFP domain** is an algebraic dcpo P such that $P^o$ is nice. So the ideal completion of a nice poset is an SFP domain, and all SFP domains arise in this way.

An equivalent definition uses the notion of MUB-closure (see Plotkin, [P]). If $X \in P$ define $MUB(X) = \{p \in P : p \text{ is a minimal upper bound of } X\}$. Also define an increasing chain $U^n(X)$ $(n \leqslant \omega)$ by: $U^o(X) = X$, $U^{n+1}(X) = \bigcup\{MUB(Y) : Y \subseteq U^n(X)\}$, $U^\omega(X) = \bigcup_{n<\omega} U^n(X)$. $U^\omega(X)$ is called the MUB-closure of X. Then it is easily seen that P is SFP iff for all finite $X \subseteq P^o$,

(i) for all $p \in P$ with $X \subseteq p\downarrow$ there is $y \in MUB(X)$ with $y \leqslant p$

(ii) $MUB(X)$ is finite

(iii) $U^\omega(X)$ is finite.

In fact, in this case $U^\omega(X) \trianglelefteq P^o$. Domains satisfying (i) and (ii) are sometimes called 2/3-SFP. Of course, (iii) implies (ii).

Examples of nice posets are any finite poset, any linear order with a least element, any Boolean algebra, and any tree with finitely many minimal elements. The restriction to finitely many minimal elements is necessary. If P is a nice poset then take finite $N \trianglelefteq P$; every $p \in P$ lies above some element of N. Then $M = \{m \in N : \neg\exists n \in N(n<m)\}$ is non-empty and finite, and every $p \in P$ lies above an element of M.

The following are the three main kinds of non-nice poset. See [Sm].

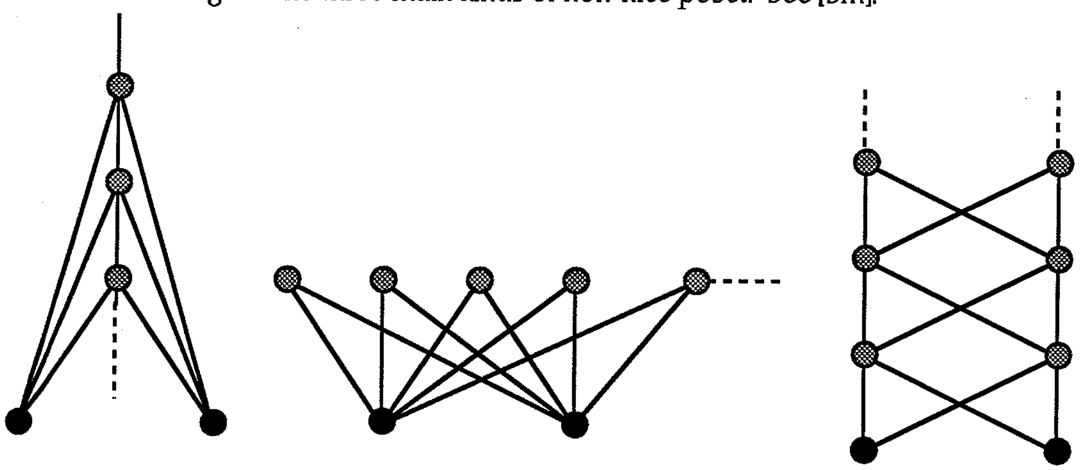

**Figure 1.2**

On the left the two black elements have no minimal upper bound, violating condition (i) above. In the centre poset they have infinitely many minimal upper bounds, violating (ii). The right-hand one satisfies (i) and (ii) but now the black elements have infinite MUB-closure.

SFP domains were introduced in [P] as those arising as inverse limits of Sequences of Finite Posets. They are of considerable interest in computer science, where they are used to provide denotational semantics for programming languages. Any domain P can be equipped with a topology (the Scott topology): $O \subseteq P$ is open iff (i) O is closed upwards, and (ii) if $D \subseteq P$

is directed and lub(D) $\in$ O then D$\cap$O $\neq \emptyset$. If D and E are domains we write [D$\to$E] for the poset of Scott-continuous functions from D to E, ordered by $f \leqslant g$ iff for all $d \in D$, $f(d) \leqslant g(d)$. In {Sm] Smyth showed amongst other things that if D is a domain with countable base, then [D$\to$D] is also a domain with countable base iff D is SFP. In this case [D$\to$D] is also SFP. The SFP domains form the largest Cartesian closed full subcategory of the category of domains with countable bases, the morphisms being the Scott-continuous maps.

### *SFP systems*

We now give our main algebraic definition. An **SFP system** is a triple $<P,\rho,\nu>$, where

(i) P is an SFP domain.

(ii) $\rho$ is a map from P into the class of rings with a 1 ($1 \neq 0$). We will write $R_p$ for $\rho(p)$, where $\rho$ is understood.

(iii) $\nu$ is a map defined on those pairs $(p,q) \in P^2$ with $p \leqslant q$. Each $\nu(p,q)$ is a ring homomorphism from $R_p$ into $R_q$. (All ring homomorphisms in this paper preserve 0 and 1.) We write $\nu(p,q)$ as $\nu_{pq}$. We require further that

- (a) $\nu_{pp}$ is the identity on $R_p$
- (b) $\nu_{qr} \circ \nu_{pq} = \nu_{pr}$ if $p \leqslant q \leqslant r$ in P
- (c) if $D \subseteq P$ is directed with least upper bound $u \in P$, then $R_u$ is the direct limit of the direct system $<R_d, \nu_{dd'} : d \leqslant d' \text{ in } D>$, and for all $d \in D$, $\nu_{du}$ is the canonical ring homomorphism from $R_d$ into $R_u$.

**Remark 1.1**

Let P be a nice poset. Suppose we have a triple $(P,\rho,\nu)$ satisfying (ii) and (iii)(a), (b). Then we can canonically complete it to an SFP system by (a) embedding P canonically into its ideal completion Q, (b) defining $R_q$ for $q \in Q \setminus P$ to be $\lim_{\to}<R_p,\nu_{pp'} : p \leqslant p' \text{ in } P \cap q{\downarrow}>$, and (c) defining $\nu_{qq'}$ for $q \leqslant q'$ in Q to be the 'limit' of the $\nu_{pp'}$ for $p, p' \in P$ with $p \leqslant q$, $p' \leqslant q'$. Moreover, all SFP systems arise in this way. So an SFP system $<P,\rho,\nu>$ is determined by its 'finite' part: on $P^0$, $R_p$ and $\nu_{pp'}$ for $p \leqslant p'$ in $P^0$.

### *Limits of SFP systems*

Let $<P,\rho,\nu>$ be an SFP system, and let $N \trianglelefteq P$. An element $r \in \Pi<R_p : p \in P>$ is said to have **support** N if for all $p \in P$, $r(p) = \nu_{(p/N),p}[r(p/N)]$. We define the **limit** of $<P,\rho,\nu>$, or **lim**$<P,\rho,\nu>$, to be the subdirect product consisting of those elements of $\Pi R_p$ that have a finite support $N \subseteq P^0$. Since P is an SFP domain, any two finite locally directed subsets of $P^0$ are contained in a third, and it follows that the limit of $<P,\rho,\nu>$ is a subring of $\Pi R_p$. Clearly it is also identifiable with a subring of $\Pi<R_p : p \in P^0>$, since $P^0$ supports any element of $R_P$.

We will generally write $R_P$ for the limit of $<P,\rho,\nu>$. Obviously, for any $p_0 \in P$ the projection $(r \mapsto r(p_0))$ of $R_P$ onto $R_{p_0}$ is a surjective ring homomorphism.

As an example, if $P = (\mathbb{Q},<)$ and all $R_p$ are $\{0,1\}$ then $R_P$ is the unique countable atomless Boolean ring. See Example 3.4 below.

*Subsystems*

Let P be an SFP domain. If $Q \subseteq P$, we write $Q \preccurlyeq P$, and say that Q is a **subdomain** of P, if

- Q is itself an SFP domain under the ordering induced from P
- $Q^o \subseteq P^o$
- Q is a locally directed subset of P
- if $D \subseteq Q$ is directed then $\mathrm{lub}_Q(D) = \mathrm{lub}_P(D)$.

Note that these conditions imply that $P^o \cap Q \subseteq Q^o$, so that we have $P^o \cap Q = Q^o$ in fact. Clearly $\preccurlyeq$ is reflexive and transitive, and if $N \subseteq P$ is finite then $N \preccurlyeq P$ iff $N \trianglelefteq P^o$.

**Proposition 1.2**

Suppose that we have an SFP system $<P,\rho,\nu>$. Let $Q \preccurlyeq P$. Then $<Q,\rho|Q,\nu|Q^2>$ is an SFP system. Moreover, its limit ring $R_Q$ is canonically isomorphic to the subring of $R_P$ consisting of those elements supported by Q.

**Proof**

To show that $<Q,\rho|Q,\nu|Q^2>$ is an SFP system we only need to check that if $D \subseteq Q$ is directed then

$$R_{\mathrm{lub}_Q(D)} = \lim_{\rightarrow}<R_d : d \in D>.$$

But this is clear, since $<P,\rho,\nu>$ is an SFP system and $\mathrm{lub}_Q(D) = \mathrm{lub}_P(D)$.

Now if $r \in R_Q$ there is finite $N \preccurlyeq Q$ supporting r. By transitivity of $\preccurlyeq$ we have $N \preccurlyeq P$, so r extends naturally to $r' \in R_P$ given by

- $r'(p) = \nu_{p/N,p}[r(p/N)]$ for $p \in P$.

The map $r \mapsto r'$ is a ring embedding from $R_Q$ into $R_P$, and clearly its image is precisely the set of elements of $R_P$ supported by a finite locally directed subset of Q - i.e. those supported by Q.

□

In future we identify $R_Q$ with the subring $(R_Q)'$ of $R_P$, whenever $Q \preccurlyeq P$.

A special case is where $Q \preccurlyeq P$ is finite - i.e. $Q = N$, a finite locally directed subset of $P^o$. Then clearly $R_N \cong \Pi<R_n : n \in N>$, a finite direct product. If $N \subseteq N'$ are finite locally directed subsets of $P^o$, then $N \preccurlyeq N'$, and so (making the identification) $R_N$ is a subring of $R_{N'}$. Since P is SFP, the following is clear:

**Proposition 1.3**

$<R_N : N \trianglelefteq P^o$ is finite$>$ is a direct system of rings under inclusion, and its direct limit is naturally isomorphic to $R_P$.

□

So all limit rings of SFP systems arise as direct limits of some direct system of rings.

**Corollary 1.4**

Let $<P,\rho,\nu>$ be an SFP system.

(i) If P has a least element, $\perp$, say, then $R_\perp$ is a subring of $R_P$.

(ii) The following classes of rings are closed under SFP systems, in the sense that if $R_p \in K$ for all $p \in P^o$ then $R_P \in K$ also:

(a) the class of commutative rings;

(b) the class of von Neumann regular rings (i.e. $R \models \forall x \exists y(xyx = x)$);

(c) the class of Boolean rings;

(d) the class of rings that are existentially closed in the class of commutative rings ('commutative e.c.');

(e) the class of existentially closed rings in the class of Boolean rings.

**Proof**

(i) Suppose that $\perp \in P$ is such that $\perp \leqslant p$ for all $p \in P$. Clearly $\{\perp\} \trianglelefteq P^o$. The result follows from (1.3) now.

(ii) By (1.3) it is enough to show that the classes cited are preserved under finite direct products and direct limits - or at least direct limits in which the morphisms of the system are injective. This is clear for (a), (b) and (c), where there is no use of injectivity. We prove (d).

Recall (e.g. from [CK]) that if L is a first order signature and $\Sigma$ is a class of L-structures that is closed under isomorphism, an L-structure $M \in \Sigma$ is said to be **existentially closed in** $\Sigma$ (**e.c.** for short) if whenever $M \subseteq N \in \Sigma$ and $\varphi(\bar{x})$ is an existential formula of L, then

- for all $\bar{a} \in M$, if $N \models \varphi(\bar{a})$ then already $M \models \varphi(\bar{a})$.

Clearly the class of e.c. structures is closed under isomorphism. By considering disjunctive normal forms we may assume that $\varphi(\bar{x})$ is of the form $\exists \bar{y} \psi(\bar{x},\bar{y})$ where $\psi$ is a conjunction of atomic and negated atomic formulas.

It is easy to see that if $\Sigma$ is closed under direct limits of the form $\lim_{\rightarrow}<M_i, \nu_{ij} : i \leqslant j \text{ in } I>$ where the $\nu_{ij}$ are injective, then a direct limit of e.c. structures is e.c.. The class of commutative rings is closed under such limits, so to prove (d) it suffices to prove that if A, B are commutative e.c. rings (i.e. they are e.c. in the class of commutative rings) then so is $A \times B$.

Suppose $C \supseteq A \times B$ is a commutative ring. Let $e_1 = (1,0)$, $e_2 = (0,1)$ in $A \times B$. Then since C is commutative, $e_1$ is a central idempotent of C. It follows that the left ideal $Ce_1$ of C is a commutative ring in its own right, with identity $e_1$; it has a subring $(A\times B)e_1$, which is isomorphic to A via $(a,b)e_1 \mapsto a$. Similarly, $Ce_2$ is a commutative ring with a subring $(A\times B)e_2 \cong B$.

Now since $e_1e_2 = 0$ and $e_1+e_2 = 1$, we have $C \cong Ce_1 \times Ce_2$ via $c \mapsto (ce_1,ce_2)$. It follows that:

($) if $\alpha(\bar{x})$ is an atomic formula of L and $\bar{c} \in C$, then $C \models \alpha(\bar{c})$ iff $Ce_i \models \alpha(\bar{c}e_i)$ for $i = 1,2$. Similarly, if $\bar{c} \in A\times B$ then $A\times B \models \alpha(\bar{c})$ iff $(A\times B)e_i \models \alpha(\bar{c}e_i)$ for $i = 1,2$.

If $\alpha$ is an atomic formula, define $\alpha^1$ to be $\alpha$ and $\alpha^0$ to be $\neg\alpha$. Let $\psi(\bar{x},\bar{y})$ above be $\bigwedge_{j<m}\alpha_j(\bar{x},\bar{y})^{n_j}$, where the $\alpha_j$ are atomic formulas of the signature $\{+,-,.,0,1\}$ of rings, and $n_j = 0$ or 1. Suppose that $C \models \psi(\bar{a},\bar{c})$ for $\bar{a} \in A\times B$, $\bar{c} \in C$. Then by ($), there are $p_j$, $q_j \in \{0,1\}$ with $p_jq_j = n_j$ $(j<m)$, such that $Ce_1 \models \bigwedge_j \alpha_j(\bar{a}e_1,\bar{c}e_1)^{p_j}$ and $Ce_2 \models \bigwedge_j \alpha_j(\bar{a}e_2,\bar{c}e_2)^{q_j}$.

As $(A\times B)e_1 \cong A$ we can identify them and regard A as a subring of $Ce_1$. Because A is e.c. there is $\bar{c}_1 \in A$ such that $A \models \bigwedge_j \alpha_j(\bar{a}e_1,\bar{c}_1)^{p_j}$; and similarly we can find $\bar{c}_2 \in B$ with analogous properties for B. Take $\bar{d} \in A\times B$ with $\bar{d}e_1 = \bar{c}_1$, $\bar{d}e_2 = \bar{c}_2$; then $A \models \bigwedge_j \alpha_j(\bar{a}e_1,\bar{d}e_1)^{p_j}$ and $B \models \bigwedge_j \alpha_j(\bar{a}e_2,\bar{d}e_2)^{q_j}$. Hence by ($) again, $A\times B \models \bigwedge_j \alpha_j(\bar{a},\bar{d})^{n_j}$.

Hence $A\times B$ is an existentially closed commutative ring, as required.

(e) - the proof is the same as (d).

□

Note that for Boolean rings, eixstentially closed is the same as atomless. See for example [Hg, 6.3.9, Ex. 6.3.2]. Since many of the SFP domains we use have a least element $\perp$, SFP systems can often be used to produce rings extending a given ring $R = R_\perp$ (1.4(i)).

A slightly more general preservation result is: if all $R_p$ satisfy $\varphi = \forall\bar{x}\exists\bar{y}(\bigwedge_i\pi_i\to\pi)$ where $\pi_i$ and $\pi$ are equations, then also $R_P$ satisfies $\varphi$. This includes (ii(a)-(c)) above; the proof is the same.

There is an easy cardinality result that also follows from (1.3).

**Proposition 1.5**

Suppose that $\langle P,\rho,\nu\rangle$ is an SFP system in which each ring $R_p$ is countable, and P is infinite. Then $|R_P| = |P^o|+\aleph_0$.

□

## 2 Ring Ideals

Here we examine the relationship between (ring) ideals of the limit ring of an SFP system $<P,\rho,\nu>$ and the underlying SFP domain of the system. The relationship is close and we will use it extensively in later sections. We isolate a special class of ideals - the *full ideals* - and show that they correspond closely to P. We also show that the ideals of the limit of the system are determined by their projections onto the components $R_p$ $(p \in P)$.

Unless otherwise stated, all ring ideals in this section will be left ideals.

### Notation

Let P be an SFP domain, and fix an SFP system $<P,\rho,\nu>$ with limit ring $R_P$. We will generally use 'J' to denote an ideal of $R_P$, and 'I' for an ideal of a component ring $R_p$ $(p \in P)$. If J is an ideal of $R_P$ and $q \in Q \preccurlyeq P$, we will write $J_Q$ for $J \cap R_Q$ and $J_Q(q)$ for the projection $\{r(q) : r \in J_Q\}$ of $J_Q$ onto the $q^{th}$ component ring $R_q$. We write simply $J(q)$ for $J_P(q)$.

First a useful lemma.

### Lemma 2.1

Let P be any finite poset and let $<P,\rho,\nu>$ be an SFP system with limit ring $R_P$. Let J be an ideal of $R_P$. Then $J = \{r \in R_P : r(p) \in J(p) \text{ for all } p \in P\}$.

### Proof

'$\subseteq$' is clear; we prove '$\supseteq$'. For each $p \in P$ define a central idempotent $e_p \in R_P$ by

$$e_p(x) = \begin{cases} 1 \text{ if } x = p \\ \\ 0 \text{ if } x \in P\backslash\{p\} \end{cases}$$

If $r(p) \in J(p)$ for all $p \in P$, then for each p there is $s_p \in J$ with $s_p(p) = r(p)$. Then $r = \Sigma_{p \in P} (e_p.s_p) \in J$, as required.

□

This essentially says that for finite P, $J \cong \Pi(J(p) : p \in P)$. We will generalise it to arbitrary P in (2.7) below.

### Definition

If $p \in P$ and I is a proper ideal of $R_p$, we write I@p for $\{r \in R_P : r(p) \in I\}$. This is a proper ideal of the limit ring $R_P$; strictly it depends on P also, and we will sometimes write "I@p in $R_P$".

Now if $p' \in P$ and $I'$ is an ideal of $R_{p'}$, then $I@p = I'@p'$ implies that $p = p'$ and $I = I'$. For if $p \neq p'$, then as P is algebraic, $p{\downarrow} \cap P^o \neq p'{\downarrow} \cap P^o$. Assume without loss that there is $q \in P^o \cap (p{\downarrow} \setminus p'{\downarrow})$. As P is an SFP domain there is finite $N \preccurlyeq P$ (i.e. $N \trianglelefteq P^o$) containing q. Hence $p/N \neq p'/N$. We can find $r \in R_N$ such that $r(p/N) = 0$ and $r(p'/N) = 1$. Then $r \in I@p \setminus I'@p'$, a contradiction. Hence $p = p'$, and it easily follows that $I = I'$.

If J is a proper ideal of $R_P$, we say that J is **full** (in $R_P$) if J is of the form $I@p$ for some p, I. Clearly I will be a proper ideal of $R_p$. Since p and I are unique, we can define $\sigma J = p$ (the **site** of J), and $\Delta J = I$ (the **defect** of J).

The 'theoretical' interest of full ideals is in their relationship with P, via their site. We will use this to show that the limit ring $R_P$ can carry ring-theoretic traces of the underlying poset P, in a form of Stone duality. The main result involved is Theorem 2.2.

**Theorem 2.2**

Let $J \subseteq R_P$ be an ideal. Then the following are equivalent:

(i) J is full in $R_P$

(ii) for each finite $N \preccurlyeq P$, $J_N$ is full in $R_N$

(iii) for each $Q \preccurlyeq P$, $J_Q$ is full in $R_Q$.

Moreover, if any of (i)-(iii) hold, and $Q \preccurlyeq P$, we have

(iv) $\sigma(J_Q) = \sigma J/Q$

(v) $\Delta(J_Q) = (\nu_{\sigma J/Q,\sigma J})^{-1}(\Delta J)$.

**Proof**

(i $\Rightarrow$ ii):

Assume that J is full in $R_P$; let $J = I@p$ (some $p \in P$, $I \subseteq R_p$). Let $N \preccurlyeq P$ be finite and let $n = p/N$. If $r \in R_N$, then $r \in J$ iff $r(p) = \nu_{np}[r(n)] \in I$ iff $r(n) \in \nu_{np}{}^{-1}(I)$ iff $r \in [\nu_{np}{}^{-1}(I)]@n$ in $R_N$. Hence $J_N = [\nu_{np}{}^{-1}(I)]@n$ in $R_N$. This proves (ii), and also (iv) and (v) in the case where Q is finite.

(ii $\Rightarrow$ iii):

Assume (ii) and take $Q \preccurlyeq P$. If $N \preccurlyeq Q$ is finite then $N \preccurlyeq P$, so $J_N$ is full in $R_N$ for all finite $N \preccurlyeq Q$.

Now if $N, N' \preccurlyeq Q$ and $N \subseteq N'$, then $N \preccurlyeq N'$. It follows from the proof of (i $\Rightarrow$ ii) that

(£) $\quad \sigma(J_N) = \sigma(J_{N'})/N \leqslant \sigma(J_{N'})$.

So as Q is SFP, the set $D = \{\sigma J_N : \text{finite } N \preccurlyeq Q\}$ is directed. Let its lub in Q be q.

<u>Claim 1</u> If $N \leqslant Q$ is finite, then $\sigma J_N = q/N$.

<u>Proof of Claim</u> Clearly $q \geqslant \sigma J_N \in N$. Hence $\sigma J_N \leqslant q/N$. For the converse inequality, note that as $q/N \leqslant q$ and $q/N$ is a finite element of Q, there is finite $N' \leqslant Q$ such that $\sigma J_{N'} \geqslant q/N$. By (£) we may assume that $N' \supseteq N$, and so $\sigma J_N = \sigma J_{N'}/N \geqslant q/N$. This proves the claim.

Now let $I = \{r(q) : r \in J_Q\}$. Clearly I is an ideal of $R_q$.

<u>Claim 2</u> $J_Q = I@q$ in $R_Q$.

<u>Proof of Claim</u> "$\subseteq$" is clear; we pass to "$\supseteq$". Let $r \in R_Q$ be such that $r(q) \in I$. So there is $s \in J_Q$ with $s(q) = r(q)$. Since $R_q = \lim_{\rightarrow}<R_{q'} : q' \in q{\downarrow} \cap Q^o>$ and Q is SFP, we can find finite $N \leqslant Q$ supporting r and s, and such that $s(q/N) = r(q/N)$. But $s \in J_N$, which by Claim 1 is full with site $q/N$. Hence $r \in J_N$ also. This proves the claim.

So by the claim $J_Q$ is full in $R_Q$, which proves (iii).

(iii $\Rightarrow$ i): is trivial.

It remains to prove (iv) and (v) for infinite $Q \leqslant P$. Let $J \subseteq R_P$ be full; let $\sigma J = p$. Then $J_Q$ is full; let it be $I@q$.

If $N \leqslant Q$ is finite, then we may apply (iv) to get $q/N = \sigma J_N$. But also $N \leqslant P$, so similarly $\sigma J_N = p/N$. Hence $p/N = q/N$ for all finite $N \leqslant Q$. Since Q is SFP, it follows that $p{\downarrow} \cap Q^o = q{\downarrow} \cap Q^o$. Taking lubs, we obtain $p/Q = q$, proving (iv).

For (v), we must show that $I = \nu_{qp}^{-1}(\Delta J)$. Take $a \in R_q$; there is finite $N \leqslant Q$ and $r \in R_Q$ supported by N, such that $r(q) = a$. By the above, $p/N = q/N$. So $r(p) = \nu_{qp}(r(q))$, and hence

$a \in I$ iff $r \in J$ iff $r(p) \in \Delta J$ iff $r(q) = a \in \nu_{qp}^{-1}(\Delta J)$.

□

Whilst $R_P$ can have many full ideals with the same site, this is not so if we restrict to the elements of $R_P$ that take values 0, 1 only. These elements are central idempotents of $R_P$. They form a Boolean algebra in the usual way by defining $a \leqslant b$ to hold iff $ab = a$; $a \wedge b$ is ab and $a \vee b$ is a+b-ab (symmetric difference).

### Definition

We write $(R_P)^*$ for the set $\{r \in R_P : \forall p \in P\ (r(p) \in \{0,1\})\}$. If $X \subseteq R_P$ we write $X^*$ for $X \cap R_P^*$.

**Proposition 2.3**

Let I, J be full ideals of $R_P$. Then $\sigma I = \sigma J$ iff $I^* = J^*$.

**Proof**

Assume that $\sigma I = \sigma J$. Then if $r \in (R_P)^*$, $r \in I$ iff $r(\sigma I) \in \Delta I$. As $\Delta I$ is a proper ideal of $R_{\sigma I}$, this holds iff $r(\sigma I) = 0$. Since the same holds for J, we have $r \in I$ iff $r \in J$, so $I^* = J^*$.

Conversely, suppose that $\sigma I \neq \sigma J$. Since P is SFP, we can find finite $N \preccurlyeq P$ such that $\sigma I/N \neq \sigma J/N$. Let $r \in (R_P)^*$ be supported by N, and given by: $\forall n \in N$, $r(n) = 0$ if $n = \sigma I/N$, and 1 otherwise. Then $r \in I^* \setminus J^*$ so that $I^* \neq J^*$.

□

In practical terms, full ideals include the maximal, prime and irreducible ideals of $R_P$. Let us say that an ideal I of a ring R is **whole** if $R \setminus I$ conains no pair of orthogonal central idempotent elements (i.e. there do not exist $x, y \in R \setminus I$, commuting multiplicatively with every element of R, and such that $x^2 = x$, $y^2 = y$, $xy = 0$).

The following is easy:

**Proposition 2.4**

If I is a maximal, prime, or irreducible left (or right) ideal of R, then I is whole. If I is a maximal two-sided ideal of R, then I is whole.

□

But now we have:

**Proposition 2.5**

If I is a proper whole ideal of $R_P$ then I is full.

**Proof**

If I is not full, then by (2.2) there is finite $N \preccurlyeq P$ such that $I_N$ is not full in $R_N$. But clearly $I_N$ is a proper ideal of $R_N$. By (2.1) there are $n \neq n'$ in N such that the projections $I_N(n)$ and $I_N(n')$ are proper ideals of $R_n$, $R_{n'}$ respectively. So we define $e_n \in R_P$ by

- $e_n$ is supported by N; and $e_n(x) = 1$ if $x = n$, and 0 if $x \in N \setminus \{n\}$,

and similarly define $e_{n'}$. Then $e_n, e_{n'} \notin I$. Clearly $e_n, e_{n'}$ are central idempotents of $R_P$ and $e_n.e_{n'} = 0 \in I$. Hence I is not whole.

□

**Remark**

Clearly, an ideal I is prime in $R_p$ iff I@p is prime in $R_P$. If J is an ideal of $R_P$, then $J \supseteq I@p$ iff J is full, $\sigma J = p$ and $\Delta J \supseteq I$. Hence I@p is maximal, maximal two-sided or irreducible in $R_P$ iff I is so in $R_p$.

**Example 2.6**

Suppose that $<P,\rho,\nu>$ is such that $R_p = \{0,1\}$ for all $p \in P$. Hence each $\nu_{pp'}$ is the identity map. Then by (1.4), $R_P$ is a Boolean ring. Here, a full ideal is determined by its site alone, as its defect must be 0. By the remark, any full ideal of $R_P$ is maximal. By (2.4) and (2.5), the converse holds also. So P is in canonical bijection with the set of maximal ideals of $R_P$.

Now the set of maximal ideals of $R_P$ is a Stone space and carries a compact Hausdorff totally disconnected topology: the clopen sets are those arising as the set of ideals containing a chosen point of the ring. Hence a homeomorphic topology is induced on P; it is in fact the 'patch' topology referred to in (e.g.) [Hc], whose construction bears some similarity to ours. We can be explicit about the topology: if $Q \trianglelefteq P$, define an equivalence relation $\sim_Q$ on P by: $p \sim_Q p'$ iff $p/Q = p'/Q$. Then a basis of open sets on P is the set C of equivalence classes of the $\sim_N$, for finite $N \preccurlyeq P$:

$$C = \bigcup\{P/\sim_N : N \text{ finite}, N \preccurlyeq P\}.$$

Each class is clopen. This is a basis since any finite intersection of elements of C is a finite union of elements of C. For any $Q \preccurlyeq P$, any $\sim_Q$-class is closed in the topology. Hence (taking Q = P) every singleton subset of P is closed: the topology is *regular*.

□

We now move from full ideals to arbitrary ideals. As before we let P be any SFP domain and $<P,\rho,\nu>$ an SFP system with limit ring $R_p$. Our first result generalises (2.1) to this situation.

**Theorem 2.7**

Let J be any left ideal of $R_P$. Then for any $r \in R_P$,

- $r \in J$ iff $r(p) \in J(p)$ for all $p \in P$.

In other words, $J = \bigcap\{J(p)@p : p \in P\}$.

**Proof**

Clearly if $r \in J$ then $r(p) \in J(p)$ for all $p \in P$. For the converse it suffices to prove:

(*) $J = \bigcap\{J' : J' \text{ a full ideal of } R_P, J' \supseteq J\}$.

For assume that $r(p) \in J(p)$ for all $p \in P$. Let I@p be any full ideal containing J. Clearly $J(p) \subseteq I$. So $r \in I@p$. Hence $r \in J'$ for all full ideals $J' \supseteq J$. Given (*) we obtain $r \in J$ as required.

We only need to prove '$\supseteq$' of (*). Let $a \in R_P \backslash J$. We show that $a \notin \bigcap\{J' : J'$ a full ideal of $R_P$, $J' \supseteq J\}$.

Using Zorn's lemma choose a left ideal $J'$ of $R_P$ that is maximal with respect to:

- $J' \supseteq J$, $a \notin J'$.

We show that $J'$ is a full ideal of $R_P$.

If not, by (2.2) there is finite $N \leqslant P$ such that $J'_N$ is not full in $R_N$. Since it is certainly proper, by (2.1) there are orthogonal central idempotents $e_1$, $e_2$ of $R_N$ such that $e_1 \notin J'$, $e_2 \notin J'$, $e_1+e_2 = 1$. By maximality of $J'$ we have

$a = j_i + r_ie_i$ for some $j_i \in J'$, $r_i \in R_P$ $(i = 1, 2)$.

So $a \;=\; e_1a + e_2a \;=\; e_1.(j_2 + r_2e_2) + e_2.(j_1 + r_1e_1) \;=\; e_1j_2 + e_2j_1 \in J'$.

This is a contradiction. Hence $J'$ is a full ideal of $R_P$, which completes the proof.

□

If $Q \leqslant P$ and $J$ is an ideal of $R_P$, we can now express the projections $J_Q(q)$ $(q \in Q)$ of $J_Q$ in $R_Q$ in terms of the projections $J(p)$ of $J$ in $R_P$. The result generalises Theorem 2.2 to arbitrary ideals.

**Theorem 2.8**

Let $Q \leqslant P$ and let $J$ be an ideal of $R_P$. Then for each $q \in Q$,

$J_Q(q) = \bigcap\{\nu_{qp}^{-1}(J(p)) \;: p \in P,\, p/Q = q\}$.

**Proof**

For $q \in Q$ define $I_q = \bigcap\{\nu_{qp}^{-1}(J(p)) \;: p \in P,\, p/Q = q\}$. Let $a \in J_Q(q)$ for some $q$. Then there is $r \in J_Q$ with $r(q) = a$. Clearly $r(p) = \nu_{qp}(a) \in J(p)$ for all $p \in P$ with $p/Q = q$. So $a \in I_q$.

Hence $J_Q(q) \subseteq I_q$ for all $q \in Q$. It is not immediate that we have equality; for example, if $q_1 \in Q\backslash Q^o$ and

$$I_q = \begin{cases} R_{q_1} & \text{if } q = q_1 \\ 0 & \text{if } q \in Q\backslash\{q_1\} \end{cases}$$

then $0$ is the only ideal $I$ of $R_Q$ with $I(q) \subseteq I_q$ for all $q$.

We prove equality as follows. Suppose for contradiction that there is $q \in Q$ and $a \in I_q \backslash J_Q(q)$. Using Zorn's Lemma as in the previous theorem take a left ideal $J' \supseteq J$ of $R_P$ that is maximal with respect to: $a \notin J'_Q(q)$.

Claim: $J'_Q$ is full in $R_Q$ and $\sigma(J'_Q) = q$.

Proof of Claim: If not, there is finite $N \leqslant Q$ such that $J'_N$ is not full in $R_N$. As before, take orthogonal idempotents $e_1, e_2 \in R_N \backslash J'$, central in $R_P$ and such that $e_1+e_2 = 1$. By maximality of $J'$ there are $j_i \in J'$, $r_i \in R_P$ such that

(*) $j_i + r_i.e_i \in R_Q$ and $(j_i + r_i.e_i)(q) = a$ $(i = 1,2)$.

Consider $s = e_1(j_2 + r_2e_2) + e_2(j_1 + r_1e_1)$. Since $e_1, e_2 \in R_Q$ we have $s \in R_Q$. Also, $s(q) = e_1(q).a + e_2(q).a = [(e_1+e_2)(q)].a = a$. But also $s = e_1j_2 + e_2j_1 \in J'$. So $s \in J'_Q$ and $s(q) = a$, a contradiction to the choice of $J'$. Hence $J'_Q$ is full in $R_Q$, and clearly $\sigma J'_Q = q$. This proves the claim.

Now by the above, $J'_Q(q_1) \subseteq \bigcap\{\nu_{q_1p}^{-1}(J'(p)) : p/Q = q_1\}$ for all $q_1 \in Q$. It follows from the claim that $J'(p) = R_p$ for all $p \in P$ with $p/Q \neq q$.

Take $r \in R_Q$ with $r(q) = a$. Then as $a \in I_q$, $r(p) \in J(p) \subseteq J'(p)$ for all $p \in P$ with $p/Q = q$. So by (2.7), $r \in J'$, a contradiction.

□

We can now determine the left ideal of $R_P$ generated by a left ideal of $R_Q$ for $Q \leqslant P$. The following result also applies if we replace 'left' by 'two-sided' throughout.

**Corollary 2.9**

Let $Q \leqslant P$ and $I$ be a left ideal of $R_Q$. Then:

(i) the left ideal $J$ of $R_P$ generated by $I$ is given by

(*) for all $p \in P$, $J(p)$ is the left ideal of $R_p$ generated by $\nu_{p/Q,p}[I(p/Q)]$.

(ii) if $I = I'@q$ in $R_Q$ (for some $q \in Q$ and left ideal $I'$ of $R_q$) and for all $p \in P$ with $p \neq q$, $p/Q = q$, the left ideal of $R_p$ generated by $\nu_{qp}(I')$ is improper, then $I$ generates the left ideal $I'@q$ in $R_P$.

**Proof**

(i) For each $p \in P$ let $J_p$ be the left ideal of $R_p$ generated by $\nu_{p/Q,p}(I(p/Q))$. Let $J = \{r \in R_P : r(p) \in J_p$ for all $p \in P\}$. Certainly $J$ is an ideal of $R_P$, and $J(p) = J_p$ for all $p \in P$. But if $J' \supseteq I$ is a left ideal of $R_P$ then $J'_Q \supseteq I$, so by (2.8), for each $p \in P$ and $q \in Q$ with $p/Q = q$ we have $\nu_{qp}^{-1}[J'(p)] \supseteq J'_Q(q) \supseteq I(q)$. Hence $J(p) \subseteq J'(p)$ for all $p$, so that $J \subseteq J'$. So $I$ generates $J$ in $R_P$.

(ii) This is a special case of (i); we will use it in Part II.

□

**Corollary 2.10**

Assume that $Q \leqslant P$ and let the left ideal I of $R_Q$ generate the left ideal J of $R_P$. Then

(i) $J(q) = I(q)$ for all $q \in Q$

(ii) $J_Q = I$.

**Proof**

(i) is a special case of Corollary 2.9(i). Hence for each $q \in Q$, $I(q) \subseteq J_Q(q) \subseteq J(q) = I(q)$, so $J_Q(q) = I(q)$. Part (ii) now follows by (2.7).

□

## 3 Densely decomposable ideals

Here we develop a way to obtain an atomless Boolean ring as the limit of an SFP system in the case where all component rings are Boolean. As in [Z] we use densely decomposable ideals to generalise the notion of *atomless* to arbitrary rings. Again, unless otherwise stated all ring ideals will be left ideals.

Recall from the introduction the definition of densely decomposable:

**Definition**

Let R be any ring, and I a proper left ideal of R. I is said to be **densely decomposable** if whenever J is a left ideal of R properly extending I, then there are left ideals $X, Y \subseteq J$ properly extending I, with $X \cap Y = I$.

**Example 3.1**

Let R be a Boolean ring. Then the ideal $\{0\}$ is densely decomposable iff R is atomless: that is, if $r \neq 0$ in R then there is $s \in R$ with $r \neq s.r = s \neq 0$. So for an ideal of a Boolean ring, being densely decomposable is the same as having atomless quotient, and is in a sense opposite to being irreducible.

We wish to find conditions for ideals of the limit ring of an SFP system to be densely decomposable.

**Definition**

Let R be a ring and $I \subseteq J$ left ideals of R. We say that J **splits over I** if there are left ideals $X, Y \subseteq J$ with $X \supset I$, $Y \supset I$, $X \cap Y = I$. If $S \supseteq I$ is any subset of R, we say that S

**strongly splits over** I if for all left ideals J with $I \subset J \subseteq S$, J splits over I.

Clearly I is a densely decomposable ideal of R iff any set $S \supseteq I$ strongly splits over I.

Let $(P,\rho,\nu)$ be an SFP system. An ideal of $R_P$ can be densely decomposable for two reasons: its projections onto the component rings $R_p$ ($p \in P$) might already make it densely decomposable, or else it can be densely decomposable because of the SFP system structure of $R_P$. We now separate the two causes. As in Section 2, if I is an ideal of $R_P$ and $n \in N \preccurlyeq P$ we write $I_N(n)$ for the projection $\{r(n) : r \in I_N\}$ of $I_N$ ($= I \cap R_N$) onto $R_n$.

**Definition**

Let I be a left ideal of $R_P$. We define $I^\wedge$ to be the set
$\{r \in R_P :$ for any finite support $N \preccurlyeq P$ of r, there is at most one $n \in N$ with $r(n) \notin I_N(n)$ $\}$.

So $I \subseteq I^\wedge$. If I is a proper ideal of $R_P$, then by (2.2) I is full iff $I^\wedge = R_P$. If $r \in R_P$ and $i \in I^\wedge$ then clearly $i.r \in I^\wedge$. Hence $I^\wedge$ is the union of the left ideals contained in it.

**Lemma 3.2**

Let I be a left ideal of $R_P$. The following are equivalent:

(i) I is a densely decomposable ideal of $R_P$

(ii) $I^\wedge$ strongly splits over I in $R_P$.

**Proof**

We only need prove that (ii) implies (i). Let $J \supset I$ be a left ideal of $R_P$; we must prove that J splits over I. If $J \subseteq I^\wedge$ this is clear by assumption. Assume not. There is $r \in J$ and a finite support $N \preccurlyeq P$ of r such that for distinct $y, z \in N$ we have $r(y), r(z) \notin I_N$. Define $e_y \in R_N$ by: $e_y(x) = 1$ if $x=y$, and 0 otherwise. Let Y be the left ideal of $R_P$ generated by I and $e_y.r$. Define $e_z$ and Z similarly. Then $e_y$ and $e_z$ are orthogonal central idempotents of $R_P$. We clearly have $I \subset Y$, Z and Y, $Z \subseteq J$.

<u>Claim</u> $Y \cap Z = I$.

<u>Proof of Claim</u> Let $s \in Y \cap Z$. So $s = i_1 + r_1(e_y.r) = i_2 + r_2(e_z.r)$ for some $i_1, i_2 \in I$ and $r_1, r_2 \in R_P$. Multiplying by $e_z$, we obtain $e_z i_1 = e_z i_2 + r_2(e_z.r)$. Hence $r_2(e_z.r) = e_z(i_1 - i_2) \in I$. Hence $s = i_2 + r_2(e_z.r) \in I$, proving the claim.

□

Hence whether I is densely decomposable depends only on $I^\wedge$. For example, if $p \in P$ and I is an ideal of $R_p$ then I@p is densely decomposable iff I is densely decomposable in $R_p$.

Clearly the smaller $I^\wedge \setminus I$ is, the more likely I is to be densely decomposable. We describe one way to force $I^\wedge \setminus I$ to be small. Suppose that P is an SFP domain such that *for every finite* $N \preccurlyeq P$ *and* $n \in N$, *there is* $p \in P \setminus N$ *with* $p/N = n$. This condition is equivalent to:

(£) for every $n \in N$, $\{p \in P : p/N = n\}$ is infinite.

Examples are the ideal completion of any dense linear ordering with a least element, or of the tree ${}^{<\omega}\omega$ (the set of finite sequences of natural numbers, ordered by 'initial segment').

### Definition

Let $I \subseteq R_P$ be a left ideal. We say that I is **locally generated** if there is finite $N \preccurlyeq P$ such that $I_N$ generates I in $R_P$.

For example, any finitely generated ideal of $R_P$ is locally generated.

### Proposition 3.3

Suppose that we have an SFP system $\langle P,\rho,\nu\rangle$ (where P satisfies the condition (£) above) in which every $\nu_{pq}$ is a ring isomorphism. Let I be any proper locally generated left ideal of $R_P$. Then I is densely decomposable.

### Proof

By Lemma 3.2 it is enough to show that $I^\wedge = I$. Without loss all $R_p$ are the ring S and each $\nu_{pq}$ is the identity map. Let $r \in R_P \setminus I$. Take finite $N \preccurlyeq P$ supporting r and such that $I_N$ generates I. As $r \in R_N$ we have $r \notin I_N$, so by (2.1) there is $n \in N$ with $r(n) \notin I_N(n)$. By (£) we can choose finite $M \preccurlyeq P$ containing N and such that the set

$n^M = \{m \in M : m \neq n \text{ and } m/N = n\} \neq \emptyset$.

By (2.10(ii)) the left ideal of $R_M$ generated by $I_N$ is in fact $I_M$. By (2.9) $I_M(m) = I_N(m/N) \subseteq S$ for all $m \in M$. Now M also supports r. Take $m \in n^M$. We have $r(n) \notin I_M(n)$, and $r(m) = r(n) \notin I_N(n) = I_M(m)$. Hence $r \notin I^\wedge$ as required.

□

### Remark

We can evidently weaken the assumption on the $\nu_{pq}$ in (3.3) to: if $N \preccurlyeq P$ is finite then for each $n \in N$ there is $m \in P^0 \setminus N$ such that $m/N = n$ and $\nu_{nm}$ is a (surjective) isomorphism.

### Example 3.4

In the case where all $R_p$ are isomorphic to a ring S and all $\nu_{pq}$ are the identity, the limit $R_P$ is determined up to isomorphism by S and P. It follows from the proposition that letting P be the ideal completion of ${}^{<\omega}\omega$ and $S = \{0,1\}$, the zero ideal of $R_P$ is densely decomposable. Hence by (1.4), (1.5) and the above, $R_P$ is the countable atomless Boolean ring B (there is a unique such ring up to isomorphism - see [CK]). Similarly, if Q is the ideal completion of the set $\mathbb{Q}_{\geqslant 0}$ of non-negative rational numbers with the usual ordering, then $R_Q \cong B$. Here, $R_Q$ is essentially the Boolean algebra of half-open intervals of $\mathbb{Q}$. (So P is homeomorphic to Q in the patch topology - as is any SFP domain X with countable base and satisfying (£).) Since P and Q are not isomorphic posets, this shows that in general we cannot recover the poset structure of an SFP domain P from a limit ring $R_P$. We will pursue this in Part II.

## 4 $L_{\infty\omega}$-equivalence of SFP systems and their limits

Here we define a canonical model-theoretic structure $M_\sigma$ from an SFP system $\sigma = \langle P,\rho,\nu\rangle$. We prove that if $\sigma_i = \langle P_i,\rho_i,\nu_i\rangle$ (i=1,2) are SFP systems and $M_{\sigma_1}$ and $M_{\sigma_2}$ are $L_{\infty\omega}$-equivalent then so are the limit rings of $\sigma_1$ and $\sigma_2$. We will also provide a simple sufficient condition for $M_{\sigma_1}$ and $M_{\sigma_2}$ to be $L_{\infty\omega}$-equivalent; namely that $(P_1)^0$ and $(P_2)^0$ are $L_{\infty\omega}$-equivalent and the $\sigma_i$ are sufficiently similar SFP systems. In Sections 6 and 7 we will construct $2^{\aleph_1}$ SFP domains $P_i$ ($i<2^{\omega_1}$) such that $(P_i)^0$ and $(P_j)^0$ are $L_{\infty\omega}$-equivalent for all $i<j<2^{\omega_1}$, and yet the limit rings of any SFP systems built on the $P_i$ are pairwise non-embeddable. Hence these limits will nonetheless be $L_{\infty\omega}$-equivalent if the SFP systems are sufficiently similar. This means crudely that though the limit rings are different, the differences are hard to detect.

Recall from e.g. [CK] the definition of $L_{\infty\omega}$-equivalence. Let L be any signature; the infinitary language $L_{\infty\omega}$ is built from L by allowing formulas with finite strings of quantifiers but conjunctions and disjunctions of arbitrary length. Two L-structures M, N are said to be **$L_{\infty\omega}$-equivalent** (written $M \equiv_{\infty\omega} N$) if they satisfy the same sentences of $L_{\infty\omega}$.

We can usefully characterise $L_{\infty\omega}$-equivalence in terms of a game between two players, '$\forall$' and '$\exists$', played on two L-structures M and N. The game G(M,N) has $\omega$ moves. At each move in a play, player $\forall$ chooses an element from one structure - M or N. Then $\exists$ completes the move by choosing an element from the other structure. After the play is over, the result is two tuples $\bar{m} \in M$, $\bar{n} \in N$ of length $\omega$, possibly with repetitions: the $i^{th}$ elements $m_i$, $n_i$ of $\bar{m}$, $\bar{n}$ respectively consist of the elements chosen in the $i^{th}$ move of the game from M, N respectively. (No record is kept of which player chose which element.) So $\bar{m}$ and $\bar{n}$ define a relation $\theta = \{(m_i,n_i) : i<\omega\} \subseteq M\times N$. $\exists$ wins the play of the game iff $\theta$ is a partial isomorphism - i.e. $\theta$ is a partial function from M to N, and for all quantifier-free first order fomulas $\varphi(\bar{x})$ of L and all $\bar{a} \in \mathrm{dom}(\theta)$, $M \models \varphi(\bar{a})$ iff $N \models \varphi(\theta(\bar{a}))$.

**Fact 4.1**

M and N are $L_{\infty\omega}$-equivalent iff $\exists$ has a winning strategy in the game G(M,N). See [Hg] or [K] for details.

**Definition**

Let $\sigma = \langle P,\rho,\nu \rangle$ be an SFP system. Define a structure $M_\sigma = (P^o,(R_p : p \in P^o))$ in the signature $\{\leqslant,\rho^*,\nu^*,+^*,-^*,0^*,1^*\}$. The domain of $M_\sigma$ is the disjoint union of $P^o$ and the $R_p$ $(p \in P^o)$. The binary relation symbol $\leqslant$ is interpreted as the partial ordering on $P^o$: $M_\sigma \models p \leqslant q$ iff $p, q \in P^o$ and $p \leqslant q$. $\rho^*$ is a binary relation symbol, and $M_\sigma \models \rho^*(p,r)$ iff $p \in P^o$, $r \in R_p$. $\nu^*$ is a binary relation symbol corresponding to $\nu$: we define $M_\sigma \models \nu^*(r,s)$ iff $r \in R_p$, $s \in R_q$ for some $p, q \in P^o$ (necessarily unique) with $p \leqslant q$, $\nu_{pq}(r) = s$. The ternary relation symbols $+^*$, $-^*$ are defined on each $R_p$ in the obvious way: $M_\sigma \models +^*(r,s,t)$ iff $r, s, t \in R_p$ for some $p \in P^o$ and $r+s=t$, and similarly for $-^*$. $0^*$ and $1^*$ are unary relation symbols and $M_\sigma \models 0^*(r)$ iff $r = 0 \in R_p$ for some $p \in P^o$ (and similarly for $1^*$).

We say that SFP systems $\sigma_1$, $\sigma_2$ are **$L_{\infty\omega}$-equivalent** if $M_{\sigma_1} \equiv_{\infty\omega} M_{\sigma_2}$.

We use this to prove the following theorem.

**Theorem 4.2**

Let $\sigma_i = \langle P_i,\rho_i,\nu_i \rangle$ be $L_{\infty\omega}$-equivalent SFP systems with limit rings $R_i$ (i=1,2). Then $R_1 \equiv_{\infty\omega} R_2$ in the signature $\{+,-,0,1\}$ of rings.

**Proof**

By hypothesis and Fact 4.1 we may take a winning strategy for $\exists$ in the game $G(M_{\sigma_1},M_{\sigma_2})$. We will describe a winning strategy for $\exists$ in the game $G(R_1,R_2)$. We use a play of $G(R_1,R_2)$ to generate a play of $G(M_{\sigma_1},M_{\sigma_2})$. $\exists$'s strategy in this game will then suggest moves for her in the main game $G(R_1,R_2)$. The method is well known.

More fully, let $\forall$ begin by choosing (without loss) $r_1 \in R_1$. $\forall$'s choice gives rise to the following finite sequence of elements of $M_{\sigma_1}$: those in an arbitrary finite support $N_1 \trianglelefteq P_1^o$ for $r_1$, listed in some arbitrary order, together with the sequence $r_1(n_1)$ of elements of the $R_{n_1}$ $(n_1 \in N_1)$. $\exists$ treats them as successive moves of $\forall$ in a play of $G(M_{\sigma_1},M_{\sigma_2})$ and uses her winning strategy in this game to choose corresponding elements of $M_{\sigma_2}$. This correspondence gives a partial isomorphism from $M_{\sigma_1}$ to $M_{\sigma_2}$. Moreover, as the strategy is winning, the elements chosen corresponding to the $n_1$ form a locally directed subset $N_2$ of $P_2^o$. Hence the elements corresponding to the $r_1(n_1)$ give rise to an element $r_2$ of $R_2$: $r_2$ is supported by $N_2$ and for each corresponding pair $n_1$, $n_2$, $r_1(n_1)$ corresponds to $r_2(n_2)$. $\exists$'s reply in the main game $G(R_1,R_2)$ is this element $r_2$.

In each subsequent move ∀'s choice generates a further finite sequence of elements of a structure $M_{\sigma_i}$ (i = 1 or 2). We can assume that the set of all elements so far chosen in each $P_i^o \subseteq M_{\sigma_i}$ (i=1,2) form a locally directed subset. On each occasion ∃ continues with her strategy to obtain corresponding elements of the other structure. Note that at each stage, *all* elements so far chosen in $M_{\sigma_1}$ are in partial isomorphism with the corresponding ones in $M_{\sigma_2}$.

After ω moves, tuples of ω elements $\bar{a}_1 \in M_{\sigma_1}$, $\bar{a}_2 \in M_{\sigma_2}$ will have been generated. The map $\bar{a}_1 \mapsto \bar{a}_2$ is a partial isomorphism from $M_{\sigma_1}$ to $M_{\sigma_2}$. It is now easy to see that the corresponding elements of the $R_i$ (i=1,2) are also in partial isomorphism. Hence the strategy described is winning for ∃. The result follows by (4.1).

□

**Corollary 4.3**

Let $P_1$ and $P_2$ be SFP domains with $P_1^o \equiv_{\infty\omega} P_2^o$. Let $R_1$, $R_2$ be $L_{\infty\omega}$-equivalent rings and define SFP systems $\sigma_i = \langle P_i, \rho_i, \nu_i \rangle$ (i=1,2) by:

- $\rho_i(p) = R_i$ for all $p \in P_i$
- $\nu_i(p,q) = id_{R_i}$ for all $p \leqslant q$ in $P_i$.

Then the limit rings $R_{P_1}$ and $R_{P_2}$ are $L_{\infty\omega}$-equivalent.

**Proof**

It is evident that $\sigma_1$ and $\sigma_2$ are $L_{\infty\omega}$-equivalent SFP systems. The result follows by (4.2).

□

This shows that with restrictions on the rings and morphisms of the SFP systems $\sigma_1$ and $\sigma_2$, to get the limit rings to be $L_{\infty\omega}$-equivalent it suffices to begin with SFP domains having $L_{\infty\omega}$-equivalent bases. We will apply this in Section 4 of Part II.

# Part II: Anti-structure theorems

We briefly sketch the 'anti-structure theorem' that will occupy this part of the paper. Our description here is not accurate in detail. The technique is well known. To simplify matters, assume that we have two SFP domains P, $P^1$ that are in fact a certain kind of tree of height $\omega_1$, that all $R_p$ ($p \in P \cup P^1$) are the trivial ring {0,1}, and that all $\nu_{pq}$ are isomorphisms. We can express P as a union $\bigcup_{i<\omega_1} P_i$, where $P_i$ is the subtree of P consisting of the elements of height $<i$. The $P_i$ are an increasing chain of SFP subdomains of P. Thus we have $R^* = \bigcup_{i<\omega_1} R_i^*$ where each $R_i^*$ is the set of elements of $R^*$ supported by a subset of $P_i$. Take a full ideal I of $R^*$; $I = \{r \in R : r(s) = 0\}$ for some $s \in P$. It turns out that $I \cap R_i^*$ is full for all $i<\omega_1$. So for each $i<\omega_1$ there is $s_i \in P_i$ such that $I \cap R_i^* = \{r \in R_i^* : r(s_i) = 0\}$, and $s_i \leqslant s_j \leqslant s$ if $i<j<\omega_1$. The same holds for $R^{1*}$, defined similarly using $P^1$.

Assume now that $\theta : R^* \to R^{1*}$ is a ring isomorphism. Then the set $C = \{i<\omega_1 : \theta(R_i^*) = R^1{}_i{}^*\}$ is a club (a large set) in $\omega_1$. Moreover, for each $i \in C$, $\theta(I \cap R_i^*)$ is a full ideal of $R^1{}_i{}^*$ (here, 'full' is the same as 'maximal'). So there are $s^1{}_i \in P^1{}_i$ ($i \in C$) such that $\theta(I \cap R_i^*) = \{r \in R^1{}_i{}^* : r(s^1{}_i) = 0\}$. Thus $\theta$ induces a partial map $\Theta$ from $P|C$ to $P^1|C$ by: $s_i \mapsto s^1{}_i$, where $P|C = \{p \in P : p \text{ has height in } C\}$ (and similarly for $P^1$). By considering all full ideals I, $\Theta$ extends to a bijection from $P|C$ to $P^1|C$, and it is order preserving. Thus the existence of an isomorphism from $R^*$ to $R^{1*}$ forces the underlying SFP domains to be closely related: there is a club $C \subseteq \omega_1$ such that $P|C \cong P^1|C$.

So in order to produce many non-isomorphic rings $R^*$ it suffices to find many trees P such that no two are isomorphic on any club. In [AS] this is done for Aronszajn trees, using the hypothesis of $2^{\aleph_0} < 2^{\aleph_1}$ (weak diamond). Our construction here is in some ways similar, but a weaker result suffices and we do not need any set-theoretic hypotheses beyond ZFC. The trees we construct are not strictly Aronszajn trees: in fact it is consistent with $MA + 2^{\aleph_0} > \aleph_1$ that any two Aronszajn trees are isomorphic on some club [AS]. However, our construction is made complicated by our considering ring embeddings $\theta$ (and not just isomorphisms) and arbitrary rings $R_p$ (not just {0,1}). In this setting $\Theta$ becomes a relation between the restricted trees.

The layout of Part II is as follows. In §1 the appropriate form of tree is defined and the relation $\Theta$ discussed. In §2 we construct many different trees using a Aronszajn-style argument, and use them to produce many different rings. Finally we establish some higher-order properties of the rings. We show that each of their full ideals can be made countably generated (§3), and that the rings themselves can be made pairwise $L_{\infty\omega}$-equivalent (§4) and to some degree rigid (§5).

# 1 Conformal relations

Here we investigate the effect of the SFP domain of an SFP system on its limit ring. We want in particular to find a way of changing the underlying domain that necessarily changes (the isomorphism type of) the ring. Our approach is to ask how much of the domain structure gets to be encoded in the limit ring, in such a way that we can recover it purely ring-theoretically. For if we use two different domains, and they are recoverable intrinsically from the two limit rings in sufficient detail to reveal their differences, then the rings must be different as rings.

Now however we may alter the domain in a system, there is no guarantee of getting different rings as limits in the countable situation - when the base of the domain and also each component ring is countable. Suppose for instance that we build atomless Boolean rings (as in Example I.3.4). Up to isomorphism there is a unique countable such ring. Hence the domain structure here cannot exert any effect.

However, things are different if we allow the base of the domain to be uncountable, as we will see.

The chief hope of recovering the domain structure lies with the full ideals of the limit rings: in §I.2 we used the notion of *site* to relate these ideals to the underlying domain. Now if the base is of cardinality $\aleph_1$, we can here express the whole SFP domain as a union of an uncountable chain of SFP subdomains with countable bases. Each subdomain induces a subsystem, and its limit is a countable subring of the main limit ring. Hence the original ring, of cardinality $\aleph_1$, can be written as the union of a chain of countable subrings, built around the subdomains.

In this situation we can take a full ideal of the main ring and look at its intersections with the subrings. They will also be full, and their sites will form a linearly ordered subset of the main domain. By considering the 'forking behaviour' of the sets that arise in this way, we can build up a picture of the original domain. This picture turns out to be substantially independent of the choice of subdomains, and has sufficient detail to distinguish different 'main' domains, which is what we wanted.

To get this idea to work we restrict our attention to domains having a certain kind of tree structure. We can construct $2^{\aleph_1}$ different such trees, each with base set of cardinality $\aleph_1$, such that if P, P' are two such, and we take any two SFP systems on P, P', with countable component rings and having limits R, R', say, then there is no ring embedding from R into R'. Moreover, by choosing the component rings and connecting maps of the systems more carefully, we can make R and R' $L_{\infty\omega}$-equivalent. In a similar way we can also ensure that R (and R') have various higher order properties, such as some rigidity: they have few automorphisms.

The actual construction of the trees is done in §2; here, we are concerned mainly with the ring theory. However, we do need to quote some combinatorics.

*Trees*

The following are generally known definitions and we include them for convenience. A **tree** is a non-empty poset $(T,\leqslant)$ such that the set $\hat{t} = \{u \in T : u < t\}$ of predecessors of any $t \in T$ is well-ordered (hence linearly ordered). We will refer to the elements of a tree as **nodes**. The **height** of a node $t \in T$, $\mathbf{ht_T(t)}$ or $\mathbf{ht(t)}$, is the order type of $\hat{t}$. If i is an ordinal, we write $\mathbf{T(i)}$ for the set of nodes of T of height i: the $\mathbf{i^{th}}$ **level** of T.

More generally, if $S \subseteq T$ is closed downwards, we write $\mathbf{S(i)}$ for $S \cap T(i)$, and $\mathbf{ht_T(S)}$ for the least ordinal i such that $S(i) = \emptyset$. If X is a set of ordinals, we define $\mathbf{S|X}$ to be $\{s \in S : ht_T(s) \in X\}$. So for example, if i is an ordinal then $S|i$ is the set of elements of S of height $<i$. (Since S if non-empty is a tree in its own right, the notations S(i) etc. would be ambiguous if S were not closed downwards in T.)

If $t, t' \in T$, $t'$ is an **immediate successor** of t if $t' > t$ and $ht(t') = ht(t)+1$. Then also t is an **immediate predecessor** of t'. A **terminal node** is one without any successors in T; a **branching node** is a node with at least two immediate successors. A node $t \in T$ is said to be **green** in T if T contains a branching node $b \geqslant t$.

A tree T is called **normal** if whenever $t, t' \in T$ have equal limit height and $u<t$ iff $u<t'$ for all $u \in T$, then $t = t'$. Our convention is that every ordinal is exactly one of: 0, successor, limit.

A **branch** of a tree T is a maximal linearly ordered subset of T. A branch $\beta$ is said (unusually) to be **cofinal in T** if every node of $\beta$ is green in T. If T is normal this means that the branching nodes are 'cofinal' in $\beta$: if $i < ht(\beta)$ then there is a branching node $b \in \beta$ of height at least i in T.

**Remark**

Let T be a tree with a least element $\perp$. Then any $S \subseteq T$ with $\perp \in S$ is locally directed in T. If T is a dcpo then T is an SFP domain, the finite elements being those not of limit height.

*Spruce trees*

We can now define the type of tree that interests us here. A **spruce tree** is a normal tree T satisfying:

(i) every branch of T has height $\omega_1$

(ii) each node of T has exactly one non-branching immediate successor

(iii) T has no cofinal branches

(iv) for all $i < j < \omega_1$ and every branching node b of height i in T, there are exactly $\aleph_0$ branching nodes of T of height j above b

(v) T(0) has just one node '$\perp$', which is a branching node; each higher level of T has exactly $\aleph_0$ branching nodes.

An example of a spruce tree is an Aronszajn tree (cf. [J2] and below) but with each branch and node extended individually by new non-branching nodes up to height $\omega_1$. In §2 the existence of many spruce trees is established.

Let T be a spruce tree. A node of T is said to be **basic** if it is a branching node and it is finite in the sense of §I.1 - that is, its height is not a limit ordinal. We write **B(T)** for the set of basic nodes of T.

Let D be a directed set of basic nodes in T. Then D is linearly ordered, and since T has no cofinal branches, D is countable. Now although T is not a dcpo, it is normal and every branch has height $\omega_1$, so D has a unique least upper bound in T. It follows that the ideal completion of the set of basic nodes embeds canonically into T. So let us call a node of T a **limit node** if it is in the image of this embedding. We write **L(T)** for the set of limit nodes; so $B(T) \subseteq L(T)$. Clearly L(T) is a dcpo, and since it is a tree with a single least element, it is in fact an SFP domain. Notice that $B(T) \trianglelefteq L(T) \trianglelefteq T$.

We will use the SFP domain L(T) to build SFP systems. The remainder of T is used to keep track of what is going on. To do this we need to deal with the subtrees of T of countable height.

Recall that if $\lambda$ is a limit ordinal and $X_i$ $(i < \lambda)$ are arbitrary sets, the $X_i$ are said to form a **continuous chain** if $X_i \subseteq X_j$ for each $i < j < \lambda$, and for each limit ordinal $j < \lambda$, $X_j = \bigcup\{X_i : i < j\}$. The **union** of the chain is defined to be $\bigcup\{X_i : i < \lambda\}$.

If $i < \omega_1$ we define $\mathbf{L(T)_i}$ to be the set of elements of L(T) with height at most i in T. Then $L(T)_i \preccurlyeq L(T)$. Similarly define $\mathbf{B(T)_i} = B(T) \cap T|i{+}1$. B(T) has no nodes of limit height, so the $B(T)_i$ $(i < \omega_1)$ form a continuous chain with union B(T). For each $i < \omega_1$, $L(T)_i \preccurlyeq L(T)$. Moreover, since $L(T) \subseteq T$ and T is spruce we have $L(T) = \bigcup_{i<\omega_1} L(T)_i$.

*Spruce trees and SFP systems*

Now take an SFP system $<L(T),\rho,\nu>$ such that each $R_t$ $(t \in B(T))$ is countable. Let its limit ring be R. Writing $R_i$ for $R_{L(T)_i}$, we see that the $R_i$ form a continuous chain of countable subrings of R, with union R.

We define for each $i < \omega_1$ a projection $\pi_i : R \to R_i$, given as follows. If $r \in R$, then by definition r is a function from L(T) into $\bigcup\{R_t : t \in L(T)\}$. Then $\pi_i(r)$ is just the restriction $r|L(T)_i$ of r to the set $L(T)_i$.

We must show that $\pi_i(r) \in R_i$. Let $N \preccurlyeq L(T)$ be a finite support of r in R and define $N' = N \cap L(T)_i$. Then since $L(T)_i$ is closed downwards in L(T), $\perp \in N'$ and so $N' \preccurlyeq L(T)_i$.

Clearly if $x \in L(T)_i$ then $x/N = x/N'$. It follows that $N'$ supports $\pi_i(r)$ in $R_i$. So $\pi_i(r) \in R_i$, as required.

Each $\pi_i$ is a surjective ring homomorphism and is the identity on $R_i$.

*Full ideals*

The notion of full ideals becomes a little more complicated in this setting, since now we have $\aleph_1$ different rings and we can no longer tell from its site which ring a full ideal lies in. So we refine the notion of site, using the part of the tree T that lies outside L(T).

Recall that each $t \in T$ has a unique non-branching immediate successor - $t^+$ say. Hence if $ht(t) = i$, we can define a node $t^{[j]}$ for each $i \leqslant j < \omega_1$, by induction on j:

- $t^{[i]} = t$
- $t^{[j+1]} = t^{[j]+}$
- if j is a limit ordinal, $t^{[j]}$ is the unique node of height j with $t^{[j]} > t^{[k]}$ for all $i \leqslant k < j$; this is well defined as T is spruce.

Note that although certainly $t^{[j]}$ is not a branching node if $j>i$ is a successor ordinal, it may be a branching node if j is limit. If $j>i$ we have $t^{[j]} \notin L(T)$.

In the light of this we can define a map $\zeta : T \to T$ by: $\zeta(t)$ is the lowest node $t' \leqslant t$ such that $t = t'^{[ht(t)]}$. We clearly have:

**Proposition 1.1**

$\zeta(T) = L(T)$ and $\zeta^2(t) = \zeta(t)$ for all $t \in T$. For all $i < \omega_1$ the restriction $\zeta|T(i) : T(i) \to L(T)_i$ is a bijection, whose inverse is given by $t \mapsto t^{[i]}$.

□

Now if $i < \omega_1$ then the set of possible sites for full ideals of $R_i$ is $L(T)_i$, and this is in bijection with $T(i)$ via $\zeta$. So if I is a full ideal of $R_i$ with site $s \in L(T)_i$, we define the **tree site** of I, $\tau I$, to be $s^{[i]} \in T(i)$.

Tree sites behave well with respect to subrings. We have:

**Proposition 1.2**

(i) If $i < j < \omega_1$ and J is a full ideal in $R_j$, then $J \cap R_i$ is full in $R_i$ and $\tau(J \cap R_i) \leqslant \tau J$. Since it has height i, $\tau(J \cap R_i)$ is determined by this inequality.

(ii) If $i < j < \omega_1$ and I is a full ideal of $R_i$, then the ideal $\pi_i^{-1}(I) \cap R_j$ is full in $R_j$, and $\tau(\pi_i^{-1}(I) \cap R_j) = (\tau I)^{[j]}$. We write $I^{[j]}$ for this ideal.

**Proof**

(i) Let $\sigma J = p \in L(T)_j$. Since $L(T)_i \leqslant L(T)_j$, by (I.2.2) we see that $J \cap R_i$ is full in $R_i$ with site

$q = p / L(T)_i$. We must show that $p^{[j]} \geqslant q^{[i]}$.

If $ht(p) \leqslant i$ then $p \in L(T)_i$ and $q = p$, so the result is clear. So suppose that $ht(p) > i$. We show that $p \geqslant q^{[k]}$ for $ht(q) \leqslant k \leqslant i$, by induction on k.

If $k = ht(q)$ or k is a limit ordinal then this is trivial. Assume that $k+1 \leqslant i$ and $p \geqslant q^{[k]}$. If $p \ngeqslant q^{[k+1]}$ then there is $b \leqslant p$, $ht(b) = k+1$, with $b \neq q^{[k+1]}$. As the immediate predecessor of b is $q^{[k]}$, b must be a branching node in T. Hence $b \in L(T)_i$ and so $q = p / L(T)_i \geqslant b$. As $b > q$, this is a contradiction. So $p \geqslant q^{[k+1]}$, completing the induction.

(ii) Let $\pi_i^{-1}(I) \cap R_j = J$. Then for all $r \in R_j$, $r \in J$ iff $r|L(T)_i \in I$ iff $r(\sigma I) \in \Delta I$. So $J = \Delta I @ \sigma I$ in $R_j$. Hence I and J have the same site and defect, though they lie in different rings. We have $\tau J = (\sigma I)^{[j]} = (\sigma I)^{[i][j]} = (\tau I)^{[j]}$.

□

*Clubs*

Let $C \subseteq \omega_1$. C is said to be a **club** (in $\omega_1$) if it is closed and unbounded in $\omega_1$. That is:

(cl) if $C_0 \subseteq C$ is countable, then $\bigcup C_0 \in C$. ($\bigcup C_0$ is of course the least ordinal i such that $i \geqslant c$ for each $c \in C_0$.)

(ub) for each $i < \omega_1$ there is $c > i$ with $c \in C$.

Examples of clubs are $\omega_1$ itself, and the set of countable limit ordinals. We can go further. If C is any subset of $\omega_1$ we write $\partial C$ for the set of limit points of C: $\partial C$ is the set of all ordinals of the form $\bigcup\{c_i : i < \omega\}$, for some strictly increasing sequence $c_i$ $(i < \omega)$ in C. So (cl) above just says that $\partial C \subseteq C$. We then have:

**FACT** If C is a club then so is $\partial C$.

Note that (ub) implies that C is uncountable. We can think of clubs as 'large' subsets of $\omega_1$. We have:

**FACT** [J1 §7] A countable intersection of clubs is a club.

We remark that if T is a spruce tree and C a club in $\omega_1$, then T|C is normal and satisfies all conditions except possibly (ii) and the first part of (v) of the definition of 'spruce'. A node of T|C is greeen in T|C iff it is green in T.

We will also use the following lemma on clubs.

**FACT** [Hg, 5.2.2]

Let $f : \omega_1 \to \omega_1$ be a map. Then $\{i < \omega_1 : \forall j < i\ (f(j) < i)\}$ is a club in $\omega_1$.

The proofs of these facts are not hard.

*Ring embeddings and onformal relations*

Now suppose that U is another spruce tree. Take an SFP system $<L(U),\rho',\nu'>$, and write $S_u$ for $\rho'(u)$ ($u \in L(U)$) and S for its limit. Suppose that each $S_u$ is countable. We have a continuous chain $S = \bigcup\{S_i : i < \omega_1\}$, as for R. We abuse notation by using the symbol $\zeta$ to refer to the maps on T and on U; we distinguish them by context. But $\pi$ always refers to R.

Recall that if $X \subseteq S$, $X^* = \{s \in X : s(u) \in \{0,1\}$ for each $u \in L(U)\}$. Clearly the $S_i^*$ ($i < \omega_1$) form a continuous chain with union $S^*$.

**Proposition 1.3**

Suppose that $\theta : S \to R$ is a ring embedding. There is a club C of limit ordinals in $\omega_1$ such that for each $i \in C$,

(i) $\theta(S_i) = R_i \cap \theta(S)$

and (ii) if $j < i$ then $\pi_j\theta(S^*) = \pi_j\theta(S_i^*)$.

**Proof**

If $j < \omega_1$, let f(j) be the least $k < \omega_1$ such that

- $\theta(S_j) \subseteq R_k$
- $R_j \cap \theta(S) \subseteq \theta(S_k)$
- $\pi_j\theta(S^*) = \pi_j\theta(S_k^*)$;

k exists since the left hand side of each of these is countable. Then by the fact above, $C' = \{i < \omega_1 : \forall j < i\ (f(j) < i)\}$ is a club in $\omega_1$. We can take $C = C' \cap \partial\omega_1$.

□

Now let $\theta$ and C satisfy the conditions of the proposition. Define a binary relation $\Theta \subseteq T \times U$ as follows. If t, u have equal height i in T, U respectively, and $i \in C$, then:

$t\Theta u$ iff there is a full left ideal I of $R_i$ such that:

- $\tau I = t$
- $J = \theta^{-1}(I \cap \theta(S_i))$ is a full ideal of $S_i$, and $\tau J = u$.

We say that the ideal I **represents** the pair (t,u). Notice that by definition of C, $J = \theta^{-1}(I)$.

**Definition**

Let T, U be arbitrary trees. A relation $\Phi \subseteq T \times U$ is said to be **height preserving** if whenever $t\Phi u$ then $ht_T(t) = ht_U(u)$. A height preserving relation $\Phi$ is said to be **homomorphic** if whenever $t\Phi u$, $t' \leqslant t$, $u' \leqslant u$ and $t'$ and $u'$ have equal heights, then $t'\Phi u'$.

Clearly, $\Theta$ is height preserving.

**Proposition 1.4**

$\Theta$ is a homomorphic relation $\subseteq T|C \times U|C$. Moreover, in the notation above, if $i < j$ in C and J is a left ideal of $R_j$ representing (t,u), then $J \cap R_i$ represents (t',u').

**Proof**

Suppose $i < j$ in C and $t \in T(j)$, $u \in U(j)$ are related by $\Theta$. Take J representing (t,u). J is full in $R_j$, and has (tree) site t. By (1.2), $J \cap R_i$ is full in $R_i$ and has site t'. Similarly, $\theta^{-1}(J) \cap S_i$ is full in $S_i$ with site u'. But as $i \in C$ we have $\theta^{-1}(J) \cap S_i = \theta^{-1}(J \cap R_i)$. Hence $t'\Theta u'$.

□

**Definition**

Let T and U be trees of height $\omega_1$. A height preserving relation $\Phi \subseteq T \times U$ is said to be **surjective** if whenever $i < \omega_1$ and $u \in U(i)$, then there is $t \in T(i)$ such that $t\Phi u$. We then write that $\Phi : T \to U$ is a surjective relation.

**Proposition 1.5**

$\Theta : T|C \to U|C$ is a surjective relation.

**Proof**

Let $u \in U(i)$ for $i \in C$; let $\zeta(u) = z$. Let I be full in $S_i$ with site z, defect 0 (i.e. $I = 0@z$ in $S_i$).

<u>Claim</u>: $\theta(I)$ generates a proper left ideal of $R_i$.

<u>Proof of claim</u>: If not, there are $n_0 < \omega$ and $a_n \in I$, $r_n \in R_i$ $(n < n_0)$ such that

- $\Sigma_{n<n_0} r_n.\theta(a_n) = 1.$

Now for each n we have $a_n(z) = 0$. We can take finite $N \leqslant L(U)_i$ such that each $a_n$ is supported by N, and $a_n(z') = 0$, where $z' = z/N$.

Define $d \in S_i$ by: d is supported by N; $d(x) = 1$ if $x \in N$, $x \neq z'$, and $d(z') = 0$. Then $d \neq 1$, but $a_n.d = a_n$ for each $n < n_0$.

Now let $e = \theta(d) \in R_i$. Since $\theta$ is an embedding, $e \neq 1$. But we have

- $e = [\Sigma r_n.\theta(a_n)].e = \Sigma r_n.(\theta(a_n.d)) = \Sigma r_n.\theta(a_n) = 1,$

a contradiction. This proves the claim.

By Zorn's Lemma there is a maximal left ideal J of $R_i$ extending $\theta(I)$. By (I.2.4) and (I.2.5), J is full in $R_i$. Then $\theta^{-1}(J)$ is a proper left ideal of $S_i$ and extends I; so it is full with site z. So if $t = \tau J$, we have $t \Theta u$, the pair (t,u) being represented by J.

□

**Definition**

Let T and U be spruce trees and C a club in $\omega_1$. A homomorphic relation $\Phi : T|C \to U|C$ is said to be **continuous** if for all $i < j$ in C and all $u \in U(i)$ there is a node $u^{[\Phi,j]} \geq u$, of height j in U, such that

- for all $t \in T(i)$ with $t\Phi u$ and $\zeta(t) < t$, if there exists $u' \in U(j)$ such that $u' > u$ and $t^{[j]}\Phi u'$ then $u' = u^{[\Phi,j]}$.

We do not require that $u^{[\Phi,j]} = u^{[j]}$.

Essentially this says that a small change in nodes in T|C (viz. going from $t^{[j]}$ to $t'^{[j]}$, where t and t' are related via $\Phi$ to the same node $u \in U|C$) results in only a small change (no change) in their $\Phi$-relatives above u in U: if $t^{[j]}$ and $t'^{[j]}$ are related to any node above u, then they are related to only one, and the same one. Hence the name 'continuity'.

**Proposition 1.6**

$\Theta : T|C \to U|C$ is continuous.

**Proof**

Suppose that $i < j$ in C, $u \in U(i)$, and let $t_1, t_2 \in T(i)$ be such that $t_\ell \Theta u$, $\zeta t_\ell < t_\ell$ ($\ell = 1, 2$). Suppose that $u_\ell \in U(j)$ with $u_\ell > u$ are such that $t_\ell^{[j]} \Theta u_\ell$ for $\ell = 1, 2$. We must show that $u_1 = u_2$.

For $\ell = 1, 2$ there is a full left ideal $J_\ell$ of $R_j$ representing the pair $(t_\ell^{[j]}, u_\ell)$. Then $\sigma J_\ell = \zeta \tau J_\ell = \zeta t_\ell < t_\ell$. As i is a limit ordinal there is $k < i$ such that $\sigma J_\ell \in T|k$. Set $K_\ell = J_\ell \cap R_k$. By (1.2), $J_\ell = \pi_k^{-1}(K_\ell) \cap R_j$.

Assume for contradiction that $u_1 \neq u_2$. By (I.2.3), $[\theta^{-1}(J_1)]^* \neq [\theta^{-1}(J_2)]^*$, so without loss there is $s \in S_j^*$ with $\theta(s) \in J_1 \setminus J_2$. Hence $\pi_k\theta(s) \in K_1 \setminus K_2$. By definition of C (cf. 1.3(ii)), there is $s' \in S_i^*$ with $\pi_k\theta(s') = \pi_k\theta(s)$. Hence $\theta(s') \in J_1 \cap R_i \setminus J_2 \cap R_i$. Hence $[\theta^{-1}(J_1 \cap R_i)]^* \neq [\theta^{-1}(J_2 \cap R_i)]^*$.

But since $u_\ell \geq u \in U(i)$, by (1.4) we see that $\theta^{-1}(J_\ell \cap R_i)$ is full in $S_i$ with tree site u, for each $\ell$. By (I.2.3) again, $[\theta^{-1}(I_1)]^* = [\theta^{-1}(I_2)]^*$. This is a contradiction. So $u_1 = u_2$, as required.

□

We now adapt surjectivity to green nodes. $\Phi : T \to U$ is said to be **surjective on green nodes** if whenever $u \in U$ is green then there is a green node $t \in T$ with $t\Phi u$.

**Proposition 1.7**

Suppose that $\Phi : T \to U$ is a homomorphic, surjective and continuous relation, where T and U are spruce trees. Then $\Phi$ is surjective on green nodes.

**Proof**

Let $ht(u) = i$. As u is green, using property (iv) of spruceness we may first choose a branching node $u'>u$ of height j in U, and then an ordinal $k > j$ and $u'' \in U(k)$ with $u'' \neq u'^{[\Phi,k]}$.

Now as $\Phi$ is surjective there is $t'' \in T(k)$ related to $u''$ via $\Phi$. Let $t'$, t be the predecessors of $t''$ of heights j, i in T respectively. As $\Phi$ is homomorphic, $t'\Phi u'$ and $t\Phi u$.

If t is not green in T, then $\zeta(t') \leqslant t < t'$ and also $t'' = t^{[k]}$. So by continuity the only node related to $t''$ is $u^{[\Phi,k]}$. This is a contradiction, proving the proposition.

□

**Definition**

A relation $\Phi : T \to U$ on spruce trees is said to be **conformal** if it is homomorphic, surjective on green nodes, and continuous.

**Examples**

Any tree isomorphism is conformal. The results above show that $\Theta : T|C \to U|C$ is conformal.

Conformal relations preserve sufficient tree structure for us to prove our anti-structure results. We will see this in the next two sections.

## 2 Aronszajn trees

See [J1] or [J2] for the classical Aronszajn tree construction to build a tree of height $\omega_1$ with countable levels but no uncountable branches. We modify it slightly to obtain a large family of 'pseudo-Aronszajn' trees such that there is no conformal relation defined on any club between any pair of the family. Hence by the results of §II.1, the limits of any SFP systems built on the trees will be pairwise non-embeddable. We also show how to make the trees fairly rigid with respect to conformal relations. In §5 we will use this to produce rings that are also fairly rigid.

The trees we build are spruce and so not strictly Aronszajn, but they retain enough 'Aronszajn-ness' to ensure that rings built on them have the Aronszajn-like property that every maximal ideal (more generally, every full ideal) is countably generated.

To make the trees different we will use the devices of 'grids' and 'nests'. We will define nests later; first to grids.

### Definition

A **grid** is a pair $\Gamma = \langle G,\gamma \rangle$, where $G \subseteq \partial\partial\omega_1$ is a set of limit limit ordinals, and $\gamma : G \times \omega \to \partial\omega_1$ is a map that provides for each $j \in G$ a strictly increasing sequence of countable limit ordinals $\gamma(j,n) = j_n$ $(n < \omega)$ with $\bigcup\{j_n : n < \omega\} = j$.

It will also be useful to define a node a of limit height i in a spruce tree A to be **cofinal** if $\hat{a}$ is a cofinal branch of $A|i$ (i.e. there are branching nodes of unbounded height in $\hat{a}$).

Our main construction now follows. The statement of the theorem contains some terms that will be defined below.

### Theorem 2.1

Let $\Gamma = (G,\gamma)$ be a grid. Then there is a spruce tree $A = A(\Gamma)$ ('A' is for 'Aronszajn') with the properties:

($\alpha$) if $i<j<\omega_1$ and $\xi \in A(i)$ is a sequence node with $\sup(\xi) < q \in \mathbb{Q}$, then there is a sequence node $\eta \in A(j)$ with $\xi < \eta$ and $\sup(\eta) < q$.

($\kappa$) if $i < \omega_1$, $A(i)$ contains at most $\aleph_0$ sequence nodes.

($\nu$) for all $i \in \partial\omega_1$ the number of distinct cofinal green nodes $a \in A(i)$ is:

- $|\gamma^{-1}(i)|.\aleph_0$ if $i \in G$
- $|\gamma^{-1}(i)| + \aleph_0$ if $i \notin G$.

So for example if $i \in G \setminus \mathrm{im}(\gamma)$ then there are no cofinal green nodes in $A(i)$.

### Proof

Unlike in the classical Aronszajn construction the nodes of A will be of two kinds:

- **sequence nodes** These are certain elements of ${}^{<\omega_1}\mathbb{Q} = \{\eta : \exists i<\omega_1(\eta : i \to \mathbb{Q})\}$. So ${}^{<\omega_1}\mathbb{Q}$ is the set of countable sequences of rationals. If $\eta \in {}^{<\omega_1}\mathbb{Q}$ we write $\mathbf{len}(\eta)$ for $\mathrm{dom}(\eta)$ and $\mathbf{sup}(\eta)$ for $\sup\{\eta(i) : i<\mathrm{len}(\eta)\} \in \mathbb{R} \cup \{\infty,-\infty\}$. Each sequence node $\eta$ will be a *bounded increasing sequence:* i.e. $\sup(\eta) < \infty$ and $\eta(i) > \sup(\eta|i)$ for all $i < \mathrm{len}(\eta)$. The letters $\eta, \xi$ will denote sequence nodes.
- **blank nodes** These are 'filler' nodes. We can increase the height of a sequence node in the tree by inserting blank nodes beneath it.

Each node of A will be either sequence or blank - not both. The sequence nodes will be precisely the branching nodes. It will be clear that if $\Gamma = (\emptyset,\emptyset)$ then deleting the blank nodes from $A(\Gamma)$ gives a classical Aronszajn tree.

We will construct A by induction on levels. We must specify which elements of A are related in the tree ordering. As in the standard Aronszajn tree, if $\xi, \eta \in A$ are sequence nodes then $\xi<\eta$ in the tree iff $\xi$ is a proper initial segment of $\eta$. However, blank nodes are not sequences and we will specify explicitly how the tree ordering relates them. Since blank nodes may occur beneath sequence nodes we will have $\mathrm{ht}(\eta) \geqslant \mathrm{len}(\eta)$ for every sequence node $\eta \in A$, whereas in the classical case we have equality.

We now begin the construction of A. We define $A(0)$, the $0^{th}$ level of A, to be $\{\langle\rangle\}$, where $\langle\rangle$ is the empty sequence, a sequence node with supremum $-\infty$. If $A(i)$ has been defined, we construct $A(i+1)$ as follows. First, for every node $a \in A(i)$ we put a single blank node $a^+$ into $A(i+1)$ above a. This gives property (ii) of the definition of 'spruce'. Then for each sequence node $\eta \in A(i)$ and every $q \in \mathbb{Q}$ with $q > \sup(\eta)$, we put the sequence node $\eta$^$q$ (the sequence $\eta$ followed by q) into $A(i+1)$. This adds countably many sequence nodes above $\eta$. Clearly ($\alpha$) and ($\kappa$) are preserved.

Now assume that $j < \omega_1$ is a limit ordinal and we have built $A(i)$ for all $i<j$. There are two cases.

Case I: $j \notin G$

In this case we follow the classical construction. So for each sequence node $\eta \in A|j$ and each rational $q > \sup(\eta)$, we choose a rational $q'$ with $q > q' > \sup(\eta)$ and a strictly increasing sequence of ordinals $i_n$ $(n < \omega)$ with $i_0 = \mathrm{ht}(\eta)$ and $\bigcup\{i_n : n < \omega\} = j$. We then define sequence nodes $\eta_n \in A(i_n)$ $(n < \omega)$ by induction on n. We set $\eta_0 = \eta$. If $\eta_n$ has been defined, we use ($\alpha$) to find a sequence node $\eta_{n+1} \in A(i_{n+1})$ with $\eta_{n+1} > \eta_n$ and $\sup(\eta_{n+1}) < q'$. Then the union $\eta_\omega$ of the sequences $\eta_n$ is an increasing sequence of rationals with supremum $\leqslant q' < q$. We put $\eta_\omega$ into $A(j)$ above the branch of $A|j$ defined by the $\eta_n$.

**Remark**

In fact, ($\alpha$) clearly ensures that there is more than one choice for $\eta_{n+1}$ at each stage. Hence there are $2^{\aleph_0}$ possible choices of $\eta_\omega$.

We then add a single blank node above each remaining branch of $A|j$. This gives amongst other things property (i) of the definition of 'spruce'.

Clearly ($\alpha$) and ($\kappa$) are preserved by the construction.

Case II: $j \in G$

Our aim is to make level j of A 'special' by using the fact that $j \in G$, whilst all the time preserving ($\alpha$). Write $j_m$ for $\gamma(j,m)$ ($m < \omega$). Let $\eta \in A|j$ be a sequence node with $\sup(\eta) < q \in \mathbb{Q}$, and let $m < \omega$ be least such that $j_m > ht(\eta)$. Since $j_m$ is a limit ordinal, we can use the argument of Case I to choose an increasing series of sequence nodes $\eta_n$ ($n < \omega$) in $A|j_m$ with $\eta_0 = \eta$, $\bigcup\{ht(\eta_n) : n<\omega\} = j_m$ and $\sup(\eta_\omega) < q$, where $\eta_\omega$ is the union of the sequences $\eta_n$.

Now by the Remark above there are $2^{\aleph_0}$ possible choices of $\eta_\omega$, so by property ($\kappa$) we can choose one such that $\eta_\omega \notin A|j$. It follows that the branch of $A|j_m$ determined by the $\eta_n$ has only blank nodes above it in $A|j$, so it determines a branch $\beta$ of $A|j$. We then put the sequence node $\eta_\omega$ into A(j) above $\beta$. We do this for all $\eta \in A|j$. This preserves ($\alpha$) and ($\kappa$).

We complete the construction by adding a single blank node above each remaining branch of $A|j$, as in Case I. ($\alpha$), ($\kappa$) remain undisturbed.

Let A be the resulting tree of height $\omega_1$. We must check that it is spruce. All clauses of the definition except perhaps (iii) are obvious. Clause (iii) follows as in the classical Aronszajn construction, for a cofinal branch of A would give rise to an uncountable strictly increasing sequence of rationals, which is impossible as $\mathbb{Q}$ is countable.

We finally check that A satisfies ($\nu$). Let $i \in \partial\omega_1$. Green nodes of A(i) can only arise in two ways. Firstly, if $i \notin G$ then Case I of the construction puts $\aleph_0$ cofinal sequence nodes $\eta$ into A(i). If $i \in G$ then by Case II, A(i) contains no cofinal sequence nodes.

Secondly, if $i = \gamma(j,n)$ for some $(j,n) \in G\times\omega$ then Case II puts $\aleph_0$ sequence nodes $\eta$ into A(j). For each such $\eta$, if $a < \eta$ has height i then a is cofinal. All nodes $a' \geqslant a$ of height $<j$ are blank nodes. So $\eta$ gives rise to a single cofinal green node a in A(i). Hence the construction of A(j) for each $(j,n) \in \gamma^{-1}(i)$ effectively changes $\aleph_0$ cofinal non-green nodes of A(i) into green nodes.

Totting up, the number of cofinal green nodes $a \in A(i)$ is $|\gamma^{-1}(i)|.\aleph_0$, plus an extra $\aleph_0$ if $i \notin G$. This proves ($\nu$) and completes the proof of Theorem 2.1.

□

We will use ($\nu$) to show that if $\Gamma$ and $\Gamma'$ are sufficiently different grids then there is no conformal relation defined on any club between $A(\Gamma)$ and $A(\Gamma')$.

Suppose that C is a club and $\Phi : A(\Gamma)|C \to A(\Gamma')|C$ is a conformal relation. We would hope that if $i \in C$ then the $i^{th}$ levels of $A(\Gamma)$ and $A(\Gamma')$ are 'similar'. For comparison we want to use the cofinal green nodes, because we can control them using ($\nu$) of (2.1). Suppose that $b \in A(\Gamma')$ is a cofinal green node of height i. As $\Phi$ is surjective on green nodes, $A(\Gamma)$ will contain a green node a of height i with $a\Phi b$, but as $\Phi$ is a relation it does not follow that a is cofinal. However, we can show that if $i \in \partial C$ and $A(\Gamma')$ contains *uncountably many* cofinal green nodes b of height i, then $A(\Gamma)$ contains *at least one* cofinal green node a of height i with $a\Phi b$ for some such b. To do this we use our second device, the nest.

**Definition**

Let T be a tree. A **nest** in T is a set N of green nodes of T such that

- $N \subseteq T(i)$ for some $i < ht(T)$
- N is uncountable
- $\{t \in T : \exists n \in N\ (n > t)\}$ is countable.

The relationship of cofinal green nodes to nests is given by the following proposition.

**Proposition 2.2**

Let T be a spruce tree. Let $i < \omega_1$ and suppose that $N \subseteq T(i)$. Then N contains a nest in T iff

(a) i is a limit ordinal,

and (b) there are uncountably many cofinal green nodes in N.

**Proof**

To prove (a) and (b) we can assume that N is already a nest. Since T is spruce, every node of T has countably many immediate successors. It follows that (a) holds. Moreover, (b) holds; for if not, uncountably many nodes $n \in N$ would be such that $\zeta(n) < n$ (cf. (1.1)). Since by (1.1) all the $\zeta(n)$ are distinct, there are uncountably many nodes lying below nodes in N, contradicting the assumption that N is a nest.

Conversely, if (a) and (b) hold then take an uncountable set N' of cofinal green nodes in N. As $T{\restriction}i$ contains only countably many branching nodes, it is easily seen that N' is a nest.

□

We now relate this to our construction.

**Definition**

A grid $\Gamma = (G,\gamma)$ is said to be **fine** if

(i) $\gamma^{-1}(i)$ is uncountable for all $i \in im(\gamma)$

(ii) $\gamma : G\times\omega \to \partial\omega_1\backslash G$ is surjective.

It is easy to see that if $(G,\gamma)$ is fine then G must be uncountable, and for any uncountable $G \subseteq \partial\partial\omega_1$ we can find a $\gamma$ such that $(G,\gamma)$ is a fine grid.

We will usually work with fine grids from now on.

**Corollary 2.3**

Let $\Gamma = (G,\gamma)$ be a fine grid and write A for $A(\Gamma)$. Let $i < \omega_1$ be a limit ordinal. Then A(i) contains a nest iff $i \notin G$.

**Proof**

A is spruce and $\Gamma$ is fine, so by (2.2) and ($\nu$) of (2.1) a limit level A(i) of A contains a nest iff $i \in \mathrm{im}(\gamma)$ iff $i \notin G$.

□

Now we can prove the preservation lemma. Note that if T is a spruce tree, C is a club in $\omega_1$ and $i \in \partial C$, then a set $N \subseteq T(i)$ is a nest in T iff N is a nest in T|C.

**Lemma 2.4**

Let T, U be spruce trees, let C be a club in $\omega_1$ and suppose that $\Phi : T|C \to U|C$ is a conformal relation. Let $i \in \partial C$ be such that there is a nest $N \subseteq U(i)$. Then T(i) contains a cofinal green node m with $m\Phi n$ for some $n \in N$.

**Proof**

The argument is similar to that of (1.6). Let $N \subseteq U(i)$ be a nest. Since there are only countably many nodes in U lying below the elements of N, the set $N^* = \{n \in N : \forall n'<n(n \neq n'^{[\Phi,i]})\}$ is also a nest.

Take $n \in N^*$. By surjectivity for green nodes there is green $m \in T(i)$ with $m\Phi n$. Suppose for contradiction that m is not cofinal in T. Thus $\zeta(m) < m$. As $i \in \partial C$ we may choose $m' \in T|C$ such that $\zeta(m) < m' < m$. Then $\zeta(m') = \zeta(m) < m'$, and $m'^{[i]} = m$.

As $\Phi$ is homomorphic we have $m'\Phi n'$ for some $n' \in U(\mathrm{ht}_T(m'))$ with $n'<n$. But $\zeta(m') < m'$, so by continuity of $\Phi$ we must have $n = n'^{[\Phi,i]} \notin N^*$. This is a contradiction, proving the lemma.

□

**Corollary 2.5**

Let $A = A(\Gamma)$, $A' = A(\Gamma')$ be spruce trees, where $\Gamma = (G,\gamma)$, $\Gamma' = (G',\gamma')$ are fine grids. Suppose that C is a club in $\omega_1$ and $\Phi : A|C \to A'|C$ is a conformal relation. Then $G \cap \partial C \subseteq G' \cap \partial C$.

**Proof**

Pick $i \in G \cap \partial C$. As $\Gamma$ is fine, $i \notin \mathrm{im}(\gamma)$. So by ($\nu$) of (2.1) there are no cofinal green nodes in A(i). Hence by (2.4) there is no nest in A'(i). As $\Gamma'$ is also fine, by (2.3) we obtain $i \in G'$.

$\square$

Recall e.g. from [J1] that a **stationary** subset of $\omega_1$ is a set that has non-empty intersection with every club in $\omega_1$. We quote:

**Fact 2.6** [J1, Theorem 85]

There exist $\aleph_1$ pairwise disjoint stationary subsets of $\omega_1$.

This is usually attributed "essentially" to Ulam, since the easiest proof uses an Ulam matrix. The theorem was later strengthened by Solovay. Clearly the intersection of a club and a stationary set is stationary. Hence we can find pairwise disjoint stationary subsets $S_k$ $(k < \omega_1)$ of $\partial\partial\omega_1$.

Now it is easy to find subsets $X_i$ $(i < 2^{\omega_1})$ of $\omega_1$ such that if $i \neq j$ then $X_i \setminus X_j$ is non-empty. Define (for each $i < 2^{\omega_1}$) $G_i = \bigcup\{S_k : k \in X_i\}$. We see that each $G_i \subseteq \partial\partial\omega_1$, and if $i \neq j$ then $G_i \setminus G_j$ is stationary. For each $G_i$ choose $\gamma_i$ such that $\Gamma_i = (G_i, \gamma_i)$ is a fine grid, and set $A_i$ to be $A(\Gamma_i)$. Write $\mathbf{LA}_i$ for $L(A_i)$.

**Theorem 2.7**

Suppose that for each $i < 2^{\omega_1}$, $\langle LA_i, \rho_i, \nu_i \rangle$ is an SFP system with each $\rho_i(a)$ a countable ring, and with limit rings $R_i$. Suppose that $i \neq j$ $(i, j < \omega_1)$. Then there is no ring embedding $\theta : R_i \to R_j$. Hence the rings $R_i$ $(i < 2^{\omega_1})$ are pairwise non-embeddable.

**Proof**

By the results of §II.1 such a $\theta$ would give rise to a conformal relation $\Theta : A_j|C \to A_i|C$ for some club $C \subseteq \omega_1$. Since $G_j \setminus G_i$ is stationary we may find $k \in (G_j \setminus G_i) \cap \partial C$. By (2.3) there is a nest in $A_i(k)$. By (2.4) there must be a nest in $A_j(k)$, so by (2.3) again we must have $k \notin G_j$. This is a contradiction.

$\square$

We will now modify (2.1) to produce a spruce tree A such that if $C \subseteq \partial\omega_1$ is a club and $\Phi : A|C \to A|C$ is a conformal relation then $a\Phi a$ for all green $a \in A|C$. (That is, A is 'conformally rigid' - but note that there may also be $b \neq a$ with $a\Phi b$ or $b\Phi a$). This is enough to produce rigid rings - see §5.

First take pairwise disjoint stationary sets $S_i$ ($i<\omega_1$) with $S_i \subseteq \partial\partial\omega_1$ (all i), and set $G = \bigcup\{S_i : i<\omega_1\}$. A is built by induction on levels. As each sequence node $\xi$ is introduced a new set $S_i$ is assigned to $\xi$. This is possible as $A|i$ contains only countably many sequence nodes for any $i<\omega_1$. We can then write this $S_i$ as $S_\xi$. By deleting elements of $S_\xi$ we can assume that $j>ht_A(\xi)$ for all $j \in S_\xi$.

When $S_\xi$ has been defined we also choose two grids $V_\xi = (S_\xi, v_\xi)$ and $W_\xi = (S_\xi, w_\xi)$. We require:

- $im(v_\xi) \cap \partial\partial\omega_1 = S_\xi$
- $v_\xi^{-1}(i)$ is uncountable for all $i \in S_\xi$
- $im(w_\xi) \subseteq \partial\omega_1 \setminus \partial\partial\omega_1$.

These conditions are easy to arrange.

The construction of A at non-limit levels is as in (2.1). We build the limit level j of A as follows. If $j \notin G$ we apply 'Case I' of (2.1) - this is the classical Aronszajn case. Suppose then that $j \in G$. Then $j \in S_\xi$ for some sequence node $\xi \in A|j$. For each sequence node $\eta \in A|j$ and rational $q>\sup(\eta)$ we want to include a sequence node $\eta'$ in $A(j)$ with $\eta'>\eta$ and $\sup(\eta')<q$. We apply Case II of (2.1), but using the grid $V_\xi$ if $\eta \geq \xi$ and $W_\xi$ otherwise.

Let A be the result of the construction. We have:

**Lemma 2.8**

Let $i \in S_\xi$ for some sequence node $\xi \in A$. Then:

(i) there is a nest in $A(i)$ above $\xi$

(ii) if $a \in A(i)$ is a cofinal green node then $a > \xi$.

**Proof**

(i) As $i \in S_\xi$, $i \in im(v_\xi)$. Hence there are uncountably many $j \in S_\xi$ and $n_j < \omega$ with $v_\xi(j,n_j) = i$. Take such a j, and choose a sequence node $\eta \geq \xi$ in $A|i$ with $ht(\eta) \geq v_\xi(j,m)$ for all $m < n_j$. By construction there is a cofinal branch $\beta$ of $A|i$ with $\eta \in \beta$, such that the sequence node

$$\bigcup\beta =_{def.} \bigcup\{\eta' : \eta' \text{ a sequence node, } \eta' \in \beta\}$$

is in $A(j)$ above $\beta$. Thus if $a_j < \bigcup\beta$ has height i, $a_j$ is cofinal and green and $a_j > \xi$. Since moreover every node $a < \bigcup\beta$ of height $\geq i$ is a blank node, the $a_j$ ($j \in S_\xi$) are all distinct. Hence by the proof of (2.2), $\{a_j : j \in S_\xi\}$ is a nest above $\xi$ in $A(i)$.

(ii) Let $a \in A(i)$ be a cofinal green node. Since $i \in G$, Case II was used to construct $A(i)$. Hence no cofinal green nodes were introduced. So a must have been made green at some later stage. That is, there are a sequence node $\eta \in A$, $\gamma \in \{v_\eta, w_\eta\}$, $j \in S_\eta$ and $n < \omega$, such that $\gamma(j,n) = i$ and the sequence node $U\hat{a}$ was put into $A(j)$ above a.

Now $i \in S_\xi \subseteq \partial\partial\omega_1$. As $\mathrm{im}(w_\eta) \cap \partial\partial\omega_1 = \emptyset$ we have $\gamma = v_\eta$. Hence by construction, $a > \eta$. But now $i \in \partial\partial\omega_1 \cap \mathrm{im}(v_\eta) = S_\eta$. As the $S_\eta$, are pairwise disjoint, $\eta = \xi$. So $\xi < a$ as required.

□

**Corollary 2.9**

Let $\Phi : A|C \to A|C$ be a conformal relation, for some club $C \subseteq \partial\omega_1$. Let $t \in A|C$ be a green node. Then $t\Phi t$.

**Proof**

Choose a sequence node $\eta \geqslant t$ in A and $i \in S_\eta \cap \partial C$. By Lemma 2.8(i) there is a nest N above $\eta$ in $A(i)$. By (2.4) there is a cofinal green node $m \in A(i)$ with $m\Phi n$ for some $n \in N$. Hence $m > \eta \geqslant t$ by Lemma 2.8(ii). Because $\Phi$ is homomorphic we obtain $t\Phi t$ as required.

□

By taking $\bigcup\{S_\eta : \eta \in A\} = G_i$, where $G_i$ is as defined after (2.6), we can combine (2.9) with (2.7) to produce $2^{\aleph_1}$ 'conformally different' rigid trees. The method is standard and we will not describe it further.

## 3 Countable generation of full ideals

In the last three sections we study in more detail the limit rings of SFP systems built on the SFP domains LA, for A as in §1. Already by (I.3.4), if each map $\nu$ of the system is an isomorphism then each of the locally generated ideals of the limit is densely decomposable. In §4 below we will show that they can all be made $L_{\infty\omega}$-equivalent, and in §5 we build in some rigidity (the rings will have few endomorphisms). For the present we show that every full ideal can be made countably generated.

Let A be a spruce tree as built in (2.1), and let $\langle LA,\rho,\nu \rangle$ be an SFP system such that $\rho(a)$ is countable for all $a \in LA$ (or equivalently for all finite elements $a \in LA$). Let the SFP system have limit ring R, and let $R_i$ be the limit of the system restricted to $LA_i$ (all $i < \omega_1$). By (I.1.5) R is uncountable, of cardinality $\aleph_1$. Nonetheless we will now use Corollary I.2.9(ii) to show that every full ideal (either left or two-sided) of R has a countable set of generators. By (I.2.4) and (I.2.5) the full ideals include the maximal, prime and irreducible left ideals and also the maximal two-sided ideals of R.

First we need a technical lemma.

**Lemma 3.1**

Let T be a spruce tree. Then every node of L(T) is either terminal in L(T) or branching with $\aleph_0$ immediate successors in L(T).

**Proof**

Assume that $t \in L(T)$ is not terminal in L(T). If $t \in B(T)$ then clearly t has $\geq\aleph_0$ immediate successors in L(T). Assume that $t \in L(T)\setminus B(T)$. There is $b \in L(T)$ with $b>t$. We can assume that $i = ht_T(b)$ is least possible, so $b \in B(T)$. Since i is a successor ordinal, b has an immediate predecessor b′ in T.

By choice of b, if x is an immediate successor of b′ in T and $x \in L(T)$ then x is an immediate successor of t in L(T). By (ii) and (iv) of 'spruce' the immediate predecessor of a branching node in T is also branching and has $\aleph_0$ immediate successors in T. Hence t has $\geq\aleph_0$ immediate successors in L(T).

It remains to prove that no $t \in L(T)$ has $>\aleph_0$ immediate successors in L(T). Assume for contradiction that $t \in L(T)$ is a counterexample. Let $ht_T(t) = i$. As T is spruce, there are arbitrarily large $j < \omega_1$ with $j>i$ such that there is an immediate successor b of t of height j in T. Clearly $j = j'+1$ for some $j'$. Let $b'<b$ have height $j'$ in T. There is no $x \in L(T)$ with $t<x\leq b'$. Hence $b'/L(T) = t$. It follows that $b' = t^{[j']}$. As above, b′ is a branching node of T. As this holds for arbitrarily large $j'$, it follows that the branch of T determined by $\{t^{[j]} : i<j<\omega_1\}$ is cofinal in T. This contradicts the spruceness of T.

□

We now get:

**Theorem 3.2**

Let J be a full left ideal of R. Then J is countably generated.

**Proof**

Suppose that $J = I@a$ in R, for some $a \in LA$ and some ideal $I \subseteq R_a$. Recall from §II.1 that $LA_i =_{def.} LA \cap A|i+1$ (all $i < \omega_1$), and that $LA = \bigcup_{i<\omega_1} LA_i$. So there is $i < \omega_1$ such that $a \in LA_i$.

By Lemma 3.1 every node of LA is either terminal in LA or is branching with $\aleph_0$ immediate successors in LA. So we can choose i so that all of the immediate successors of a in

LA (if any) are already in $LA_i$. It follows that:

- $LA_i \preccurlyeq LA$
- $\{a' \in LA : a'/LA_i = a\} = \{a\}$.

By (I.2.2) $J_{LA_i} = I@a$ in $R_i$. By (I.2.9) $J_{LA_i}$ generates J in R. The result follows by (I.1.5), as $R_i$ is countable.

□

## 4 $L_{\infty\omega}$-equivalence

Here we prove that if two grids $\Gamma^1, \Gamma^2$ are 'sparse' enough then $BA(\Gamma^1)$ and $BA(\Gamma^2)$ are $L_{\infty\omega}$-equivalent trees (recall from §1 above that $BA = (LA)^0$). Since the rings of (2.7) are the limits of SFP systems of the form $\langle LA(\Gamma),\rho,\nu\rangle$, this will allow us to strengthen (2.7) so that under the conditions of (I.4.3) say, the rings $R_i$ ($i < 2^{\omega_1}$) of the conclusion are all $L_{\infty\omega}$-equivalent.

We mentioned $L_{\infty\omega}$-equivalence in §I.4. There is another characterisation of $L_{\infty\omega}$-equivalence in terms of **back-and-forth systems.** A **back-and-forth system** between M and N is a set $\Theta$ of partial isomorphisms from M to N such that:

- $\emptyset \in \Theta$
- if $\theta \in \Theta$ and $a \in M$ then there is $b \in N$ such that $\theta \cup \{(a,b)\} \in \Theta$
- if $\theta \in \Theta$ and $b \in N$ then there is $a \in M$ such that $\theta \cup \{(a,b)\} \in \Theta$.

**Fact 4.1** (Karp's theorem, [K])
M and N are $L_{\infty\omega}$-equivalent iff there is a back-and-forth system $\Theta$ between M and N.

We will show that $LA(\Gamma^1)$ and $LA(\Gamma^2)$ are $L_{\infty\omega}$-equivalent for sparse $\Gamma^t$, by finding a back-and-forth system between them. It will then follow that $BA(\Gamma^1) \equiv_{\infty\omega} BA(\Gamma^2)$. Though LA is definable in A by a first order formula, $A(\Gamma^1)$ and $A(\Gamma^2)$ will not in general be $L_{\infty\omega}$-equivalent. (If they were, then for all $i < \omega_1$, if $A(\Gamma^1)(i)$ contains a green node a with $\hat{a}$ a cofinal branch of $A(\Gamma^1)|i$ then so does $A(\Gamma^2)(i)$.) So we must work directly with the LA, remembering that if $t \in LA$ then maybe $ht_{LA}(t) < ht_A(t)$, and t may be a branching node of LA without being branching in A (though it will be green in A).

**Definition**

An uncountable set $C \subseteq \omega_1$ is said to be **sparse** if for each $i \in C$, $\min\{j \in C : j>i\} > i+i$ $(= i.2)$.

Clearly an uncountable subset of a sparse set is also sparse. If we define an $\omega_1$-sequence $z_j$ $(j<\omega_1)$ inductively by: $z_0 = \omega$, $z_{j+1} = z_j.2 + 1$, $z_\delta = \bigcup\{z_j : j<\delta\}$ for limit $\delta < \omega_1$, then $Z = \{z_j : j<\omega_1\}$ is a sparse club. Hence if $S \subseteq \omega_1$ is stationary then $S \cap Z$ is sparse and stationary. It follows that in (2.6) we can assume that the $S_i$ are subsets of Z, so that the $G_i$ defined prior to (2.7) are sparse.

Sparseness is used in the following lemma.

**Lemma 4.2**

Let G be a sparse subset of $\partial\partial\omega_1$, and let $\Gamma = (G,\gamma)$ be a fine grid. Write A for $A(\Gamma)$. Let b be a branching node of LA with $ht_{LA}(b) = i$. Then for all ordinals j with $i<j<\omega_1$ and all $q \in \mathbb{Q}$, there is a branching node $c > b$ of LA with $ht_{LA}(c) = j$ and $\sup\{\sup(\eta) : \eta$ a sequence node, $\eta \leqslant c\} > q$.

**Proof**

Since b is a branching node of LA, b is green in A. Let $\eta$ be the lowest sequence node in A with $\eta \geqslant b$. By construction we may take a sequence node $\eta' \in A$ such that $\eta'$ is an immediate successor of $\eta$ in A and $\sup(\eta') > q$. Then $\eta' \in LA$ and $ht_{LA}(\eta') = i+1$. This proves the lemma in case $j = i+1$.

Assume that $j>i+1$. With $\eta'$ as above, any sequence node $>\eta'$ already has supremum $>q$. So replacing b by $\eta'$, it is enough to find a sequence node above b and of height $\geqslant j$ in LA.

Let g be the least element of G such that $g > j$, $g > ht_A(b)$. By ($\alpha$) of (2.1) we can find a sequence node $\xi \in A(g)$ with $\xi>b$. We will not have $\xi \in LA$. If $ht_{LA}(\hat{\xi} \cap LA) \geqslant j$ then choose $\xi' \in LA$ so that $\xi' \leqslant \xi$, $ht_{LA}(\xi') = j$. Clearly $\xi'$ is not a terminal node of LA, so by (3.1) it is a branching node, and we are done.

Suppose on the other hand that $ht_{LA}(\hat{\xi} \cap LA) = k < j$. Let $k' < \omega_1$ be such that $k+k' = j$, and set $j' = g+k'$. Using ($\alpha$) choose a sequence node $\xi' \in A(j')$ with $\xi' > \xi$. Now $k' \leqslant j < g$. As G is sparse, there is no $g' \in G$ with $g < g' \leqslant j'$. By construction it follows that every node t of A with $\xi \leqslant t \leqslant \xi'$ is a sequence node, so $t \in LA$ for all such t. Hence $ht_{LA}(\xi') = ht_{LA}(\xi) + k' = j$, and we can take c to be $\xi'$.

□

**Corollary 4.3**

Under the assumptions of (4.2), for all limit ordinals j with $i<j<\omega_1$ there is a terminal node t of LA with $t>b$, $ht_{LA}(t) = j$.

**Proof**

Take a strictly increasing sequence of successor ordinals $j_n$ ($n<\omega$) with $j_0 > i$ and $\bigcup_{n<\omega} j_n = j$. As each $j_n$ is a successor ordinal we may use (4.2) to define sequence nodes $\xi_n \in LA$ by induction, with $ht_{LA}(\xi_n) = j_n$, $\xi_0 > b$, $\xi_{n+1} > \xi_n$ and $\sup(\xi_n) > n$ (all $n<\omega$).

Let $t = lub_{LA}\{\xi_n : n<\omega\}$. Then $ht_{LA}(t) = j$. Further, $\sup\{\sup(\eta) : \eta$ a sequence node, $\eta < t\} = \infty$. Hence there can be no sequence node $\eta \geqslant t$ in A, so t must be a terminal node of LA.

□

For the rest of this section let $G^1$, $G^2$ be sparse stationary subsets of $\omega_1$, and let $\Gamma^\iota = (G^\iota, \gamma^\iota)$ be fine grids ($\iota = 1,2$). Consider the trees $A^1 = A(\Gamma^1)$, $A^2 = A(\Gamma^2)$ constructed in Theorem 2.1. By (3.1) every node of $LA^\iota$ ($\iota$=1,2) is either a branching node (with infinitely many immediate successors) or is terminal.

We will prove:

**Theorem 4.4**

(i) $LA^1$ and $LA^2$ are $L_{\infty\omega}$-equivalent in the signature $L = \{=,<\}$.

(ii) $BA^1$ and $BA^2$ are also $L_{\infty\omega}$-equivalent.

Part (i) of the theorem will follow immediately from (4.5) below. Part (ii) follows from part (i) here, since there is a first order L-formula $\varphi(x)$ such that for any A as in (2.1), $\{a \in LA : LA \models \varphi(a)\} = BA$. $\varphi$ simply says that x does not have limit height in LA. Part (ii) is what is required for $L_{\infty\omega}$-equivalence of the limit rings.

We begin the proof of the theorem with a definition.

**Definition**

Let T be any tree. If $U \subseteq T$, U is said to be a **full subtree** of T if U is non-empty and closed downwards in T. If $S \subseteq T$, we write $\langle S\rangle$ for the full subtree $\{t \in T : t \leqslant s$ for some $s \in S\}$ of T generated by S. T is said to be **finitely generated** if $T = \langle S\rangle$ for some finite $S \subseteq T$. Note that no branch of a finitely generated tree can have limit height.

If $U^\iota$ is a full subtree of $T^\iota$ ($\iota$=1,2), a map $\theta : U^1 \to U^2$ is said to be a **strong isomorphism** if $\theta$ is bijective and preserves <, and each $u \in U^1$ is a branching node of $T^1$ iff $\theta(u)$ is a branching node of $T^2$.

For example, writing $\perp$ for the unique least element of $T^1$ and $T^2$, $\{\perp\}$ is a finitely generated full subtree of each $T^\iota$, and $\{(\perp,\perp)\}$ is a strong isomorphism. Hence the set $\Theta$ of subsets of strong isomorphisms between finitely generated full subtrees of $LA^1$, $LA^2$ is non-empty. The next lemma shows that $\Theta$ is a back-and-forth system between $LA^1$ and $LA^2$, and so proves Theorem 4.4.

**Lemma 4.5**

Let $T^\iota$ be finitely generated full subtrees of $LA^\iota$ ($\iota$=1,2), and suppose that $\theta : T^1 \to T^2$ is a strong isomorphism from $T^1$ to $T^2$.

(i) Let $t^1 \in LA^1$. Then there is $t^2 \in LA^2$ such that $\theta \cup \{(t^1,t^2)\}$ extends to a strong isomorphism from $\langle T^1 \cup \{t^1\}\rangle$ to $\langle T^2 \cup \{t^2\}\rangle$.

(ii) Similarly, exchanging the indices '1' and '2' in (i).

**Proof**

We will only prove (i); (ii) is similar. So let $T^1$, $T^2$, $\theta$ be given, and let $t^1 \in LA^1$. We can assume that $t^1 \notin T^1$ - the result is trivial otherwise. Now $T^1$ has no branches of limit height. So if $T^1 \neq \emptyset$ there is a unique largest node $v \in T^1$ with $v < t^1$; in fact we have $T^1 \trianglelefteq LA^1$ and $v = t^1/T^1$ in the notation of §I.1. Let $\theta(v) = w \in T^2$ and let $ht_{LA^1}(t^1) = h < \omega_1$. It suffices to prove the following:

Claim There is $t^2 \in LA^2$ with $t^2/T^2 = w$, $ht_{LA^2}(t^2) = h$, and such that $t^1$ is a terminal node of $LA^1$ iff $t^2$ is a terminal node of $LA^2$.

Given the claim, we can finish as follows. Let $T^{\iota\prime}$ be the full subtree of $LA^\iota$ generated by $T^\iota \cup \{t^\iota\}$ ($\iota$=1,2). Since $ht(t^2) = h$ we can extend $\theta$ canonically to an order-preserving bijection $\theta' : T^{1\prime} \to T^{2\prime}$. Since by (3.1) every node of each $LA^\iota$ is either branching or terminal, $\theta'$ will be a strong isomorphism.

Proof of Claim Since $t^1 \notin T^1$, v is not terminal, so v is a branching node of $LA^1$. As $\theta$ is strong, w is also branching in $LA^2$, with infinitely many immediate successors. As $T^2$ is finitely generated we can take an immediate successor $w'$ of w in $LA^2$ with $w' \notin T^2$. It suffices to find $t^2 \geqslant w'$ in $LA^2$ with the required properties.

If $t^1$ is a branching node of $LA^1$ then by (4.2) there is a branching node $t^2$ of $LA^2$ above $w'$ and of height h in $LA^2$. If $t^1$ is terminal in $LA^1$ then h must be a limit ordinal, so we can use (4.3) to choose a terminal node $t^2 \in LA^2$ above $w'$ of height h. This proves the claim and with it the lemma.

□

Theorem 4.4 is proved.

## 5 Rigidity

By imposing restrictions on the homomorphisms $\nu_{pq}$ in the SFP systems and using (2.9), the $2^{\aleph_1}$ limit rings of (2.7) can be made somewhat rigid. To conclude our survey we will prove a sample result for Boolean rings. We will define an SFP system with Boolean limit ring R having no non-trivial injective endomorphisms. The example will also illustrate the use of SFP systems in which not all $\nu_{pq}$ are isomorphisms.

There are further cases in [Hk]. For example, we may set up an SFP system with limit ring R so that any injective endomorphism $\theta : R \to R$ satisfies $\theta^{-1}(I) = I$ for every maximal two-sided ideal I of R. If R is Boolean this implies that $\theta = id_R$.

Take any countable Boolean ring S, and fix a countable set $\Xi$ of maximal ideals of S such that any proper finitely generated ideal I of S is contained in some $K \in \Xi$.

Suppose that A is a 'conformally rigid' tree considered in (2.8). We build an SFP system $\sigma$ on LA as follows. First we partition $\mathbb{Q}$ into sets $\mathbb{Q}_K$ ($K \in \Xi$) such that each $\mathbb{Q}_K$ is dense in $\mathbb{Q}$. For each sequence node $\eta \in BA$ we define $R_\eta$ to be S. For each $q \in \mathbb{Q}$ with $q > \sup(\eta)$ we define $\ker(\nu_{\eta,\eta\wedge q}) = K$ where $q \in \mathbb{Q}_K$. Since $S/K \cong \{0,1\}$ this defines $\nu_{\eta,\eta\wedge q}$ completely. Hence if $a \in LA\backslash BA$ we will have $R_a = \lim_{\to}(R_\eta, \nu_{\eta,\eta'} : \eta < \eta' \text{ in } BA\cap\hat{a}) = \{0,1\}$, and if $b > a$ then $\nu_{ab}$ must be the unique embedding of $\{0,1\}$ into $R_b$. We have now defined $\sigma$ completely.

Let $R = \lim(\sigma)$. Then R is an uncountable Boolean ring. If $i < \omega_1$ write $R_i$ for $\lim\langle LA_i,\rho',\nu'\rangle$ as usual, where $\rho'$ and $\nu'$ are the appropriate restrictions. Let $\theta : R \to R$ be a ring embedding. As in (1.3) we can find a club $C \subseteq \partial\omega_1$ so that $\theta$ induces a conformal relation $\Theta : A|C \to A|C$.

We claim that $\theta = id_R$. Suppose not. There is $r \in R$ such that $\theta(r) = r' \neq r$. There is $i \in C$ such that $r, r' \in R_i$. As $R_i$ is Boolean it is easily seen that at least one of the sets $\{r,1-r'\}$, $\{1-r,r'\}$ generates a proper ideal of $R_i$ containing just one of r, r'. Assume without loss that $\{r,1-r'\}$ has this property. Take a finite support $N \leqslant BA_i$ for $\{r,1-r'\}$. The ideal of $R_N$ generated by $\{r,1-r'\}$ is proper, so there is $\eta \in N$ and a proper finitely generated ideal H of $R_\eta = S$ with $r(\eta), 1-r'(\eta) \in H$. (Note that $\eta$ is a finite element of LA and hence a sequence node in A.) There is $K \in \Xi$ containing H. Choose $q \in \mathbb{Q}_K$ with $q > \sup(\eta)$ and $\eta\wedge q \notin N$, and then choose a green node (say a sequence node) $a \in A(i)$ such that $a > \eta\wedge q$, $a/N = \eta$. Write z for $\zeta a \in LA$. Then $r(z) = \nu_{\eta z}(r(\eta)) = \nu_{\eta\wedge q,z}.\nu_{\eta,\eta\wedge q}(r(\eta)) = 0$. Similarly, $1-r'(z) = 0$.

Now by (2.9) we have $a\Theta a$. Hence by definition of $\Theta$ there are proper ideals I, J of $R_z$ such that in $R_i$, $\theta^{-1}(I@z) = J@z$. Hence $\theta(J@z) \subseteq I@z$.

Since $r(z) = 0$, $r \in J@z$ in $R_i$. Hence $r' \in I@z$. But $r'(z) = 1$, contradicting the assumption that I is proper. Hence $\theta = \mathrm{id}_R$ as claimed.

□

So any injective endomorphism of R is the identity map. We also have all the standard properties: every maximal ideal of R is countably generated (3.2), and by the argument of (I.3.3) R is easily seen to be atomless and hence existentially closed. The atomless property can equally be obtained by taking S to be atomless, or we can include the zero ideal 0 in $\Xi$, require that $\nu_{\eta,\eta\wedge q} = \mathrm{id}_S$ whenever $q \in \mathbb{Q}_0$, and use the Remark following (I.3.3). We can combine the construction of A with the techniques of (2.1) and §4 to produce $2^{\aleph_1}$ pairwise non-embeddable $L_{\infty\omega}$-equivalent such R.

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