The McKinsey–Lemmon logic is barely canonical

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Abstract

We study a canonical modal logic introduced by Lemmon, and axiomatised by an infinite sequence of axioms generalising McKinsey's formula. We prove that the class of all frames for this logic is not closed under elementary equivalence, and so is non-elementary. We also show that any axiomatisation of the logic involves infinitely many non-canonical formulas.

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1 Introduction

Our story starts with *McKinsey's formula,*[1]

$$M: \qquad \Box\Diamond p \to \Diamond\Box p. \tag{1}$$

M has long been studied by modal logicians. On the one hand, the normal modal logic $K4M$ (also known as $K4.1$) axiomatised by M together with the transitivity axiom $\Box p \to \Box\Box p$ is a well-behaved logic. It is canonical (i.e., valid in its own canonical frame), and hence Kripke complete. The class of all frames validating $K4M$ is elementary: it is the class of transitive frames such that colloquially, every world sees a world that can see at most itself (see, e.g., [2, proposition 3.46] or [1, example 3.57]).

On the other hand, M itself is rather wild. The logic KM that M axiomatises alone is determined by its finite frames [3], and so it is Kripke complete. However, the class of all frames validating KM is not elementary [5], and not even closed under elementary equivalence [19].[2] KM is not the logic of *any* elementary class of frames [5], and is not canonical [6]. M is often called the simplest formula not equivalent to a Sahlqvist formula (see [1, §3.6] or [2, §10.3] for details of Sahlqvist formulas).

KM was cited by Lemmon in [16] as a logic that had not yielded to the 'canonical model' completeness method expounded in that work. Lemmon then generalised M to an infinite sequence of formulas

$$M_k: \qquad \Diamond\big((\Diamond p_1 \to \Box p_1) \wedge \ldots \wedge (\Diamond p_k \to \Box p_k)\big), \quad \text{for } k \geq 1. \tag{2}$$

[1]McKinsey actually studied (in [17]) the system $S4$ augmented with $\Box\Diamond p \wedge \Box\Diamond q \to \Diamond(p \wedge q)$, but Smoryński showed in [18] that this is the same system as $S4 + M$. For this and further discussion, see [8].

[2]Actually, the class of frames validating any modal logic is elementary iff it is closed under elementary equivalence [20].

M is equivalent (in the basic normal modal logic K) to M_1. It may help to observe that since $\Diamond p \to \Box p$ is equivalent to $\Box\neg p \vee \Box p$, we can rewrite M_k equivalently as

$$M_0 = \top, \qquad M_k = \Diamond \bigwedge_{i<k} (\Box p_i \vee \Box\neg p_i) \quad \text{for } k \geq 1, \tag{3}$$

where for later convenience we use the propositional variables $p_0, \ldots, p_{k-1}$. We will use this form of the M_k throughout the paper. Now we can see that the validity of M_1 in a Kripke frame $\mathcal{F}$ says that for any partition of the worlds of $\mathcal{F}$ into at most two sets (corresponding to the interpretations of p and $\neg p$ in a Kripke model over $\mathcal{F}$), any world sees a world whose successors all lie in a single partition set. M_k says the same as M_1 but for a partition into at most 2^k sets. Clearly, $M_{k+1} \vdash M_k$ for all $k \geq 1$. Lemmon showed by a short proof-theoretic argument that assuming transitivity, all the M_k are equivalent to M_1.

Lemmon defined KM^∞ to be the modal logic axiomatised by the axioms in (2). This logic, standing between KM and $K4M$, is the subject of our paper. Lemmon proved that it is the logic of the class of Kripke frames satisfying

$$m^\infty : \qquad \forall x \exists y \big(R(x,y) \wedge \forall z\, z'(R(y,z) \wedge R(y,z') \to z = z')\big). \tag{4}$$

This condition says that every world sees a world with at most one successor. By considering partitions as above, it is easily seen that KM^∞ is valid in all frames with this property. Lemmon proved completeness by a compactness argument that showed that the canonical frame for KM^∞ satisfies m^∞. This means that KM^∞ is canonical. The logic obtained from KM^∞ by adding the transitivity axiom is $K4M$, so since the transitivity axiom is also canonical, this gives another proof of the canonicity of $K4M$. Since KM^∞ is the logic of an elementary class of frames — those satisfying m^∞ — its canonicity also follows from *Fine's theorem* that the modal logic of an elementary class of frames is canonical [4]. However, the proof by compactness is different, and the method applies in some cases where Fine's result does not [10, 9]. A similar argument was used by Hughes in [13]. [11] derives these results from a more general perspective, obtaining (3) effectively from a formulation of (4) in hybrid logic.

In [14, §6], Jónsson showed using new algebraic proofs that the M_k are theorems of $K4M$, that KM^∞ is canonical, and hence that $K4M$ is canonical.

Our paper Here, we add to the impression that KM^∞ lies somewhat nearer to KM than to $K4M$. First, we show that, just as for KM, the class of all frames for KM^∞ is non-elementary, and not even closed under elementary equivalence (theorem 2.2 below). The proof is similar to that of [19] for KM. In remark 3.9, we show that the class of frames for KM^∞ is not closed under ultraproducts.

We also study the canonicity of KM^∞. We have seen that it shares canonicity with $K4M$. But we will show that it is *only barely canonical.* A formula is said to be *canonical* if the logic that it axiomatises is canonical. We prove:

- (Theorem 4.3) For no $k \geq 1$ is M_k canonical. This generalises the result of [6] that M is not canonical.

- KM^∞ cannot be axiomatised by canonical formulas. Hence, it is not axiomatisable by Sahlqvist formulas.

- (Theorem 4.4) Indeed, any axiomatisation of KM^∞ has infinitely many non-canonical axioms.

It follows by Fine's theorem that KM^∞ is *only barely the logic of an elementary class of frames.* No M_k (for any $k \geq 1$) axiomatises the logic of any elementary class of frames; and any axiomatisation of KM^∞ contains infinitely many axioms that, taken individually, fail to axiomatise the logic of any elementary class of frames. But KM^∞ itself is the logic of a finitely axiomatisable elementary class of frames.

Thus, the canonicity of KM^∞, and its being the logic of an elementary class of frames, do not arise from properties of any finite number of axioms. They only emerge in the limit when all the axioms are taken together. This striking phenomenon has been seen before. In an algebraic setting, [12] showed that the variety RRA of representable relation algebras, and also the variety of modal algebras of 'infinite chromatic number', cannot be axiomatised by finitely many non-canonical axioms plus arbitrarily many canonical ones. Analogous results on elementary frame classes then follow from Fine's theorem as above. We use the same proof methods here. For each k, l with $2 \leq l \leq k < \omega$, we construct an inverse system of finite Kripke frames validating M_k, whose inverse limit is a frame that validates M_l but not M_{l+1}. The frames are based on those used in the proof in [6] of non-canonicity of KM. We can then deduce the third result above by first-order compactness.

Organisation of paper In section 2 we prove that the class of frames for KM^∞ is non-elementary (theorem 2.2). In section 3, we introduce some particular frames, and determine which M_k they validate. They will be put to use in section 4, where we show that no M_k is canonical (theorem 4.3), and that any axiomatisation of KM^∞ involves infinitely many non-canonical axioms (theorem 4.4).

Notation Let $f : X \to Y$ be a map. We write $\operatorname{dom} f$ and $\operatorname{rng} f$ for the domain and range of f, respectively. If $S \subseteq X$, we write $f \restriction S$ denote the restriction f to S. If $S \subseteq Y$, we write $f^{-1}[S]$ for $\{x \in X : f(x) \in S\}$. If $y \in Y$, we write $f^{-1}[y]$ for $f^{-1}[\{y\}]$. For sets X_i $(i \in I)$, we write $\prod_{i \in I} X_i$ for the set of maps $\eta : I \to \bigcup_{i \in I} X_i$ such that $\eta(i) \in X_i$ for each i. We often write $\eta(i)$ as η_i in this case.

Natural numbers will be regarded as ordinals. So for a natural number $n < \omega$, we identify n with $\{0, 1, \ldots, n-1\}$. For an ordinal α, we write ${}^\alpha 2$ for the set of maps $f : \alpha \to 2$, and ${}^{<\omega}2$ for $\bigcup_{n<\omega} {}^n 2$. Given a set X and a cardinal κ, the expression $[X]^{\geq \kappa}$ denotes $\{Y \subseteq X : |Y| \geq \kappa\}$.

Kripke semantics We set up our notation for this. A *(Kripke) frame* $\mathcal{F} = (W, R)$ consists of a non-empty set W of 'worlds', together with a binary 'accessibility' relation R on W. We will write $\operatorname{dom} \mathcal{F}$ for W, and write $R(x, y)$ to indicate that $(x, y) \in R$. An *R-successor* (respectively, *R-predecessor*) of $w \in W$ is a world $x \in W$ satisfying $R(w, x)$ (respectively, $R(x, w)$). We may indicate informally that $R(w, x)$ by saying that w sees x, or that x is accessible from w. We will write R^w for the set of all R-successors of w.

We fix a countably infinite set $V = \{p_0, p_1, \ldots\}$ of propositional variables. An *assignment* into $\mathcal{F}$ is a map $h : V \to \wp(W)$, the power set of W. The pair $(\mathcal{F}, h)$ is called a *(Kripke) model.* We evaluate modal formulas at worlds of Kripke models in the usual way: for $p \in V$, $(\mathcal{F}, h), w \models p$ iff $w \in h(p)$; booleans as usual; and $(\mathcal{F}, h), w \models \Diamond\phi$ (respectively, $(\mathcal{F}, h), w \models \Box\phi$) iff $(\mathcal{F}, h), x \models \phi$ for some (respectively, all) $x \in R^w$. A modal formula ϕ is *valid* at a world w of a frame $\mathcal{F}$ if $(\mathcal{F}, h), w \models \phi$ for every assignment h into $\mathcal{F}$. ϕ is *valid* in a frame $\mathcal{F}$, written $\mathcal{F} \models \phi$, if it is valid at every world of $\mathcal{F}$.

A frame (W, R) is a *generated subframe* of another, (W', R'), if $W \subseteq W'$ and $R = R' \cap (W \times W')$. In this case, it is known (e.g., from [1, proposition 2.6] or [2, theorem 2.7]) that for any $h' : V \to \wp(W')$, if $h : V \to \wp(W)$ is given by $h(p) = h'(p) \cap W$ for $p \in V$, then

$$(W, R, h), w \models \phi \iff (W', R', h'), w \models \phi$$

for every $w \in W$ and every modal formula ϕ. Hence, validity is preserved under generated subframes.

We will assume familiarity with basic notions of modal logic, such as canonical models. See [1, 2] for guidance if required.

2 The frames for KM^∞ are non-elementary

Using a result of Goldblatt and an argument along the lines of van Benthem's proof for KM in [19], we can establish our first result. We first quote theorem 1 of [6] (reproduced as [7, theorem 10.1]).

Theorem 2.1 *Let $\mathcal{F} = (W, R)$ be a frame. Suppose that W contains a point r with the property that $|R^m| \geq |R^r| + \omega$ for every $m \in R^r$. Then no M_k $(k \geq 1)$ is valid in $\mathcal{F}$.*

Proof. Let $|R^r| = \kappa$, and put $R^r = \{m_i : i < \kappa\}$. Define distinct points $x_i, y_i \in W$ for $i < \kappa$ by induction as follows. If $i < \kappa$ and x_j, y_j have been defined for all $j < i$, we define x_i, y_i to be any distinct points of $R^{m_i} \setminus \{x_j, y_j : j < i\}$. This is possible because $|R^{m_i}| \geq \kappa + \omega$, so $R^{m_i} \setminus \{x_j, y_j : j < i\}$ is infinite. Now define a Kripke model $\mathcal{M}$ over $\mathcal{F}$ by making p true at precisely $\{y_i : i < \kappa\}$. If $\mathcal{M}, r \models \Diamond(\Box p \vee \Box\neg p)$, then there is $i < \kappa$ such that $\mathcal{M}, m_i \models \Box p \vee \Box\neg p$. But $x_i, y_i \in R^{m_i}$, $\mathcal{M}, x_i \models \neg p$, and $\mathcal{M}, y_i \models p$, so this is impossible. Hence, $\mathcal{M}, r \not\models M_1$. Since $M_{k+1} \vdash M_k$ for $k \geq 1$, no M_k for any $k \geq 1$ is valid in $\mathcal{F}$. $\Box$

Theorem 2.2 *For each $k \geq 1$, the class of frames that validate M_k is not closed under elementary substructures, and hence is not elementary. The same holds for the class of frames validating KM^∞.*

Proof. Let $\mathcal{F} = (W, R)$ be the Kripke frame with $W = \{r\} \cup [\omega]^{\geq\omega} \cup \omega$, where $r \notin [\omega]^{\geq\omega} \cup \omega$ is arbitrary, and R is given by:

$$\begin{array}{rcll} R^r & = & [\omega]^{\geq\omega}, & \\ R^S & = & S & \text{for each } S \in [\omega]^{\geq\omega}, \\ R^n & = & \{n\} & \text{for each } n \in \omega. \end{array}$$

(So for $n \in \omega$ and $S \in [\omega]^{\geq\omega}$, $R(S, n)$ holds iff $n \in S$.) Then $\mathcal{F}$ is a frame for KM^∞. For, given any $k \geq 1$ and any assignment $h : \{p_0, \ldots, p_{k-1}\} \to \wp(W)$, there is $S \in [\omega]^{\geq\omega}$ such that for all $x, y \in S$ and $i < k$, we have $(\mathcal{F}, h), x \models p_i$ iff $(\mathcal{F}, h), y \models p_i$. Then $(\mathcal{F}, h), S \models \bigwedge_{i<k}(\Box p_i \vee \Box\neg p_i)$, so $(\mathcal{F}, h), r \models M_k$. Validity of M_k at all other points in $\mathcal{F}$ is clear, as they have a successor (an element of ω) related only to itself. So $\mathcal{F} \models M_k$ for each k, and $\mathcal{F}$ validates KM^∞.

But in any countable elementary substructure $\mathcal{F}_0 = (W_0, R_0)$ of $\mathcal{F}$, it is easy to check that $r \in W_0$, and $|R_0^r| \leq \omega = |R_0^S|$ for each $S \in R_0^r$. (For example, this follows from the preservation under elementary substructures of the formulas $\exists x \forall y \neg R(y,x)$ and $\forall x(R(r,x) \to \exists_{\geq n} y\, R(x,y))$ for each finite n.) By theorem 2.1, $\mathcal{F}_0 \not\models M_k$ for every $k \geq 1$.

Hence, the class of frames validating KM^∞ is not closed under elementary substructures and so cannot be elementary. Also, for each $k \geq 1$, the class of all frames validating M_k is not closed under elementary substructures (since $\mathcal{F}$ validates M_k but $\mathcal{F}_0$ does not). □

3 Frames validating some M_k and not others

In this section, we study the canonicity properties of KM^∞ and its axiomatisations, using special frames based on those in [6].

3.1 Squat frames

Definition 3.1 Let $\mathcal{F} = (W, R)$ be a frame. A world of W is called a *root* of $\mathcal{F}$ if it has no R-predecessors, a *leaf* of $\mathcal{F}$ if it has no R-successors other than itself, and a *midpoint,* otherwise. A world is *reflexive* if it is R-related to itself, and *irreflexive* otherwise.

$\mathcal{F}$ is said to be *squat* ([6] uses the term 'trellis-like') if it has a unique root, say r; r is not a leaf; all successors of r are midpoints; and all successors of midpoints are reflexive leaves.

For example, the frame in theorem 2.2 is squat.

Remark 3.2 We will often use the obvious fact that each M_k is valid at every world of a squat frame except perhaps the root. This is because each non-root has a reflexive leaf among its successors, and a reflexive leaf must clearly validate $\bigwedge_{i<k}(\Box p_i \vee \Box \neg p_i)$ for any $k \geq 1$.

Definition 3.3 Let $I \neq \emptyset$, and for each $i \in I$ let $\mathcal{F}_i$ be a squat frame. We write $\sum_{i \in I} \mathcal{F}_i$ for the squat frame consisting of a copy of each $\mathcal{F}_i$ ($i \in I$), the copies being disjoint except that their roots are identified.

Formally, if $\mathcal{F}_i = (W_i, R_i)$ then $\sum_{i \in I} \mathcal{F}_i = (W, R)$, where $W = (\bigcup_{i \in I} W_i \times \{i\})/{\sim}$, the equivalence relation $\sim$ is given by $(w,i) \sim (w',i')$ iff $(w,i) = (w',i')$ or w, w' are the roots of $\mathcal{F}_i, \mathcal{F}_{i'}$ respectively, and $R = \{((w,i)/{\sim}, (w',i)/{\sim}) : i \in I,\ R_i(w,w')\}$, where $(w,i)/{\sim}$ denotes the $\sim$-class of (w,i). If $I = \{i_1, \ldots, i_m\}$, we write the sum as $\mathcal{F}_{i_1} + \cdots + \mathcal{F}_{i_m}$.

We will usually identify each non-root world w of each $\mathcal{F}_i$ with its 'copy' $(w,i)/{\sim}$ in $\sum_{i \in I} \mathcal{F}_i$.

Lemma 3.4 *Let $\mathcal{F}_i$ ($i \in I \neq \emptyset$) be squat frames. For any $k < \omega$, we have $\sum_{i \in I} \mathcal{F}_i \models M_k$ iff $\mathcal{F}_i \models M_k$ for some $i \in I$.*

Proof. Recall that $M_0 = \top$ and $M_k = \Diamond \bigwedge_{j<k}(\Box p_j \vee \Box \neg p_j)$ for $k \geq 1$. The result is trivial for M_0. Let $k \geq 1$. Write r_i for the root of $\mathcal{F}_i$ (each i), and r for the root of $\mathcal{F} = \sum_{i \in I} \mathcal{F}_i$. Note that roots are by definition irreflexive. We will use remark 3.2 without explicit mention. We write $\alpha = \bigwedge_{j<k}(\Box p_j \vee \Box \neg p_j)$, so that $M_k = \Diamond \alpha$.

$\Rightarrow$: If $\mathcal{F}_i \not\models M_k$ for each $i \in I$, then for each i there is an assignment h_i into $\mathcal{F}_i$ such that $(\mathcal{F}_i, h_i), r_i \not\models M_k$. Let h be an assignment into $\mathcal{F}$ such that for each i, h agrees with h_i on the non-root worlds of $\mathcal{F}_i$. Assume for contradiction that $(\mathcal{F}, h), r \models M_k$. Pick a successor s of r with $(\mathcal{F}, h), s \models \alpha$. Since r is irreflexive, $s \neq r$. Suppose that s is in $\mathcal{F}_i$, say. Then $s \in R_i^{r_i}$, where R_i is the accessibility relation of $\mathcal{F}_i$. Now it is clear that the subframe of $\mathcal{F}_i$ based on

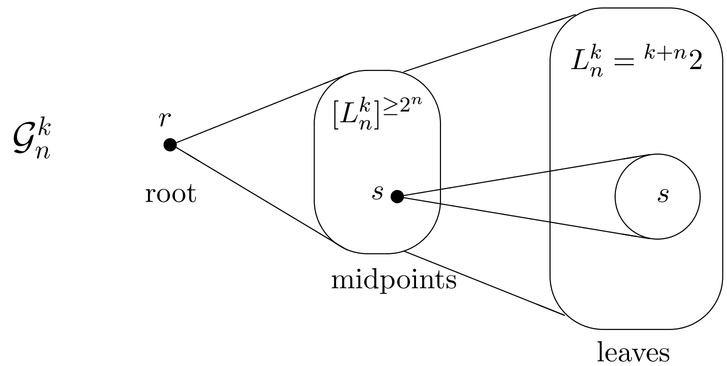

Figure 1: The squat frame $\mathcal{G}_n^k$

$\{s\} \cup R_i^s$ is a generated subframe of both $\mathcal{F}$ and $\mathcal{F}_i$. It follows that $(\mathcal{F}_i, h_i), s \models \alpha$, and hence $(\mathcal{F}_i, h_i), r_i \models M_k$, contradicting the choice of h_i. So $(\mathcal{F}, h), r \not\models M_k$, and M_k is not valid in $\mathcal{F}$.

$\Leftarrow$: Suppose that $i \in I$ and $\mathcal{F}_i \models M_k$. Let h be any assignment into $\mathcal{F}$. We show that $(\mathcal{F}, h), r \models M_k$. Let h_i be the 'restriction' of h to $\mathcal{F}_i$. By assumption, $(\mathcal{F}_i, h_i), r_i \models M_k$, so there is a successor s of r_i in $\mathcal{F}_i$ with $(\mathcal{F}_i, h_i), s \models \alpha$. By definition of $\mathcal{F}$, s is a successor of r in $\mathcal{F}$. As before, $(\mathcal{F}, h), s \models \alpha$. So $(\mathcal{F}, h), r \models M_k$ as required. $\square$

3.2 Special squat frames

The following squat frames are modifications of frames used in [6] to prove non-canonicity of M. We will use them to study the canonicity of KM^∞.

Definition 3.5 For each $k, n < \omega$, we define $\mathcal{G}_n^k$ to be the squat frame with a root r, a set $L_n^k = {}^{k+n}2$ of leaves, and a set $[L_n^k]^{\geq 2^n} = \{Y \subseteq L_n^k : |Y| \geq 2^n\}$ of midpoints. See figure 1. The accessibility relation R on $\mathcal{G}_n^k$ is given by:

- $R^r = [L_n^k]^{\geq 2^n}$,
- $R^s = s$ for each $s \in [L_n^k]^{\geq 2^n}$,
- $R^x = \{x\}$ for each $x \in L_n^k$.

Fix $k < \omega$. The following lemmas determine which M_l $(l < \omega)$ are valid in which $\mathcal{G}_n^k$.

Lemma 3.6 *$\mathcal{G}_0^k$ validates M_l for every $l < \omega$.*

Proof. $M_0 = \top$ is valid, so suppose $l \geq 1$. All singleton subsets of L_0^k are midpoints of $\mathcal{G}_0^k$. So each of these midpoints has a unique successor. But any point with at most one successor validates $\bigwedge_{i<l}(\Box p_i \vee \Box \neg p_i)$. Since the root sees all midpoints, M_l is valid at the root, and hence (remark 3.2) valid in $\mathcal{G}_0^k$. $\square$

Lemma 3.7 *For each $n < \omega$, M_k is valid in $\mathcal{G}_n^k$.*

Proof. Certainly, M_0 is valid. Assume that $k \geq 1$. By remark 3.2, we only need check that M_k is valid at the root. Let $h : V \to \wp(\operatorname{dom} \mathcal{G}_n^k)$ be an arbitrary assignment. Then h induces a partition of L_n^k into at most 2^k sets, namely, the equivalence classes of the equivalence relation on L_n^k given by $x \sim y$ iff $x \in h(p_i) \iff y \in h(p_i)$ for each $i < k$. Since $|L_n^k| = 2^{k+n}$, at least one partition set s must have cardinality at least 2^n, and so is in $[L_n^k]^{\geq 2^n}$. Then $(\mathcal{G}_n^k, h), s \models \Box p_i \vee \Box\neg p_i$ for each $i < k$. As s is accessible from the root, we see that $(\mathcal{G}_n^k, h), r \models M_k$. Since h was arbitrary, the proof is complete. □

Lemma 3.8 *If $n \geq 1$ then M_{k+1} is not valid in $\mathcal{G}_n^k$.*

Proof. As $n \geq 1$, we may assign truth values to the variables $p_0, \ldots, p_k$ at points in L_n^k by: p_i is true at $\eta \in {}^{k+n}2$ iff $\eta(i) = 1$. Let $s \subseteq L_n^k$ and suppose that for each $i \leq k$, p_i has the same truth value on every element of s. Define $\xi \in {}^{k+1}2$ by: for each $i \leq k$, $\xi(i) = 1$ iff p_i is true at every element of s. Then $\xi = x\restriction(k+1)$ for every $x \in s$. It follows that $|s| \leq 2^{k+n-(k+1)} = 2^{n-1}$. As all midpoints of $\mathcal{G}_n^k$ have at least 2^n elements, s cannot be in $\mathcal{G}_n^k$. So $\bigwedge_{i \leq k}(\Box p_i \vee \Box\neg p_i)$ is false at every midpoint in $\mathcal{G}_n^k$, and therefore M_{k+1} is false at the root under this assignment. □

Note that the accessibility relation of $\mathcal{G}_n^k$ is not transitive. This is essential. For as we mentioned in the introduction, $M_{k+1} \vdash M_k$ for all k, and any transitive frame validating M_1 actually validates all the M_k. It follows that any transitive frame validating M_k (for any $k \geq 1$) must also validate M_{k+1}. So transitivity would violate the lemmas. They show that $M_k \nvdash M_{k+1}$. So in the absence of transitivity, the M_k are strictly increasing in strength.

Remark 3.9 These results will show that the class of frames that validate KM^∞ is not closed under ultraproducts, thereby reproving theorem 2.2. For each $n < \omega$, let $\mathcal{F}_n = \sum_{k<\omega} \mathcal{G}_n^k$. By lemma 3.7, $\mathcal{G}_n^k$ validates M_k for each k. By lemma 3.4, $\mathcal{F}_n$ also validates M_k for each k. Now consider a non-principal ultraproduct $\mathcal{F}$ of the $\mathcal{F}_n$. Every midpoint of $\mathcal{F}_n$ has at least 2^n successors. By standard saturation properties of ultraproducts, or by direct inspection, each midpoint of $\mathcal{F}$ has 2^ω successors, and $|\operatorname{dom} \mathcal{F}| = 2^\omega$ as well. By theorem 2.1, $\mathcal{F}$ validates no M_k for any $k \geq 1$. Our result now follows from the fact that any elementary class is closed under ultraproducts. The same argument shows that for any $k \geq 1$, the class of frames validating M_k is not closed under ultraproducts.

3.3 Descriptive frames and inverse limits

We wish to apply a result of Goldblatt on inverse limits of families of descriptive frames, so we will recall what these are.

Definition 3.10 A *general frame* is a triple (W, R, P), where (W, R) is a Kripke frame, and $P \subseteq \wp(W)$ is non-empty and closed under intersection, complement, and the map $l_R : S \mapsto \{x \in W : \forall y(R(x, y) \to y \in S)\}$ (for $S \subseteq W$).

A general frame (W, R, P) is said to be a *descriptive frame* if

1. If $x, y \in W$ are distinct, then there is some $S \in P$ with $x \in S$ and $y \notin S$.
2. If $x, y \in W$ and $\neg R(x, y)$, then there is some $S \in P$ with $x \in l_R(S)$ and $y \notin S$.

3. $\bigcap \mu \neq \emptyset$ for every 'ultrafilter' $\mu \subseteq P$ — i.e., a subset of P satisfying, for all $S, S' \in P$, (i) $S' \supseteq S \in \mu \Rightarrow S' \in \mu$, (ii) $S, S' \in \mu \Rightarrow S \cap S' \in \mu$, and (iii) $S \in \mu \iff (W \setminus S) \notin \mu$.

For information about descriptive frames, see, e.g., [7, §§1.9–1.11], [2, §8.4], and [1, §5.5].

Definition 3.11

1. If $\mathcal{F} = (W, R)$ is a Kripke frame, we write $\mathcal{F}^+$ for $(W, R, \wp(W))$. ([7, 1.3.5] uses this notation in a different way.) Clearly, if $\mathcal{F}$ is finite (i.e., W is finite), then $\mathcal{F}^+$ is a descriptive frame.

2. If $\mathcal{F} = (W, R, P)$ is a descriptive frame, we write $\mathcal{F}_+$ for its underlying Kripke frame (W, R). (We will not use this notation for non-descriptive general frames because it would clash with well known algebraic notation.)

Definition 3.12 Let $\mathcal{F} = (W, R, P)$ be a general frame and ϕ a modal formula. We say that ϕ is *valid in $\mathcal{F}$,* written $\mathcal{F} \models \phi$, if $(W, R, h), w \models \phi$ for every assignment $h : V \to P$ and every $w \in W$.

Clearly, ϕ is valid in a Kripke frame $\mathcal{F}$ iff it is valid in the general frame $\mathcal{F}^+$:

$$\mathcal{F} \models \phi \iff \mathcal{F}^+ \models \phi. \tag{5}$$

We will also need the notions of bounded morphism, frame homomorphism, and inverse family.

Definition 3.13 Recall that given frames $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$, a map $f : W \to W'$ is said to be a *bounded morphism* from $\mathcal{F}$ to $\mathcal{F}'$ if for all $w \in W$ and $v' \in W'$, we have $R'(f(w), v')$ iff there is $v \in R^w$ with $f(v) = v'$.

We remark that the generated subframes of a frame $\mathcal{F}$ are precisely the ranges of bounded morphisms into $\mathcal{F}$.

Definition 3.14 [7, definition 1.5.1] Let $\mathcal{F} = (W, R, P)$ and $\mathcal{F}' = (W', R', P')$ be general frames. We say that $f : \mathcal{F} \to \mathcal{F}'$ is a *frame homomorphism* if $f : (W, R) \to (W', R')$ is a bounded morphism and $f^{-1}[S'] \in P$ for every $S' \in P'$.

Clearly, if $f : \mathcal{F} \to \mathcal{F}'$ is a bounded morphism between Kripke frames, then $f : \mathcal{F}^+ \to \mathcal{F}'^+$ is a frame homomorphism.

Definition 3.15 [7, definition 1.11.1] An *inverse family* of descriptive frames is an object

$$\mathcal{I} = \big((I, \leq), (\mathcal{F}_i : i \in I), (f_{ij} : i \geq j \text{ in } I)\big),$$

where $(I, \leq)$ is an upwards-directed partial order ('upwards-directed' means that any finite subset of I has an upper bound in I), $\mathcal{F}_i = (W_i, R_i, P_i)$ is a descriptive frame for each $i \in I$, and for each $i, j \in I$ with $i \geq j$, $f_{ij} : \mathcal{F}_i \to \mathcal{F}_j$ is a frame homomorphism such that (a) f_{ii} is the identity map on W_i, and (b) $f_{jk} \circ f_{ij} = f_{ik}$ whenever $k \leq j \leq i$ in I.

The *inverse limit* $\lim_{\leftarrow} \mathcal{I}$ of $\mathcal{I}$ is defined to be $\mathcal{F} = (W, R, P)$, where

$$\begin{aligned} W &= \{x \in \textstyle\prod_{i \in I} W_i : f_{ij}(x_i) = x_j \text{ for each } i \geq j \text{ in } I\}, \\ R &= \{(x, y) \in W : R_i(x_i, y_i) \text{ for each } i \in I\}, \\ P &= \{f_i^{-1}[S] : i \in I,\ S \in P_i\}. \end{aligned}$$

In the third line, for each $i \in I$, $f_i : W \to W_i$ is the projection given by $f_i(x) = x_i$.

The main fact we need about inverse limits is:

Fact 3.16 **[7, 1.11.2(8), 1.11.4]** *In the above notation, the inverse limit $\mathcal{F}$ of $\mathcal{I}$ is itself a descriptive frame. Moreover, for any modal formula ϕ, if ϕ is valid in $\mathcal{F}_i$ for every $i \in I$, then ϕ is valid in $\mathcal{F}$.*

3.4 Inverse limits of the squat frames

We will now apply this to our squat frames $\mathcal{G}_n^k$.

Definition 3.17 Let $k < \omega$ and $n \leq m < \omega$. We define $\pi_{mn}^k : \operatorname{dom} \mathcal{G}_m^k \to \operatorname{dom} \mathcal{G}_n^k$ as follows:

- It takes the root of $\mathcal{G}_m^k$ to the root of $\mathcal{G}_n^k$.
- $\pi_{mn}^k(x) = x \restriction (k+n)$ for each leaf $x \in L_m^k = {}^{k+m}2$.
- π_{mn}^k maps a set $s \in [L_m^k]^{\geq 2^m}$ to the set $\{\pi_{mn}^k(x) : x \in s\}$.
  (It is clear that $|\pi_{mn}^k(s)| \geq |s|/2^{m-n}$, so that indeed, $\pi_{mn}^k(s) \in [L_n^k]^{\geq 2^n}$.)

Lemma 3.18 *Let $k < \omega$ and $n \leq m \leq l < \omega$. Then $\pi_{mn}^k : \mathcal{G}_m^k \to \mathcal{G}_n^k$ is a surjective bounded morphism, π_{nn}^k is the identity on $\operatorname{dom} \mathcal{G}_n^k$, and $\pi_{ln}^k = \pi_{mn}^k \circ \pi_{lm}^k$.*

Proof. Straightforward. □

We need a little notation: if $\mathcal{F}, \mathcal{F}', \mathcal{G}, \mathcal{G}'$ are squat frames and $f : \mathcal{F} \to \mathcal{F}'$, $g : \mathcal{G} \to \mathcal{G}'$ are bounded morphisms taking roots to roots, then we define $f + g : \mathcal{F} + \mathcal{G} \to \mathcal{F}' + \mathcal{G}'$ to be the map (clearly a well defined bounded morphism) taking the root of $\mathcal{F} + \mathcal{G}$ to the root of $\mathcal{F}' + \mathcal{G}'$, and given on the remaining worlds x by

$$(f+g)(x) = \begin{cases} f(x), & \text{if } x \in \operatorname{dom} \mathcal{F}, \\ g(x), & \text{otherwise.} \end{cases}$$

Until §4, fix $k, l < \omega$. We will define two inverse families of descriptive frames made from finite squat frames (for π_{mn}^k and $-^+$ see definitions 3.17 and 3.11):

1. $\mathcal{I}^k = \big((\omega, <), ((\mathcal{G}_n^k)^+ : n < \omega), (\pi_{mn}^k : n \leq m < \omega)\big)$,
2. $\mathcal{J}^{k,l} = \big((\omega, <), ((\mathcal{G}_n^k + \mathcal{G}_1^l)^+ : n < \omega), (\pi_{mn}^k + \iota : n \leq m < \omega)\big)$, where ι is the identity map on $\operatorname{dom} \mathcal{G}_1^l$.

The general frames here are descriptive frames because they are of the form $\mathcal{F}^+$ for a finite Kripke frame $\mathcal{F}$. We are interested in the inverse limits of these families. For short, write

$$\begin{aligned} \mathcal{G}_\infty &= \big(\varprojlim(\mathcal{I}^k)\big)_+ \\ \mathcal{F}_\infty &= \big(\varprojlim(\mathcal{J}^{k,l})\big)_+ \end{aligned} \tag{6}$$

Lemma 3.19 $\mathcal{F}_\infty \cong \mathcal{G}_\infty + \mathcal{G}_1^l$.

Proof. Let r be the root of $\mathcal{F}_\infty$ and r' the root of $\mathcal{G}_\infty + \mathcal{G}^l_1$. By definition, for $\eta \in \mathcal{F}_\infty$ we have $\eta_n \in \operatorname{dom}(\mathcal{G}^k_n + \mathcal{G}^l_1)$ for each $n < \omega$. Define

$$\eta' = \begin{cases} r', & \text{if } \eta = r, \\ \eta, & \text{if } \eta \neq r \text{ and } \eta_n \in \operatorname{dom} \mathcal{G}^k_n \text{ for each } n < \omega, \\ \eta_0, & \text{if } \eta \neq r \text{ and } \eta_n \in \operatorname{dom} \mathcal{G}^l_1 \text{ for each } n < \omega. \end{cases}$$

It can be checked that $(\eta \mapsto \eta') : \mathcal{F}_\infty \to \mathcal{G}_\infty + \mathcal{G}^l_1$ is well defined and is the required isomorphism. □

$\mathcal{F}_\infty$ is the underlying Kripke frame of the inverse limit of an inverse family of descriptive frames whose underlying Kripke frames all validate $M_{\max(k,l)}$ (by lemmas 3.7 and 3.4). Now $k, l < \omega$ are arbitrary, and it could be that $k \gg l$. Nevertheless, and perhaps surprisingly, $\mathcal{F}_\infty$ need not validate M_k. Indeed, we will show that $\mathcal{F}_\infty \models M_l$ but $\mathcal{F}_\infty \not\models M_{l+1}$.

This will be proved by showing that $\mathcal{G}_\infty \not\models M_n$ for any $n \geq 1$. The proof will need some technical lemmas. The first one is almost immediate from the definition of $\mathcal{G}_\infty$:

Lemma 3.20 *$\mathcal{G}_\infty$ is a squat frame with at most 2^ω worlds.*

Proof (sketch). The maps π^k_{nm} take roots to roots, midpoints to midpoints, and leaves to leaves. So each element in $\operatorname{dom} \mathcal{G}_\infty$ is a sequence in $\prod_{n<\omega} \operatorname{dom} \mathcal{G}^k_n$ consisting entirely of roots, entirely of midpoints, or entirely of reflexive leaves. It is not so hard to see that such a sequence is a root, midpoint, or reflexive leaf of $\mathcal{G}_\infty$, respectively. (In particular, because the maps π^k_{mn} are bounded morphisms, we can inductively construct a sequence of leaves that is a successor in $\mathcal{G}_\infty$ of any given sequence in $\mathcal{G}_\infty$ consisting of midpoints. We will prove a stronger result in corollary 3.25 below.) It follows easily that $\mathcal{G}_\infty$ is squat. Since the $\mathcal{G}^k_n$ are finite, $|\operatorname{dom} \mathcal{G}_\infty| \leq \left|\prod_{n<\omega} \operatorname{dom} \mathcal{G}^k_n\right| = 2^\omega$. □

The next fact we need — that each midpoint of $\mathcal{G}_\infty$ has 2^ω successors — is a little harder to prove. Let $\mathcal{G}_\infty = (W, R)$, say. Fix an arbitrary midpoint $s = (s_n : n < \omega)$ of $\mathcal{G}_\infty$. So (i) each s_n is a midpoint of $\mathcal{G}^k_n$, and (ii) $\pi^k_{mn}(s_m) = s_n$ whenever $n \leq m$. By the definitions, this says:

(i) $s_n \subseteq {}^{k+n}2$ and $|s_n| \geq 2^n$ for each $n < \omega$,

(ii) $s_n = \{x \restriction (k+n) : x \in s_m\}$ whenever $n \leq m < \omega$.

An element $x = (x_n : n < \omega)$ of $\mathcal{G}_\infty$ is a leaf of $\mathcal{G}_\infty$ iff $x_n \in {}^{k+n}2$ for each n. In this case, $x_n = x_m \restriction (k+n)$ for each $n \leq m < \omega$, and $x \in R^s$ iff $x_n \in s_n$ for all n.

Definition 3.21 Let $n < \omega$ and $x \in s_n$.

1. For $n \leq m < \omega$, write $s^x_m = \{y \in s_m : y \restriction (k+n) = x\}$.
2. For $c < \omega$, we say that x is *c-big* if $|s^x_m| \geq 2^{m-n-c}$ for every $m \geq n$.

Since $s^x_m \subseteq {}^{k+m}2$ and $x \in {}^{k+n}2$, we see that

$$|s^x_m| \leq 2^{m-n} \text{ for any } n \leq m < \omega \text{ and } x \in s_n. \tag{7}$$

Clearly,

$$s^x_l = \bigcup \{s^y_l : y \in s^x_m\}, \text{ whenever } n \leq m \leq l < \omega \text{ and } x \in s_n. \tag{8}$$

Also note that 'c-big' gets *weaker* as c grows: any c-big element is $(c+1)$-big.

Lemma 3.22 *If $x \in s_n$ is not c-big, then for all large enough $l \geq n$ we have $|s^x_l| < 2^{l-n-c}$.*

Proof. By assumption, there is $m \geq n$ such that $|s^x_m| < 2^{m-n-c}$. Take any $l \geq m$. By (7), $|s^y_l| \leq 2^{l-m}$ for each $y \in s^x_m$. So by (8), $|s^x_l| \leq 2^{l-m} \cdot |s^x_m| < 2^{l-m} \cdot 2^{m-n-c} = 2^{l-n-c}$. □

Corollary 3.23 *There is some k-big $x \in s_0$.*

Proof. If not, then since s_0 is finite, by the preceding lemma we may choose large enough $n < \omega$ such that $|s^x_n| < 2^{n-k}$ for every $x \in s_0$. Now $s_0 \subseteq {}^k2$, so $|s_0| \leq 2^k$. By (ii) above, $s_n = \bigcup\{s^x_n : x \in s_0\}$. Hence, $|s_n| < 2^{n-k} \cdot |s_0| \leq 2^n$, contradicting (i) above. □

Proposition 3.24 *For any $n, c < \omega$ and any c-big $x \in s_n$, there is some $m > n$ such that s^x_m contains at least two c-big elements.*

Proof. By induction on c. If $c = 0$, then $|s^x_m| \geq 2^{m-n}$ for all $m \geq n$. But (cf. (7)) we have $s^x_m \subseteq \{y \in {}^{k+m}2 : y\restriction(k+n) = x\}$, and the right-hand size has cardinality 2^{m-n}. So in fact,

$$s^x_m = \{y \in {}^{k+m}2 : y\restriction(k+n) = x\} \quad \text{for all } m \geq n. \tag{9}$$

Now, for $l \geq m \geq n$ and any $y \in s^x_m$, we have

$$\begin{aligned} s^y_l &= \{z \in s_l : z\restriction(k+m) = y\} && \text{by definition of } s^y_l, \\ &= \{z \in s^x_l : z\restriction(k+m) = y\} && \text{since } y\restriction(k+n) = x, \\ &= \{z \in {}^{k+l}2 : z\restriction(k+m) = y\} && \text{by (9).} \end{aligned}$$

So $|s^y_l| = 2^{l-m}$, and hence every element of every s^x_m $(m \geq n)$ is 0-big.

Suppose that $1 \leq c < \omega$, and inductively assume the proposition for smaller c. Let $x \in s_n$ be c-big. There are two cases.

Case 1: some element of $\bigcup_{m \geq n} s^x_m$ is $(c-1)$-big. Suppose that $m \geq n$ and $y \in s^x_m$ is $(c-1)$-big. By the inductive hypothesis, there is $l > m$ such that s^y_l (and hence s^x_l) contains at least two $(c-1)$-big (and hence c-big) elements, as required.

Case 2: otherwise. So x itself is not $(c-1)$-big, and by lemma 3.22 we may take $m \geq c+n$ such that $|s^x_m| < 2^{m-n-(c-1)}$. We show that there are at least two c-big elements of s^x_m.

Assume for contradiction that s^x_m has at most one c-big element. By (ii) above, $s^x_m \neq \emptyset$. So we may take $y \in s^x_m$ such that $s^x_m \setminus \{y\}$ contains no c-big elements. By the case assumption, y is not $(c-1)$-big. So using lemma 3.22 repeatedly, there is large enough $l > m$ such that

$$\begin{aligned} &|s^y_l| \leq 2^{l-m-c+1} - 1, \\ &|s^z_l| \leq 2^{l-m-c} - 1 \qquad \text{for all } z \in s^x_m \setminus \{y\}. \end{aligned}$$

Now by (8), $s^x_l = s^y_l \cup \bigcup\{s^z_l : z \in s^x_m \setminus \{y\}\}$. We know that $|s^x_m| < 2^{m-n-(c-1)}$, so

$$|s^x_m \setminus \{y\}| \leq 2^{m-n-c+1} - 2.$$

Because x is c-big, we have $|s^x_l| \geq 2^{l-n-c}$. We conclude that

$$\begin{aligned} 2^{l-n-c} \;\leq\; |s^x_l| \;\leq\; & |s^y_l| + \sum_{z \in s^x_m \setminus \{y\}} |s^z_l| \\ \leq\; & 2^{l-m-c+1} - 1 + (2^{m-n-c+1} - 2)(2^{l-m-c} - 1) \\ =\; & 2^{l-m-c+1} - 1 + 2^{l-n-2c+1} - 2^{l-m-c+1} - 2^{m-n-c+1} + 2 \\ =\; & 2^{l-n-2c+1} - 2^{m-n-c+1} + 1 \\ <\; & 2^{l-n-2c+1} \quad (\text{since } m \geq c+n, \text{ so } m-n-c+1 \geq 1). \end{aligned}$$

Hence $l-n-c < l-n-2c+1$, and so $c < 1$, contradicting our assumption. So again, s^x_m has at least two c-big elements, as required.

This completes the induction and the proof. □

Corollary 3.25 $|R^s| \geq 2^\omega$.

Proof. For each $\sigma \in {}^n2$ (each $n < \omega$), we will choose k-big $\widehat{\sigma} \in s_m$ for some $m \geq n$ by induction on n, such that $\widehat{\sigma\hat{\ }0}, \widehat{\sigma\hat{\ }1}$ are distinct elements of $s^{\widehat{\sigma}}_l$ for some $l > m$. Here, we write $\sigma\hat{\ }i$ for the map $\tau \in {}^{n+1}2$ given by $\tau{\restriction}n = \sigma$ and $\tau(n) = i$ (for $i = 0, 1$).

We have ${}^02 = \{\emptyset\}$. Let $\widehat{\emptyset}$ be any k-big element of s_0; by corollary 3.23, such an element exists. Inductively, if k-big $\widehat{\sigma} \in s_m$ has been chosen, by proposition 3.24 we can choose $l > m$ and distinct k-big $\widehat{\sigma\hat{\ }0}, \widehat{\sigma\hat{\ }1} \in s^{\widehat{\sigma}}_l$.

Now, for each $\eta \in {}^\omega 2$, $\{\widehat{\eta{\restriction}n} : n < \omega\}$ generates a leaf $\lambda(\eta) = \big((\widehat{\eta{\restriction}n}){\restriction}(k+n) : n < \omega\big) \in R^s$, and the $\lambda(\eta)$ for distinct η are pairwise distinct. So $\lambda : {}^\omega 2 \to R^s$ is one-one, and hence $|R^s| \geq 2^\omega$. □

The corollary holds for any midpoint s of $\mathcal{G}_\infty$. This completes our analysis of the structure of $\mathcal{G}_\infty$. The underlying 'combinatorial principle' we used is that for any $k < \omega$, any subtree of the infinite binary tree whose nth level has at least 2^{n-k} nodes, for each n, has 2^ω branches.

It now follows that:

Proposition 3.26 *$\mathcal{G}_\infty \not\models M_n$ for every $n \geq 1$.*

Proof. Let r be the root of $\mathcal{G}_\infty$. Then R^r is the set of midpoints of $\mathcal{G}_\infty$. By lemma 3.20 and corollary 3.25, for any $s \in R^r$ we have $|R^r| + \omega \leq 2^\omega = |R^s|$. The result follows by theorem 2.1. □

We can now prove what we wanted.

Corollary 3.27 *$\mathcal{F}_\infty \models M_l$ but $\mathcal{F}_\infty \not\models M_{l+1}$.*

Proof. Recall from lemma 3.19 that $\mathcal{F}_\infty \cong \mathcal{G}_\infty + \mathcal{G}^l_1$. By lemma 3.7, $\mathcal{G}^l_1 \models M_l$, so by lemma 3.4, $\mathcal{F}_\infty \models M_l$ as well. By proposition 3.26, $\mathcal{G}_\infty \not\models M_{l+1}$; and by lemma 3.8, $\mathcal{G}^l_1 \not\models M_{l+1}$. So by lemma 3.4, $\mathcal{F}_\infty \not\models M_{l+1}$. □

We summarise our conclusions in the following

Theorem 3.28 *Let $k, l < \omega$.*

1. *There is an inverse family $\mathcal{J}^{k,l}$ of finite descriptive frames validating $M_{\max(k,l)}$, such that if $\mathcal{F}_\infty = (\lim_{\leftarrow} \mathcal{J}^{k,l})_+$ is the underlying Kripke frame of the inverse limit of $\mathcal{J}^{k,l}$, then $\mathcal{F}_\infty \models M_l$ but $\mathcal{F}_\infty \not\models M_{l+1}$.*

2. *There is a descriptive frame $\mathcal{D}^{k,l} = (W, R, P)$ with $|P| = \omega$, such that $\mathcal{D}^{k,l} \models M_{\max(k,l)}$, $\mathcal{D}^{k,l}_+ \models M_l$, and $\mathcal{D}^{k,l}_+ \not\models M_{l+1}$.*

Proof. The first part has already been established. For the second part, take $\mathcal{D}^{k,l} = \lim_{\leftarrow} \mathcal{J}^{k,l} = (W, R, P)$, say. It is clear from definition 3.15 that P is countably infinite. By fact 3.16, $\mathcal{D}^{k,l} \models M_{\max(k,l)}$. The rest is as in the first part. □

4 Canonical axioms and KM^∞

A modal formula ϕ is said to be *canonical* if it is valid in the canonical frame of the normal modal logic axiomatised by ϕ. The following is more convenient here, and is well known to be equivalent to this:

Definition 4.1 A modal formula ϕ is said to be *countably d-persistent* if whenever it is valid in a descriptive frame $\mathcal{F} = (W, R, P)$ with P countable, it is also valid in its underlying Kripke frame $\mathcal{F}_+$.

Lemma 4.2 *Any canonical formula is countably d-persistent, and conversely.*

Proof (sketch). We only sketch the proof, because it is well known (see, e.g., [15, p. 221]). We assume familiarity with canonical models; see [1, 2] or any modal logic text for details. Write L for the set of all modal formulas written using only propositional variables from our countable set V. Let $\mathcal{F} = (W, R, P)$ be a descriptive frame such that P is countable, and suppose that $\phi \in L$ is canonical and valid in $\mathcal{F}$. Let Λ be the modal logic axiomatised by ϕ, and let $\mathcal{M} = (W^*, R^*, h^*)$ be its canonical model — so W^* is the set of all maximal Λ-consistent subsets of L.

Since P is countable, we may choose a surjective assignment $h : V \to P$. For $w \in W$, put $\Gamma_w = \{\psi \in L : (\mathcal{F}, h), w \models \psi\}$. Since ϕ is valid in $\mathcal{F}$, Λ is also valid in $\mathcal{F}$, and it follows that each Γ_w is maximal Λ-consistent (i.e., in W^*). Using that $\mathcal{F}$ is a descriptive frame and that h is surjective, it can be checked that the map $f : W \to W^*$ given by $f(w) = \Gamma_w$ is a one-one bounded morphism. Since ϕ is assumed canonical, it is valid in (W^*, R^*), and so also in its generated subframe based on $\operatorname{rng} f$. But f is an isomorphism from (W, R) onto this. So ϕ is valid in (W, R), as required.

Conversely, if ϕ is countably d-persistent then of course it is canonical, because the canonical model of the logic axiomatised by ϕ can be viewed as a descriptive frame with countable 'P'-part (namely, the truth sets of formulas in L), and ϕ is valid in it. □

We can now prove our second main result. The case $k = 1$ was proved in [6].

Theorem 4.3 *For no $k \geq 1$ is M_k canonical.*

Proof. Let $k \geq 1$; we prove that M_k is not countably d-persistent. For each n, $\mathcal{G}^k_n \models M_k$ by lemma 3.7. By fact 3.16, $\lim_{\leftarrow} \mathcal{I}^k \models M_k$ as well. By definition, the 'P'-part of $\lim_{\leftarrow} \mathcal{I}^k$ is countable. But by proposition 3.26, $(\lim_{\leftarrow} \mathcal{I}^k)_+ = \mathcal{G}_\infty \not\models M_k$. □

It follows that no M_k ($k \geq 1$) is d-persistent (this stronger notion is defined as in definition 4.1 but without the cardinality restriction).

To prove our third result, we want to use first-order compactness. To do this, we view a general frame (W, R, P) as a first-order structure whose domain is the disjoint union of W and P, with unary relations picking out W and P, and binary relations $R \subseteq W \times W$ and $\in \subseteq W \times P$ interpreted in the natural way. It is easy to write down a finite set Δ of first-order sentences expressing that a structure (W, R, P) for this signature is a general frame.

As is well known (see, e.g., [1, definition 2.45]), every modal formula ϕ has a *standard translation* to a formula $ST_x(\phi)$ of first-order logic, with a free variable x. We modify this here by regarding propositional variables as first-order variables. For a propositional variable p, we define $ST_x(p)$ to be $x \in p$. We put $ST_x(\top) = \top$, etc., $ST_x(\phi \wedge \psi) = ST_x(\phi) \wedge ST_x(\psi)$ and similarly for negation, and $ST_x(\Box\phi) = \forall y(R(x,y) \to ST_y(\phi))$ and $ST_x(\Diamond\phi) = \exists y(R(x,y) \wedge ST_y(\phi))$. Here, y is a new variable. For a formula $\phi(p_1, \ldots, p_n)$, we write $ST(\phi)$ for the universal closure $\forall x \in W\, \forall p_1 \ldots p_n \in P\, ST_x(\phi)$. For a set X of formulas, we write $ST(X)$ for $\{ST(\phi) : \phi \in X\}$. Clearly, a modal formula ϕ is valid in a general frame $\mathcal{G}$ iff $ST(\phi)$ is true in it in first-order semantics:

$$\mathcal{G} \models \phi \iff \mathcal{G} \models ST(\phi). \tag{10}$$

Hence (cf. (5)), ϕ is valid in a Kripke frame $\mathcal{F}$ iff $ST(\phi)$ is true in $\mathcal{F}^+$ in first-order semantics:

$$\mathcal{F} \models \phi \iff \mathcal{F}^+ \models ST(\phi). \tag{11}$$

With these preliminaries in hand, we can prove our third theorem.

Theorem 4.4 *Any axiomatisation of the logic KM^∞ has infinitely many non-canonical axioms.*

Proof. Suppose on the contrary that (without loss of generality) KM^∞ is axiomatised by a single axiom B together with a set Σ of canonical formulas. Since $\Sigma \cup \{B\}$ and $\{M_k : k < \omega\}$ axiomatise the same logic, the two first-order theories

$$\Delta \cup ST(\Sigma \cup \{B\}),$$
$$\Delta \cup \{ST(M_k) : k < \omega\}$$

have the same models. (Here, Δ is as above.) Therefore, bearing in mind that for $m > n$, $M_m \vdash M_n$ and hence $\Delta \cup ST(M_m) \models ST(M_n)$, first-order compactness yields:

(a) there is $l < \omega$ such that $\Delta \cup ST(M_l) \models ST(B)$,

(b) there is a finite subset $X \subseteq \Sigma$ such that $\Delta \cup ST(X \cup \{B\}) \models ST(M_{l+1})$,

(c) there is finite k such that $\Delta \cup ST(M_k) \models ST(X)$. (Necessarily, $k > l$.)

Let $\mathcal{D} = \mathcal{D}^{k,l}$ be the descriptive frame of theorem 3.28(2). The 'P'-part of $\mathcal{D}$ is countable, $\mathcal{D} \models M_{\max(k,l)}$, $\mathcal{D}_+ \models M_l$, and $\mathcal{D}_+ \not\models M_{l+1}$.

We have $\mathcal{D} \models M_k$. Plainly, $\mathcal{D} \models \Delta$. Now, by (c) and (10), we obtain $\mathcal{D} \models X$. The formulas in X are assumed canonical, so by lemma 4.2, $\mathcal{D}_+ \models X$ as well. By (11), $(\mathcal{D}_+)^+ \models ST(X)$.

As $\mathcal{D}_+ \models M_l$, (11) gives $(\mathcal{D}_+)^+ \models ST(M_l)$. Clearly, $(\mathcal{D}_+)^+ \models \Delta$. So by (a), $(\mathcal{D}_+)^+ \models ST(B)$.

Now we have $(\mathcal{D}_+)^+ \models \Delta \cup ST(X \cup \{B\})$, so by (b) and (11), we arrive at $\mathcal{D}_+ \models M_{l+1}$, a contradiction. $\Box$

Of course, it follows that KM^∞ is not finitely axiomatisable.

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