A construction of many uncountable rings using SFP domains

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Abstract

The paper in in two parts. In Part I we describe a construction of a certain kind of subdirect product of a family of rings. We endow the index set of the family with the partial order structure of an SFP domain, as introduced by Plotkin, and provide a commuting system of homomorphisms between those rings whose indices are related in the ordering. We then take the subdirect product consisting of those elements of the direct product having finite support in the sense of this domain structure. We examine the properties of rings obtainable in this way.

In Part II we prove an 'anti-structure theorem' by exhibiting $2^{\aleph_1}$ pairwise non-embeddable rings of cardinality $\aleph_1$ with various higher-order properties. The construction uses Aronszajn trees.

Introduction

This paper presents a blend of ideas from ring theory, set-theoretic combinatorics and computer science. It is divided into two parts: part I will perhaps be of more interest to algebraists, and part II to logicians.

In part I we develop a method of constructing a subdirect product of certain families of rings. To do this we impose a partial order structure on the index set of the family. We will take this poset structure to be that of an SFP domain, a notion introduced in [P] and well known to domain theorists in computer science. We will analyse the behaviour of the ideals of the resulting subdirect product and show that *inter alia* they carry information about the underlying poset structure on the index set. We can then exert control over the subdirect product by purely partial order-theoretic means.

We exploit this in part II. Using a variant of the construction of Aronszajn trees in set theory we will construct, using ZFC only, $2^{\aleph_1}$ SFP domains such that, assuming that all component rings are countable, any subdirect products obtained with them will be pairwise non-embeddable rings. We can impose conditions on the component rings themselves to obtain

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stronger results.

A typical product is:

THEOREM: *Let S be a countable Boolean ring. There are $2^{\aleph_1}$ $L_{\infty\omega}$-equivalent pairwise non-embeddable Boolean rings R_i ($i<2^{\aleph_1}$) of cardinality $\aleph_1$ extending S. Each R_i is existentially closed and rigid, and each of its maximal ideals has a countable set of generators.*

This suggests that there are too many such rings to classify fully. It is thus an *anti-structure theorem* in the spirit of, for example, the result of [Sh] that if T is a first order non-superstable complete first order theory of cardinality κ then there are 2^{λ} pairwise non-elementarily embeddable models of T of cardinality λ for all regular $\lambda>\kappa$.

The technique tends to produce rings with many orthogonal central idempotents, so is most at home when constructing Boolean or von Neumann regular rings.

The work in this paper simplifies the construction of the doctoral thesis [Hk] of the first author, which also uses the continuum hypothesis. The argument there is more complicated and less general because SFP domains are not used. The motivation for [Hk] came from the paper of Ziegler [Z]. If I is a left ideal of a ring R, we say that I is *densely decomposable* if whenever A is a left ideal properly extending I then there are left ideals X, Y $\subseteq$ A properly extending I but with X$\cap$Y = I (see Section 3 of Part I). If R is countable, commutative and von Neumann regular then a proper ideal I of R is densely decomposable iff the ring R/I is atomless (has no principal maximal ideals), iff the injective hull of the left R-module R/I has no indecomposable direct summand. If R is additionally assumed to be countable and atomless then R has $2^{\aleph_0}$ (i.e. $2^{|R|}$) maximal ideals; this was generalised to arbitrary countable rings in [Z] (7.1(1), 7.2, 8.3). Our initial objective was to show that the result fails for $|R| = \aleph_1$. This is established by the theorem quoted above. Every maximal ideal of each R_i of the theorem is countably generated, so they are at most $2^{\aleph_0}$ in number - this can be less than $2^{\aleph_1} = 2^{|R_i|}$. The construction in [BK] gives an atomless Boolean ring of cardinality $\aleph_1$ also illustrating this, but Jensen's $\diamond$ (diamond) is used. On the other hand, unlike the construction in [BK], an atomless Boolean algebra constructed by the methods we give will generally have an uncountable set of pairwise incomparable elements.

It would be interesting to prove an intrinsic characterisation theorem for rings arising by our construction, analogous to that for varieties and reduced products. Possibly the work of Smyth [Sm] would be relevant.

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Part I: SFP systems

This part of the paper contains the results of a more algebraic nature. We will define the notion of an SFP system of rings, and study some of the properties of its limit.

Let us describe the approach in rather more detail than above. Let $(P,\leqslant)$ be a poset such that for every $p \in P$ we have a ring R_p. Suppose further that for every $p \leqslant q$ in P we have a ring homomorphism $\nu_{pq} : R_p \to R_q$. We require that the ν_{pq} ($p \leqslant q$ in P) form a commuting system in the usual sense.

Assume that P has a least element $\perp$. Then the presence of the maps ν allow us to embed the ring $R_\perp$ diagonally into the direct product $\Pi(R_p : p \in P)$, via $r \mapsto (\nu_{\perp p}(r) : p \in P)$ for $r \in R_\perp$. We would like to generalise this as follows. Let $N \subseteq P$ be finite. Can we embed the finite direct product $\Pi(R_n : n \in N)$ diagonally into the full direct product?

So let $r \in \Pi(R_n : n \in N)$. We need to define its image r' in $\Pi(R_p : p \in P)$. By analogy with the case $N = \{\perp\}$, for each $p \in P$ we would like to define r'(p) to be $\nu_{np}(r(n))$, where n is an appropriate element of N, depending on p. To force a unique choice of n we will assume that N satisfies: for all $p \in P$ there is a unique maximal element of $\{n \in N : n \leqslant p\}$. This would hold if for example N is linearly ordered. We write this maximal element as $\mathbf{p/N}$. We can now define r' to be $(\nu_{p/N,p}(r(p/N)) : p \in P)$. Then N is in effect a *finite support* of r' in $\Pi(R_p : p \in P)$.

So we consider the set R^* of all elements of $\Pi(R_p : p \in P)$ having a finite support in this sense. We require that R^* be a subring of $\Pi(R_p : p \in P)$. To obtain closure under + and - we will need any two finite supports to be contained in a third, and to avoid redundancy of any R_p we will formally require that (*) *any finite subset of P extends to a finite support $N \subseteq P$.* For example, if P is linearly ordered with a least element this is trivially true. So we could take P to be $(\mathbb{Q}\cup\{-\infty\},<)$, each R_p to be $\{0,1\}$ and all ν_{pq} to be the identity map. In this case R^* turns out to be the countable atomless Boolean ring (see Example 3.4). However, the condition (*) holds in much more general cases and is closely related to the SFP domains of Plotkin [P]. Any such P extends canonically to an SFP domain by adding where necessary a least upper bound h for each directed subset D of P. These extra points h turn out to be very useful: $\langle R_d, \nu_{dd'} : d \leqslant d' \text{ in } D\rangle$ forms a direct system and it is technically convenient to define R_h to be its direct limit, and extend ν accordingly. Hence we will work with SFP domains throughout.

It is easy to show that if the 'component rings' R_p ($p \in P$) have various properties then so does R^*. Examples of properties preserved in this way are: commutative; Boolean; von Neumann regular; existentially closed commutative. The cardinality of R^* is also related to $|P|$ and the $|R_p|$. We also show that the $L_{\infty\omega}$-theory of R^* is determined by the $L_{\infty\omega}$-theory of P together with the R_p and the maps ν_{pq}.

So far the construction could be undertaken for any model-theoretic structure. We consider rings because we can fruitfully study their ideals. (Generalisations to structures such as lattices are probably possible here.) An important class of ideals arises as follows. If I is a (left) ideal of R_S for some $s \in P$ then $I^* = \{r \in R^* : r(s) \in I\}$ is a left ideal of R^*. Ideals of this form are called *full ideals:* they are in a sense 'locally determined'. We can recover I and s from I^*, so the full ideals are closely related to the poset structure of P. They are a kind of basis for the set of all ideals of R^*. Using the extra elements h of P we can show that any maximal, prime or irreducible ideal of R^* must be full, and every ideal of R^* is the intersection of the full ideals that contain it.

The layout of this part of paper is as follows. In Section 1 we discuss SFP domains and formally lay out the subdirect product construction. In Section 2 we discuss ideals of R^* and use the results in the next section to enforce that R^* has a property generalising 'atomlessness' in Boolean algebras. Finally, in Section 4 we discuss $L_{\infty\omega}$-equivalence.

1 SFP systems

In this section we give most of the definitions that we will need, plus some examples and useful lemmas for illustration.

Algebraic dcpos

Recall that a partially ordered set, or **poset**, is a (usually non-empty) set equipped with a reflexive transitive binary relation, written here as '$\leqslant$'. A poset $(D,\leqslant)$ is **directed** if D is non-empty and whenever $d_1, d_2 \in D$, then there is $d_3 \in D$ with $d_3 \geqslant d_1, d_2$.

A non-empty poset P is said to be **directed complete** (a 'dcpo') if whenever $D \subseteq P$ is directed then D has a least upper bound in P. We write this bound as **lub(D)**, or more explicitly $\mathbf{lub_P(D)}$; it is necessarily unique.

An element p of a dcpo P is said to be **finite** if whenever $D \subseteq P$ is directed and $p \leqslant lub(D)$ then $p \leqslant d$ for some $d \in D$. We write $\mathbf{P^0}$ for the set of finite elements of P; P^0 is called the **base** of P. P is said to be **algebraic** if for all $p \in P$, the set $\mathbf{p\!\downarrow} = \{q \in P : q \leqslant p\}$ is such that $p\!\downarrow \cap P^0$ is directed and $lub(p\!\downarrow \cap P^0) = p$. That is, p is the lub of the set of finite elements beneath it, and we can usually replace p by this set. It follows that in this case P is determined by its base (see below). Algebraic dcpos P with countable base are usually called **domains** in the computer science literature.

Examples of algebraic dcpos are all finite (non-empty) posets and all successor ordinals. If X is a non-empty set then $\wp(X)$, ordered by inclusion, is an algebraic dcpo, and the finite elements are just the finite subsets of X - hence the name. The half-open real interval (0,1] has no finite elements and shows that a dcpo need not be algebraic, as does the following dcpo:

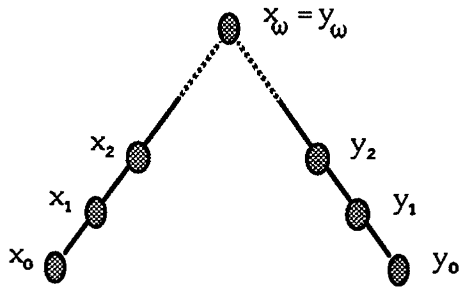

Figure 1.1

Ideals

Let P be any poset. An **ideal** of P is a subset I of P that is closed downwards (i.e. if $x \leqslant y \in I$ then $x \in I$) and directed. Clearly if $p \in P$ then $p{\downarrow}$ is an ideal; ideals of this form are said to be **principal**. It is well known that if P is an arbitrary non-empty poset, the set of ideals of P, ordered by inclusion, forms an algebraic dcpo whose finite elements are just the principal ideals; these are in order-isomorphism with P. Hence any P can be 'completed' to an algebraic dcpo by taking this **'ideal completion'**. Moreover, any algebraic dcpo P is isomorphic to the ideal completion of its base P^0. We will often identify a poset P with the set of finite elements of its ideal completion. A similar ideal completion can be undertaken for preorders also.

Locally directed sets

Now let P be a poset. A subset N of P is said to be **locally directed in P** (written $N \trianglelefteq P$) if for all $p \in P$, $p{\downarrow} \cap N$ is directed. Equivalently, $N \trianglelefteq P$ iff $N \cap I$ is directed for all ideals I of P. For example, if P is an algebraic dcpo then $P^0 \trianglelefteq P$. If P contains a least element $\perp$, then any linearly ordered subset N of P with $\perp \in N$ is locally directed in P. $N \subseteq \wp X$ is locally directed in $(\wp X, \subseteq)$ iff N is closed under finite (including empty) unions. Since $P \trianglelefteq P$ for any P, locally directed does not imply directed. The converse also fails, as if $\perp$ is the least element of P then $N \trianglelefteq P \Rightarrow \perp \in N$.

It is easily seen that $\trianglelefteq$ is a reflexive and transitive relation on posets, and that if $N \trianglelefteq P$ and $N \subseteq Q \subseteq P$ then $N \trianglelefteq Q$.

Now assume that P is a dcpo. If $N \trianglelefteq P$ and $p \in P$, we write **p/N** for $\mathrm{lub}_P(p{\downarrow} \cap N)$; this exists since P is a dcpo, and indeed if N is finite, or more generally a dcpo such that $\mathrm{lub}_P(D) = \mathrm{lub}_N(D)$ for all directed $D \subseteq N$, then $p/N \in N$. We can view p/N as N's best approximation to p. We have $p/N \leqslant p$ for all p; further, P is algebraic iff $P^0 \trianglelefteq P$ and $p/P^0 = p$ for all $p \in P$. If $N \trianglelefteq P$ we can define an equivalence relation $\sim_N$ on P by $x \sim_N y$ iff $x/N = y/N$. We will see in Section 2 that the equivalence classes are related to the well known 'patch' topology on P.

SFP domains

We can now define the strain of poset of interest to us here. A poset P is said to be **nice** if any finite subset $X \subseteq P$ can be extended to a finite locally directed subset of P. An **SFP domain** is an algebraic dcpo P such that P^o is nice. So the ideal completion of a nice poset is an SFP domain, and all SFP domains arise in this way.

An equivalent definition uses the notion of MUB-closure (see Plotkin, [P]). If $X \in P$ define $MUB(X) = \{p \in P : p \text{ is a minimal upper bound of } X\}$. Also define an increasing chain $U^n(X)$ $(n \leqslant \omega)$ by: $U^o(X) = X$, $U^{n+1}(X) = \bigcup\{MUB(Y) : Y \subseteq U^n(X)\}$, $U^\omega(X) = \bigcup_{n<\omega} U^n(X)$. $U^\omega(X)$ is called the MUB-closure of X. Then it is easily seen that P is SFP iff for all finite $X \subseteq P^o$,

(i) for all $p \in P$ with $X \subseteq p{\downarrow}$ there is $y \in MUB(X)$ with $y \leqslant p$

(ii) MUB(X) is finite

(iii) $U^\omega(X)$ is finite.

In fact, in this case $U^\omega(X) \trianglelefteq P^o$. Domains satisfying (i) and (ii) are sometimes called 2/3-SFP. Of course, (iii) implies (ii).

Examples of nice posets are any finite poset, any linear order with a least element, any Boolean algebra, and any tree with finitely many minimal elements. The restriction to finitely many minimal elements is necessary. If P is a nice poset then take finite $N \trianglelefteq P$; every $p \in P$ lies above some element of N. Then $M = \{m \in N : \neg\exists n \in N(n<m)\}$ is non-empty and finite, and every $p \in P$ lies above an element of M.

The following are the three main kinds of non-nice poset. See [Sm].

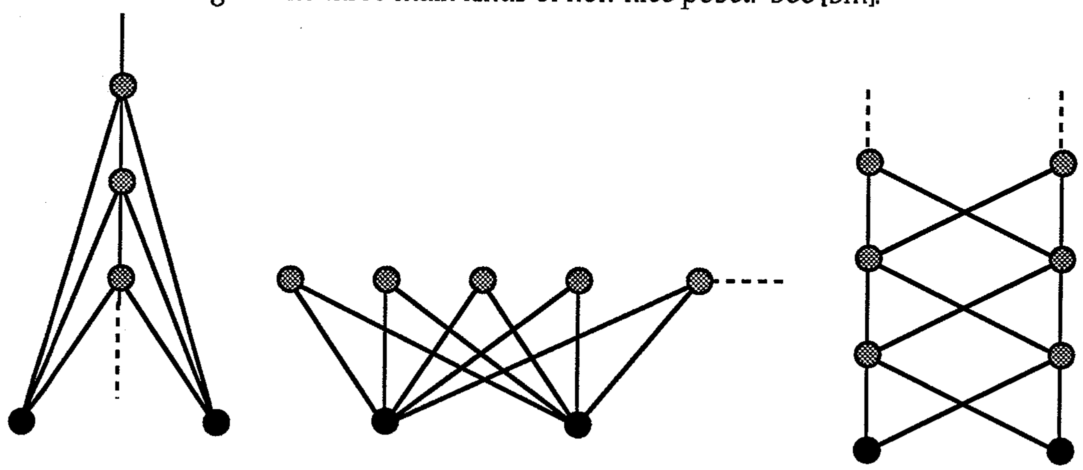

Figure 1.2

On the left the two black elements have no minimal upper bound, violating condition (i) above. In the centre poset they have infinitely many minimal upper bounds, violating (ii). The right-hand one satisfies (i) and (ii) but now the black elements have infinite MUB-closure.

SFP domains were introduced in [P] as those arising as inverse limits of Sequences of Finite Posets. They are of considerable interest in computer science, where they are used to provide denotational semantics for programming languages. Any domain P can be equipped with a topology (the Scott topology): $O \subseteq P$ is open iff (i) O is closed upwards, and (ii) if $D \subseteq P$

is directed and lub(D) $\in$ O then $D \cap O \neq \emptyset$. If D and E are domains we write $[D \to E]$ for the poset of Scott-continuous functions from D to E, ordered by $f \leqslant g$ iff for all $d \in D$, $f(d) \leqslant g(d)$. In {Sm] Smyth showed amongst other things that if D is a domain with countable base, then $[D \to D]$ is also a domain with countable base iff D is SFP. In this case $[D \to D]$ is also SFP. The SFP domains form the largest Cartesian closed full subcategory of the category of domains with countable bases, the morphisms being the Scott-continuous maps.

SFP systems

We now give our main algebraic definition. An **SFP system** is a triple $<P,\rho,\nu>$, where

(i) P is an SFP domain.

(ii) ρ is a map from P into the class of rings with a 1 ($1 \neq 0$). We will write R_p for $\rho(p)$, where ρ is understood.

(iii) ν is a map defined on those pairs $(p,q) \in P^2$ with $p \leqslant q$. Each $\nu(p,q)$ is a ring homomorphism from R_p into R_q. (All ring homomorphisms in this paper preserve 0 and 1.) We write $\nu(p,q)$ as ν_{pq}. We require further that

 (a) ν_{pp} is the identity on R_p

 (b) $\nu_{qr} \circ \nu_{pq} = \nu_{pr}$ if $p \leqslant q \leqslant r$ in P

 (c) if $D \subseteq P$ is directed with least upper bound $u \in P$, then R_u is the direct limit of the direct system $<R_d, \nu_{dd'} : d \leqslant d' \text{ in } D>$, and for all $d \in D$, ν_{du} is the canonical ring homomorphism from R_d into R_u.

Remark 1.1

Let P be a nice poset. Suppose we have a triple (P,ρ,ν) satisfying (ii) and (iii)(a), (b). Then we can canonically complete it to an SFP system by (a) embedding P canonically into its ideal completion Q, (b) defining R_q for $q \in Q \setminus P$ to be $\lim_{\to} <R_p, \nu_{pp'} : p \leqslant p' \text{ in } P \cap q\downarrow>$, and (c) defining $\nu_{qq'}$ for $q \leqslant q'$ in Q to be the 'limit' of the $\nu_{pp'}$ for $p, p' \in P$ with $p \leqslant q$, $p' \leqslant q'$. Moreover, all SFP systems arise in this way. So an SFP system $<P,\rho,\nu>$ is determined by its 'finite' part: on P^o, R_p and $\nu_{pp'}$ for $p \leqslant p'$ in P^o.

Limits of SFP systems

Let $<P,\rho,\nu>$ be an SFP system, and let $N \trianglelefteq P$. An element $r \in \Pi<R_p : p \in P>$ is said to have **support** N if for all $p \in P$, $r(p) = \nu_{(p/N),p}[r(p/N)]$. We define the **limit** of $<P,\rho,\nu>$, or **lim**$<P,\rho,\nu>$, to be the subdirect product consisting of those elements of ΠR_p that have a finite support $N \subseteq P^o$. Since P is an SFP domain, any two finite locally directed subsets of P^o are contained in a third, and it follows that the limit of $<P,\rho,\nu>$ is a subring of ΠR_p. Clearly it is also identifiable with a subring of $\Pi<R_p : p \in P^o>$, since P^o supports any element of R_P.

We will generally write R_P for the limit of $<P,\rho,\nu>$. Obviously, for any $p_o \in P$ the projection $(r \mapsto r(p_o))$ of R_P onto R_{p_o} is a surjective ring homomorphism.

As an example, if $P = (\mathbb{Q},<)$ and all R_p are $\{0,1\}$ then R_P is the unique countable atomless Boolean ring. See Example 3.4 below.

Subsystems

Let P be an SFP domain. If $Q \subseteq P$, we write $Q \preccurlyeq P$, and say that Q is a **subdomain** of P, if

- Q is itself an SFP domain under the ordering induced from P
- $Q^o \subseteq P^o$
- Q is a locally directed subset of P
- if $D \subseteq Q$ is directed then $lub_Q(D) = lub_P(D)$.

Note that these conditions imply that $P^o \cap Q \subseteq Q^o$, so that we have $P^o \cap Q = Q^o$ in fact. Clearly $\preccurlyeq$ is reflexive and transitive, and if $N \subseteq P$ is finite then $N \preccurlyeq P$ iff $N \trianglelefteq P^o$.

Proposition 1.2

Suppose that we have an SFP system $<P,\rho,\nu>$. Let $Q \preccurlyeq P$. Then $<Q,\rho|Q,\nu|Q^2>$ is an SFP system. Moreover, its limit ring R_Q is canonically isomorphic to the subring of R_P consisting of those elements supported by Q.

Proof

To show that $<Q,\rho|Q,\nu|Q^2>$ is an SFP system we only need to check that if $D \subseteq Q$ is directed then

$$R_{lub_Q(D)} = \lim_{\rightarrow}<R_d : d \in D>.$$

But this is clear, since $<P,\rho,\nu>$ is an SFP system and $lub_Q(D) = lub_P(D)$.

Now if $r \in R_Q$ there is finite $N \preccurlyeq Q$ supporting r. By transitivity of $\preccurlyeq$ we have $N \preccurlyeq P$, so r extends naturally to $r' \in R_P$ given by

- $r'(p) = \nu_{p/N,p}[r(p/N)]$ for $p \in P$.

The map $r \mapsto r'$ is a ring embedding from R_Q into R_P, and clearly its image is precisely the set of elements of R_P supported by a finite locally directed subset of Q - i.e. those supported by Q.

□

In future we identify R_Q with the subring $(R_Q)'$ of R_P, whenever $Q \preccurlyeq P$.

A special case is where $Q \preccurlyeq P$ is finite - i.e. $Q = N$, a finite locally directed subset of P^o. Then clearly $R_N \cong \Pi<R_n : n \in N>$, a finite direct product. If $N \subseteq N'$ are finite locally directed subsets of P^o, then $N \preccurlyeq N'$, and so (making the identification) R_N is a subring of $R_{N'}$. Since P is SFP, the following is clear:

Proposition 1.3

$<R_N : N \trianglelefteq P^o$ is finite$>$ is a direct system of rings under inclusion, and its direct limit is naturally isomorphic to R_P.

□

So all limit rings of SFP systems arise as direct limits of some direct system of rings.

Corollary 1.4

Let $<P,\rho,\nu>$ be an SFP system.

(i) If P has a least element, $\bot$, say, then $R_\bot$ is a subring of R_P.

(ii) The following classes of rings are closed under SFP systems, in the sense that if $R_p \in K$ for all $p \in P^o$ then $R_P \in K$ also:

(a) the class of commutative rings;

(b) the class of von Neumann regular rings (i.e. $R \models \forall x \exists y (xyx = x)$);

(c) the class of Boolean rings;

(d) the class of rings that are existentially closed in the class of commutative rings ('commutative e.c.');

(e) the class of existentially closed rings in the class of Boolean rings.

Proof

(i) Suppose that $\bot \in P$ is such that $\bot \leqslant p$ for all $p \in P$. Clearly $\{\bot\} \trianglelefteq P^o$. The result follows from (1.3) now.

(ii) By (1.3) it is enough to show that the classes cited are preserved under finite direct products and direct limits - or at least direct limits in which the morphisms of the system are injective. This is clear for (a), (b) and (c), where there is no use of injectivity. We prove (d).

Recall (e.g. from [CK]) that if L is a first order signature and Σ is a class of L-structures that is closed under isomorphism, an L-structure $M \in \Sigma$ is said to be **existentially closed in** Σ (**e.c.** for short) if whenever $M \subseteq N \in \Sigma$ and $\varphi(\bar{x})$ is an existential formula of L, then

- for all $\bar{a} \in M$, if $N \models \varphi(\bar{a})$ then already $M \models \varphi(\bar{a})$.

Clearly the class of e.c. structures is closed under isomorphism. By considering disjunctive normal forms we may assume that $\varphi(\bar{x})$ is of the form $\exists \bar{y} \psi(\bar{x},\bar{y})$ where ψ is a conjunction of atomic and negated atomic formulas.

It is easy to see that if Σ is closed under direct limits of the form $\lim_{\rightarrow}<M_i, \nu_{ij} : i \leqslant j$ in I$>$ where the ν_{ij} are injective, then a direct limit of e.c. structures is e.c.. The class of commutative rings is closed under such limits, so to prove (d) it suffices to prove that if A, B are commutative e.c. rings (i.e. they are e.c. in the class of commutative rings) then so is $A \times B$.

Suppose $C \supseteq A \times B$ is a commutative ring. Let $e_1 = (1,0)$, $e_2 = (0,1)$ in $A \times B$. Then since C is commutative, e_1 is a central idempotent of C. It follows that the left ideal Ce_1 of C is a commutative ring in its own right, with identity e_1; it has a subring $(A\times B)e_1$, which is isomorphic to A via $(a,b)e_1 \mapsto a$. Similarly, Ce_2 is a commutative ring with a subring $(A\times B)e_2 \cong B$.

Now since $e_1e_2 = 0$ and $e_1+e_2 = 1$, we have $C \cong Ce_1 \times Ce_2$ via $c \mapsto (ce_1,ce_2)$. It follows that:

($) if $\alpha(\bar{x})$ is an atomic formula of L and $\bar{c} \in C$, then $C \models \alpha(\bar{c})$ iff $Ce_i \models \alpha(\bar{c}e_i)$ for $i = 1,2$. Similarly, if $\bar{c} \in A\times B$ then $A\times B \models \alpha(\bar{c})$ iff $(A\times B)e_i \models \alpha(\bar{c}e_i)$ for $i = 1,2$.

If α is an atomic formula, define α^1 to be α and α^0 to be $\neg\alpha$. Let $\psi(\bar{x},\bar{y})$ above be $\bigwedge_{j<m}\alpha_j(\bar{x},\bar{y})^{n_j}$, where the α_j are atomic formulas of the signature $\{+,-,\cdot,0,1\}$ of rings, and $n_j = 0$ or 1. Suppose that $C \models \psi(\bar{a},\bar{c})$ for $\bar{a} \in A\times B$, $\bar{c} \in C$. Then by ($), there are $p_j, q_j \in \{0,1\}$ with $p_jq_j = n_j$ $(j<m)$, such that $Ce_1 \models \bigwedge_j \alpha_j(\bar{a}e_1,\bar{c}e_1)^{p_j}$ and $Ce_2 \models \bigwedge_j \alpha_j(\bar{a}e_2,\bar{c}e_2)^{q_j}$.

As $(A\times B)e_1 \cong A$ we can identify them and regard A as a subring of Ce_1. Because A is e.c. there is $\bar{c}_1 \in A$ such that $A \models \bigwedge_j \alpha_j(\bar{a}e_1,\bar{c}_1)^{p_j}$; and similarly we can find $\bar{c}_2 \in B$ with analogous properties for B. Take $\bar{d} \in A\times B$ with $\bar{d}e_1 = \bar{c}_1$, $\bar{d}e_2 = \bar{c}_2$; then $A \models \bigwedge_j \alpha_j(\bar{a}e_1,\bar{d}e_1)^{p_j}$ and $B \models \bigwedge_j \alpha_j(\bar{a}e_2,\bar{d}e_2)^{q_j}$. Hence by ($) again, $A\times B \models \bigwedge_j \alpha_j(\bar{a},\bar{d})^{n_j}$.

Hence $A\times B$ is an existentially closed commutative ring, as required.

(e) - the proof is the same as (d).

□

Note that for Boolean rings, eixstentially closed is the same as atomless. See for example [Hg, 6.3.9, Ex. 6.3.2]. Since many of the SFP domains we use have a least element $\perp$, SFP systems can often be used to produce rings extending a given ring $R = R_\perp$ (1.4(i)).

A slightly more general preservation result is: if all R_p satisfy $\varphi = \forall\bar{x}\exists\bar{y}(\bigwedge_i\pi_i\to\pi)$ where π_i and π are equations, then also R_P satisfies φ. This includes (ii(a)-(c)) above; the proof is the same.

There is an easy cardinality result that also follows from (1.3).

Proposition 1.5

Suppose that $\langle P,\rho,\nu\rangle$ is an SFP system in which each ring R_p is countable, and P is infinite. Then $|R_P| = |P^0|+\aleph_0$.

□

2 Ring Ideals

Here we examine the relationship between (ring) ideals of the limit ring of an SFP system $<P,\rho,\nu>$ and the underlying SFP domain of the system. The relationship is close and we will use it extensively in later sections. We isolate a special class of ideals - the *full ideals* - and show that they correspond closely to P. We also show that the ideals of the limit of the system are determined by their projections onto the components R_p ($p \in P$).

Unless otherwise stated, all ring ideals in this section will be left ideals.

Notation

Let P be an SFP domain, and fix an SFP system $<P,\rho,\nu>$ with limit ring R_P. We will generally use 'J' to denote an ideal of R_P, and 'I' for an ideal of a component ring R_p ($p \in P$). If J is an ideal of R_P and $q \in Q \preccurlyeq P$, we will write J_Q for $J \cap R_Q$ and $J_Q(q)$ for the projection $\{r(q) : r \in J_Q\}$ of J_Q onto the q^{th} component ring R_q. We write simply $J(q)$ for $J_P(q)$.

First a useful lemma.

Lemma 2.1

Let P be any finite poset and let $<P,\rho,\nu>$ be an SFP system with limit ring R_P. Let J be an ideal of R_P. Then $J = \{r \in R_P : r(p) \in J(p) \text{ for all } p \in P\}$.

Proof

'$\subseteq$' is clear; we prove '$\supseteq$'. For each $p \in P$ define a central idempotent $e_p \in R_P$ by

$$e_p(x) = \begin{cases} 1 \text{ if } x = p \\ 0 \text{ if } x \in P\backslash\{p\} \end{cases}$$

If $r(p) \in J(p)$ for all $p \in P$, then for each p there is $s_p \in J$ with $s_p(p) = r(p)$. Then $r = \Sigma_{p \in P} (e_p.s_p) \in J$, as required.

□

This essentially says that for finite P, $J \cong \Pi(J(p) : p \in P)$. We will generalise it to arbitrary P in (2.7) below.

Definition

If $p \in P$ and I is a proper ideal of R_p, we write $I@p$ for $\{r \in R_P : r(p) \in I\}$. This is a proper ideal of the limit ring R_P; strictly it depends on P also, and we will sometimes write "$I@p$ in R_P".

Now if $p' \in P$ and I' is an ideal of $R_{p'}$, then $I@p = I'@p'$ implies that $p = p'$ and $I = I'$. For if $p \neq p'$, then as P is algebraic, $p{\downarrow} \cap P^o \neq p'{\downarrow} \cap P^o$. Assume without loss that there is $q \in P^o \cap (p{\downarrow} \setminus p'{\downarrow})$. As P is an SFP domain there is finite $N \preccurlyeq P$ (i.e. $N \trianglelefteq P^o$) containing q. Hence $p/N \neq p'/N$. We can find $r \in R_N$ such that $r(p/N) = 0$ and $r(p'/N) = 1$. Then $r \in I@p \setminus I'@p'$, a contradiction. Hence $p = p'$, and it easily follows that $I = I'$.

If J is a proper ideal of R_P, we say that J is **full** (in R_P) if J is of the form $I@p$ for some p, I. Clearly I will be a proper ideal of R_p. Since p and I are unique, we can define $\sigma J = p$ (the **site** of J), and $\Delta J = I$ (the **defect** of J).

The 'theoretical' interest of full ideals is in their relationship with P, via their site. We will use this to show that the limit ring R_P can carry ring-theoretic traces of the underlying poset P, in a form of Stone duality. The main result involved is Theorem 2.2.

Theorem 2.2

Let $J \subseteq R_P$ be an ideal. Then the following are equivalent:

(i) J is full in R_P

(ii) for each finite $N \preccurlyeq P$, J_N is full in R_N

(iii) for each $Q \preccurlyeq P$, J_Q is full in R_Q.

Moreover, if any of (i)-(iii) hold, and $Q \preccurlyeq P$, we have

(iv) $\sigma(J_Q) = \sigma J/Q$

(v) $\Delta(J_Q) = (\nu_{\sigma J/Q,\sigma J})^{-1}(\Delta J)$.

Proof

(i ⇒ ii):

Assume that J is full in R_P; let $J = I@p$ (some $p \in P$, $I \subseteq R_p$). Let $N \preccurlyeq P$ be finite and let $n = p/N$. If $r \in R_N$, then $r \in J$ iff $r(p) = \nu_{np}[r(n)] \in I$ iff $r(n) \in \nu_{np}^{-1}(I)$ iff $r \in [\nu_{np}^{-1}(I)]@n$ in R_N. Hence $J_N = [\nu_{np}^{-1}(I)]@n$ in R_N. This proves (ii), and also (iv) and (v) in the case where Q is finite.

(ii ⇒ iii):

Assume (ii) and take $Q \preccurlyeq P$. If $N \preccurlyeq Q$ is finite then $N \preccurlyeq P$, so J_N is full in R_N for all finite $N \preccurlyeq Q$.

Now if $N, N' \preccurlyeq Q$ and $N \subseteq N'$, then $N \preccurlyeq N'$. It follows from the proof of (i ⇒ ii) that

(£) $\sigma(J_N) = \sigma(J_{N'})/N \leqslant \sigma(J_{N'})$.

So as Q is SFP, the set $D = \{\sigma J_N : \text{finite } N \preccurlyeq Q\}$ is directed. Let its lub in Q be q.

Claim 1 If $N \leqslant Q$ is finite, then $\sigma J_N = q/N$.

Proof of Claim Clearly $q \geqslant \sigma J_N \in N$. Hence $\sigma J_N \leqslant q/N$. For the converse inequality, note that as $q/N \leqslant q$ and q/N is a finite element of Q, there is finite $N' \leqslant Q$ such that $\sigma J_{N'} \geqslant q/N$. By (£) we may assume that $N' \supseteq N$, and so $\sigma J_N = \sigma J_{N'}/N \geqslant q/N$. This proves the claim.

Now let $I = \{r(q) : r \in J_Q\}$. Clearly I is an ideal of R_q.

Claim 2 $J_Q = I@q$ in R_Q.

Proof of Claim "$\subseteq$" is clear; we pass to "$\supseteq$". Let $r \in R_Q$ be such that $r(q) \in I$. So there is $s \in J_Q$ with $s(q) = r(q)$. Since $R_q = \lim_{\rightarrow}<R_{q'} : q' \in q\downarrow \cap Q^o>$ and Q is SFP, we can find finite $N \leqslant Q$ supporting r and s, and such that $s(q/N) = r(q/N)$. But $s \in J_N$, which by Claim 1 is full with site q/N. Hence $r \in J_N$ also. This proves the claim.

So by the claim J_Q is full in R_Q, which proves (iii).

(iii $\Rightarrow$ i): is trivial.

It remains to prove (iv) and (v) for infinite $Q \leqslant P$. Let $J \subseteq R_P$ be full; let $\sigma J = p$. Then J_Q is full; let it be $I@q$.

If $N \leqslant Q$ is finite, then we may apply (iv) to get $q/N = \sigma J_N$. But also $N \leqslant P$, so similarly $\sigma J_N = p/N$. Hence $p/N = q/N$ for all finite $N \leqslant Q$. Since Q is SFP, it follows that $p\downarrow \cap Q^o = q\downarrow \cap Q^o$. Taking lubs, we obtain $p/Q = q$, proving (iv).

For (v), we must show that $I = \nu_{qp}^{-1}(\Delta J)$. Take $a \in R_q$; there is finite $N \leqslant Q$ and $r \in R_Q$ supported by N, such that $r(q) = a$. By the above, $p/N = q/N$. So $r(p) = \nu_{qp}(r(q))$, and hence

$$a \in I \quad \text{iff} \quad r \in J \quad \text{iff} \quad r(p) \in \Delta J \quad \text{iff} \quad r(q) = a \in \nu_{qp}^{-1}(\Delta J).$$

□

Whilst R_P can have many full ideals with the same site, this is not so if we restrict to the elements of R_P that take values 0, 1 only. These elements are central idempotents of R_P. They form a Boolean algebra in the usual way by defining $a \leqslant b$ to hold iff $ab = a$; $a \wedge b$ is ab and $a \vee b$ is $a+b-ab$ (symmetric difference).

Definition

We write $(R_P)^*$ for the set $\{r \in R_P : \forall p \in P\ (r(p) \in \{0,1\})\}$. If $X \subseteq R_P$ we write X^* for $X \cap R_P^*$.

Proposition 2.3

Let I, J be full ideals of R_P. Then $\sigma I = \sigma J$ iff $I^* = J^*$.

Proof

Assume that $\sigma I = \sigma J$. Then if $r \in (R_P)^*$, $r \in I$ iff $r(\sigma I) \in \Delta I$. As ΔI is a proper ideal of $R_{\sigma I}$, this holds iff $r(\sigma I) = 0$. Since the same holds for J, we have $r \in I$ iff $r \in J$, so $I^* = J^*$.

Conversely, suppose that $\sigma I \neq \sigma J$. Since P is SFP, we can find finite $N \preccurlyeq P$ such that $\sigma I/N \neq \sigma J/N$. Let $r \in (R_P)^*$ be supported by N, and given by: $\forall n \in N$, $r(n) = 0$ if $n = \sigma I/N$, and 1 otherwise. Then $r \in I^* \setminus J^*$ so that $I^* \neq J^*$.

□

In practical terms, full ideals include the maximal, prime and irreducible ideals of R_P. Let us say that an ideal I of a ring R is **whole** if $R\setminus I$ conains no pair of orthogonal central idempotent elements (i.e. there do not exist $x, y \in R\setminus I$, commuting multiplicatively with every element of R, and such that $x^2 = x$, $y^2 = y$, $xy = 0$).

The following is easy:

Proposition 2.4

If I is a maximal, prime, or irreducible left (or right) ideal of R, then I is whole. If I is a maximal two-sided ideal of R, then I is whole.

□

But now we have:

Proposition 2.5

If I is a proper whole ideal of R_P then I is full.

Proof

If I is not full, then by (2.2) there is finite $N \preccurlyeq P$ such that I_N is not full in R_N. But clearly I_N is a proper ideal of R_N. By (2.1) there are $n \neq n'$ in N such that the projections $I_N(n)$ and $I_N(n')$ are proper ideals of R_n, $R_{n'}$ respectively. So we define $e_n \in R_P$ by

- e_n is supported by N; and $e_n(x) = 1$ if $x = n$, and 0 if $x \in N\setminus\{n\}$,

and similarly define $e_{n'}$. Then $e_n, e_{n'} \notin I$. Clearly e_n, $e_{n'}$ are central idempotents of R_P and $e_n . e_{n'} = 0 \in I$. Hence I is not whole.

□

Remark

Clearly, an ideal I is prime in R_p iff I@p is prime in R_P. If J is an ideal of R_P, then $J \supseteq I@p$ iff J is full, $\sigma J = p$ and $\Delta J \supseteq I$. Hence I@p is maximal, maximal two-sided or irreducible in R_P iff I is so in R_p.

Example 2.6

Suppose that $\langle P,\rho,\nu\rangle$ is such that $R_p = \{0,1\}$ for all $p \in P$. Hence each $\nu_{pp'}$ is the identity map. Then by (1.4), R_P is a Boolean ring. Here, a full ideal is determined by its site alone, as its defect must be 0. By the remark, any full ideal of R_P is maximal. By (2.4) and (2.5), the converse holds also. So P is in canonical bijection with the set of maximal ideals of R_P.

Now the set of maximal ideals of R_P is a Stone space and carries a compact Hausdorff totally disconnected topology: the clopen sets are those arising as the set of ideals containing a chosen point of the ring. Hence a homeomorphic topology is induced on P; it is in fact the 'patch' topology referred to in (e.g.) [Hc], whose construction bears some similarity to ours. We can be explicit about the topology: if $Q \trianglelefteq P$, define an equivalence relation $\sim_Q$ on P by: $p \sim_Q p'$ iff $p/Q = p'/Q$. Then a basis of open sets on P is the set C of equivalence classes of the $\sim_N$, for finite $N \preccurlyeq P$:

$$C = \bigcup\{P/\sim_N : N \text{ finite}, N \preccurlyeq P\}.$$

Each class is clopen. This is a basis since any finite intersection of elements of C is a finite union of elements of C. For any $Q \preccurlyeq P$, any $\sim_Q$-class is closed in the topology. Hence (taking $Q = P$) every singleton subset of P is closed: the topology is *regular.*

□

We now move from full ideals to arbitrary ideals. As before we let P be any SFP domain and $\langle P,\rho,\nu\rangle$ an SFP system with limit ring R_p. Our first result generalises (2.1) to this situation.

Theorem 2.7

Let J be any left ideal of R_P. Then for any $r \in R_P$,

- $r \in J$ iff $r(p) \in J(p)$ for all $p \in P$.

In other words, $J = \bigcap\{J(p)@p : p \in P\}$.

Proof

Clearly if $r \in J$ then $r(p) \in J(p)$ for all $p \in P$. For the converse it suffices to prove:

(*) $J = \bigcap\{J' : J' \text{ a full ideal of } R_P, J' \supseteq J\}$.

For assume that $r(p) \in J(p)$ for all $p \in P$. Let I@p be any full ideal containing J. Clearly $J(p) \subseteq I$. So $r \in I@p$. Hence $r \in J'$ for all full ideals $J' \supseteq J$. Given (*) we obtain $r \in J$ as required.

We only need to prove '$\supseteq$' of (*). Let $a \in R_P \backslash J$. We show that $a \notin \bigcap\{J' : J'$ a full ideal of R_P, $J' \supseteq J\}$.

Using Zorn's lemma choose a left ideal J' of R_P that is maximal with respect to:

- $J' \supseteq J$, $a \notin J'$.

We show that J' is a full ideal of R_P.

If not, by (2.2) there is finite $N \leqslant P$ such that J'_N is not full in R_N. Since it is certainly proper, by (2.1) there are orthogonal central idempotents e_1, e_2 of R_N such that $e_1 \notin J'$, $e_2 \notin J'$, $e_1+e_2 = 1$. By maximality of J' we have

$a = j_i + r_i e_i$ for some $j_i \in J'$, $r_i \in R_P$ $(i = 1, 2)$.

So $a = e_1 a + e_2 a = e_1.(j_2 + r_2 e_2) + e_2.(j_1 + r_1 e_1) = e_1 j_2 + e_2 j_1 \in J'$.

This is a contradiction. Hence J' is a full ideal of R_P, which completes the proof.

□

If $Q \leqslant P$ and J is an ideal of R_P, we can now express the projections $J_Q(q)$ $(q \in Q)$ of J_Q in R_Q in terms of the projections $J(p)$ of J in R_P. The result generalises Theorem 2.2 to arbitrary ideals.

Theorem 2.8

Let $Q \leqslant P$ and let J be an ideal of R_P. Then for each $q \in Q$,

$J_Q(q) = \bigcap\{\nu_{qp}^{-1}(J(p)) \ : p \in P,\ p/Q = q\}$.

Proof

For $q \in Q$ define $I_q = \bigcap\{\nu_{qp}^{-1}(J(p)) \ : p \in P,\ p/Q = q\}$. Let $a \in J_Q(q)$ for some q. Then there is $r \in J_Q$ with $r(q) = a$. Clearly $r(p) = \nu_{qp}(a) \in J(p)$ for all $p \in P$ with $p/Q = q$. So $a \in I_q$.

Hence $J_Q(q) \subseteq I_q$ for all $q \in Q$. It is not immediate that we have equality; for example, if $q_1 \in Q \backslash Q^0$ and

$$I_q = \begin{cases} R_{q_1} & \text{if } q = q_1 \\ 0 & \text{if } q \in Q\backslash\{q_1\} \end{cases}$$

then 0 is the only ideal I of R_Q with $I(q) \subseteq I_q$ for all q.

We prove equality as follows. Suppose for contradiction that there is $q \in Q$ and $a \in I_q \backslash J_Q(q)$. Using Zorn's Lemma as in the previous theorem take a left ideal $J' \supseteq J$ of R_P that is maximal with respect to: $a \notin J'_Q(q)$.

Claim: J'_Q is full in R_Q and $\sigma(J'_Q) = q$.

Proof of Claim: If not, there is finite $N \leqslant Q$ such that J'_N is not full in R_N. As before, take orthogonal idempotents $e_1, e_2 \in R_N \backslash J'$, central in R_P and such that $e_1+e_2 = 1$. By maximality of J' there are $j_i \in J'$, $r_i \in R_P$ such that

(*) $j_i + r_i.e_i \in R_Q$ and $(j_i + r_i.e_i)(q) = a$ $(i = 1,2)$.

Consider $s = e_1(j_2 + r_2e_2) + e_2(j_1 + r_1e_1)$. Since $e_1, e_2 \in R_Q$ we have $s \in R_Q$. Also, $s(q) = e_1(q).a + e_2(q).a = [(e_1+e_2)(q)].a = a$. But also $s = e_1j_2 + e_2j_1 \in J'$. So $s \in J'_Q$ and $s(q) = a$, a contradiction to the choice of J'. Hence J'_Q is full in R_Q, and clearly $\sigma J'_Q = q$. This proves the claim.

Now by the above, $J'_Q(q_1) \subseteq \bigcap\{\nu_{q_1p}{}^{-1}(J'(p)) : p/Q = q_1\}$ for all $q_1 \in Q$. It follows from the claim that $J'(p) = R_p$ for all $p \in P$ with $p/Q \neq q$.

Take $r \in R_Q$ with $r(q) = a$. Then as $a \in I_q$, $r(p) \in J(p) \subseteq J'(p)$ for all $p \in P$ with $p/Q = q$. So by (2.7), $r \in J'$, a contradiction.

□

We can now determine the left ideal of R_P generated by a left ideal of R_Q for $Q \leqslant P$. The following result also applies if we replace 'left' by 'two-sided' throughout.

Corollary 2.9

Let $Q \leqslant P$ and I be a left ideal of R_Q. Then:

(i) the left ideal J of R_P generated by I is given by

(*) for all $p \in P$, $J(p)$ is the left ideal of R_p generated by $\nu_{p/Q,p}[I(p/Q)]$.

(ii) if $I = I'@q$ in R_Q (for some $q \in Q$ and left ideal I' of R_q) and for all $p \in P$ with $p \neq q$, $p/Q = q$, the left ideal of R_p generated by $\nu_{qp}(I')$ is improper, then I generates the left ideal $I'@q$ in R_P.

Proof

(i) For each $p \in P$ let J_p be the left ideal of R_p generated by $\nu_{p/Q,p}(I(p/Q))$. Let $J = \{r \in R_P : r(p) \in J_p \text{ for all } p \in P\}$. Certainly J is an ideal of R_P, and $J(p) = J_p$ for all $p \in P$. But if $J' \supseteq I$ is a left ideal of R_P then $J'_Q \supseteq I$, so by (2.8), for each $p \in P$ and $q \in Q$ with $p/Q = q$ we have $\nu_{qp}{}^{-1}[J'(p)] \supseteq J'_Q(q) \supseteq I(q)$. Hence $J(p) \subseteq J'(p)$ for all p, so that $J \subseteq J'$. So I generates J in R_P.

(ii) This is a special case of (i); we will use it in Part II.

□

Corollary 2.10

Assume that $Q \leqslant P$ and let the left ideal I of R_Q generate the left ideal J of R_P. Then

(i) $J(q) = I(q)$ for all $q \in Q$

(ii) $J_Q = I$.

Proof

(i) is a special case of Corollary 2.9(i). Hence for each $q \in Q$, $I(q) \subseteq J_Q(q) \subseteq J(q) = I(q)$, so $J_Q(q) = I(q)$. Part (ii) now follows by (2.7).

□

3 Densely decomposable ideals

Here we develop a way to obtain an atomless Boolean ring as the limit of an SFP system in the case where all component rings are Boolean. As in [Z] we use densely decomposable ideals to generalise the notion of *atomless* to arbitrary rings. Again, unless otherwise stated all ring ideals will be left ideals.

Recall from the introduction the definition of densely decomposable:

Definition

Let R be any ring, and I a proper left ideal of R. I is said to be **densely decomposable** if whenever J is a left ideal of R properly extending I, then there are left ideals $X, Y \subseteq J$ properly extending I, with $X \cap Y = I$.

Example 3.1

Let R be a Boolean ring. Then the ideal {0} is densely decomposable iff R is atomless: that is, if $r \neq 0$ in R then there is $s \in R$ with $r \neq s.r = s \neq 0$. So for an ideal of a Boolean ring, being densely decomposable is the same as having atomless quotient, and is in a sense opposite to being irreducible.

We wish to find conditions for ideals of the limit ring of an SFP system to be densely decomposable.

Definition

Let R be a ring and $I \subseteq J$ left ideals of R. We say that J **splits over I** if there are left ideals $X, Y \subseteq J$ with $X \supset I$, $Y \supset I$, $X \cap Y = I$. If $S \supseteq I$ is any subset of R, we say that S