Fine's canonicity theorem and its converse

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Thanks to the organisers for inviting me.
Congratulations and best wishes to Wilfrid on his retirement.

aim of talk

to show how the beautiful finite graphs constructed by Erdős (1959) can be used to study canonicity of modal logics

part 1: crash course in modal logic

- syntax and Kripke semantics
- canonical modal logics
- Fine's theorem: *the modal logic of any elementary class of Kripke frames is canonical.*

part 2: converse of Fine's theorem?

- Erdős graphs and applications

part 1: crash course in modal logic

- Propositional logic with an extra unary connective $\Box$.
- Formulas: $p \mid \varphi \wedge \psi \mid \neg\varphi \mid \Box\varphi$ (p taken from countable set)
- Intuitive meaning of $\Box\varphi$:
  φ is known/believed/provable/will always be true/...
- Prior to $\sim$1960, modal logics mainly given axiomatically:
  1. all propositional tautologies
  2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ ('normality')
  3. possibly more axioms: e.g., $\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$, etc
  4. rules: modus ponens, substitution, and $\varphi/\Box\varphi$.
- A *modal logic* is *by definition* a set of formulas containing 1, 2, and closed under 4. E.g., the theorems of the system.

Kripke semantics $\sim$1960

Idea: $\Box\varphi$ means *φ is true in all possible worlds.*

Which worlds are possible depends on the current world.

- *Kripke frame* (W, R), where $R \subseteq W \times W$ is the *'accessibility relation'* for $\Box$.
- $h : \{\text{atoms}\} \to \wp(W)$ *truth assignment. Kripke model* (W, R, h).
- $(W, R, h), w \models p$ iff $w \in h(p)$
- $(W, R, h), w \models \Box\varphi$ iff $(W, R, h), v \models \varphi$ for all $v \in W$ with wRv.

Kripke semantics gave intuitive meaning to many existing modal logics.

It has been of lasting significance.

Kripke completeness

φ is *valid* in a Kripke frame (W, R) iff $(W, R, h), w \models \varphi$ for all h, w.

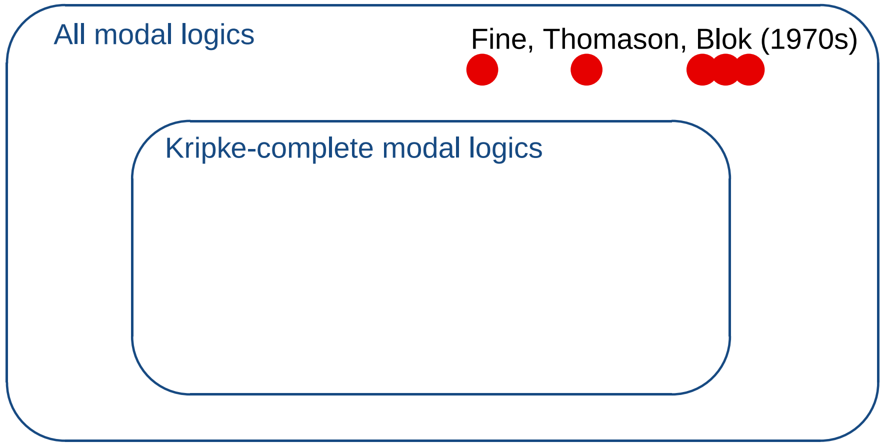
Any class $\mathcal{C}$ of Kripke frames determines a modal logic, 'the logic of $\mathcal{C}$': namely, the modal formulas valid in all frames in $\mathcal{C}$.

Such a modal logic is called *(Kripke) complete.*

- The basic modal logic is the logic of (the class of) all frames.
- With $\Box\varphi \to \varphi$, we get the logic of all *reflexive* frames.
- With $\Box\varphi \to \Box\Box\varphi$, we get the logic of all *transitive* frames.
- And so on.

Are *all* modal logics complete? How were these results proved?

Sadly, not all modal logics are complete



So when *is* a modal logic complete?

canonical frame/model — Lemmon–Scott, ~1966.

Also (independently) Cresswell, Makinson; Jónsson–Tarski (1951).

A widely used technique to show completeness of a modal logic.

Build the so-called *'canonical'* Kripke frame $\mathcal{F} = (W, R)$:

- W consists of all *maximal consistent sets* of the logic.
- $\Gamma R \Delta$ iff $\Box\varphi \in \Gamma \Rightarrow \varphi \in \Delta$ (for all φ).

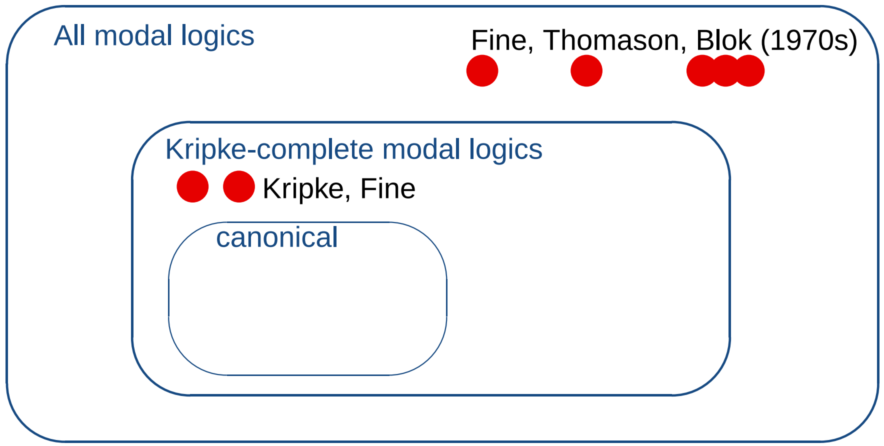
Can add the *canonical assignment:* $h(p) = \{\Gamma : p \in \Gamma\}$.
This assignment invalidates all non-theorems of the logic.

So *if all theorems of the logic are valid in $\mathcal{F}$,* we have a completeness theorem: *the canonical frame $\mathcal{F}$ determines the logic all by itself!*

A logic is said to be *canonical* if its theorems are valid in its canonical frame. All canonical logics are complete: we get a free completeness theorem.

complete $\Rightarrow$ canonical?

Sadly, not all complete modal logics are canonical.
Examples found by Kripke (1967), Fine (1974), etc.



So is canonicity useful at all?

Yes! Many logics are canonical!

- the basic modal logic is valid in all frames.
- $\Box\varphi \to \varphi$ is valid in reflexive frames, and the canonical frame turns out to be reflexive.
- $\Box\varphi \to \Box\Box\varphi$ is valid in transitive frames, and the canonical frame turns out to be transitive.

Many natural modal logics Λ are like this.
For some elementary class $\mathcal{C}$ of Kripke frames:

1. all axioms (and hence all theorems) of Λ are valid in $\mathcal{C}$
2. the canonical frame of Λ is in $\mathcal{C}$.

So Λ is the logic of $\mathcal{C}$.

canonicity and elementarity — Fine's theorem

By 1975 (and later),

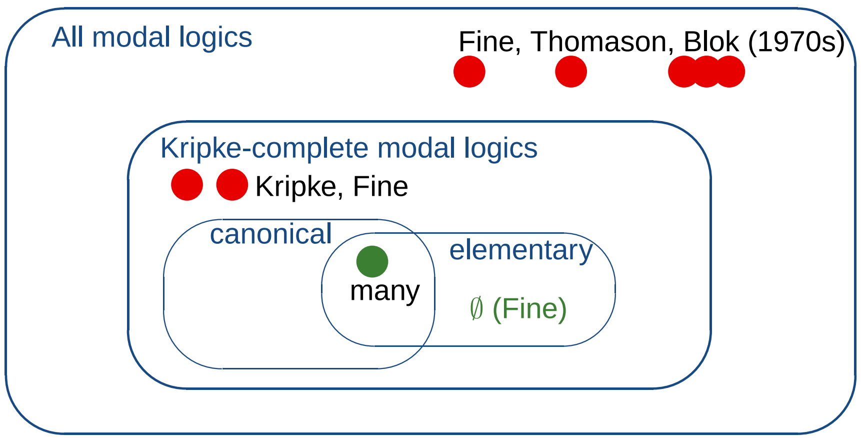
- every known canonical logic was the logic of some elementary class of frames.
- every known non-canonical logic was not the logic of any elementary class of frames.

What would you guess?

And in 1975 Fine proved that *the logic of any elementary class of frames is canonical.*

Seminal result relating two conceptually different notions.
Proofs use saturation, algebraic methods, 'forcing'.
Developed by van Benthem (1979), Goldblatt (1989–), Gehrke–Harding–Venema (2006).

story so far



We call a modal logic *'elementary'* if it is the set of formulas valid in some elementary class of frames.

part 2: converse of Fine's theorem?

Is every canonical modal logic elementary?

We can use graphs constructed by Erdős to solve this problem.

graphs

Here, a *graph* is a structure $G = (V, E)$ where $V \neq \emptyset$ ('vertices', 'nodes') and $E \subseteq V \times V$ ('edges').
A graph is the same thing as a Kripke frame.

Undirected loop-free graphs (E symmetric and irreflexive) will be called *standard.*

cycles

For $k \geq 3$, a *cycle of length k* in G is (here) a sequence $v_1, \ldots, v_k \in V$ of distinct nodes with $v_1 E v_2, v_2 E v_3, \ldots, v_{k-1} E v_k, v_k E v_1$.

chromatic number

Let $G = (V, E)$ be a graph.

A subset $X \subseteq V$ is *independent* if $E \cap (X \times X) = \emptyset$.

For $k < \omega$, a *k-colouring* of G is a partition of V into k independent sets.
The *chromatic number* $\chi(G)$ of G is the least $k < \omega$ such that G has a k-colouring, and ∞ if there is no such k.

facts

- If G has a reflexive node (xEx), then $\chi(G) = \infty$.
- A standard graph has a 2-colouring iff it has no cycles of odd length.

Erdős graphs

In spite of above, chromatic number is a 'global' property of graphs:

Theorem 1 (Erdős, 1959) *For every $n < \omega$, there is a finite standard graph G_n with*

- $\chi(G_n) > n$
- *G_n has no cycles of length $\leq n$.*

So all subgraphs of G_n of size $\leq n$ are 2-colourable.

Proof was radical: probabilistic. Lovász (1968): explicit construction.

We will fix Erdős graphs G_n $(n < \omega)$ with wlog. $|G_0| < |G_1| < \cdots$

Very suggestive to logicians... take G to be a 'limit' (by compactness or ultraproducts) of the G_n. What would $\chi(G)$ be?

chromatic number via modal logic

Blackburn–de Rijke–Venema 2001: Modal languages are simple yet expressive languages for talking about relational structures (graphs).

Can we talk about chromatic number with modal logic?

Yes (Hughes, 1990).

We will need global expressivity. So we add a *universal modality* A (a second box). Put $E = \neg A \neg$.

Frames have the form $F = (W, R_\Box, R_A)$, and usually, $R_A = W \times W$.

Semantics (usually): $F, h, w \models A\varphi$ iff $F, h, v \models \varphi$ for all worlds v of F.

Extra axioms: $A\varphi \to \Box\varphi$, $A\varphi \to \varphi$, $E\varphi \to AE\varphi$.

axiomatising chromatic number

We can regard a graph $G = (W, R)$ as a frame $(W, R, W \times W)$, which we also denote by G.

Conversely: any frame $(W, R_\Box, R_A)$ has a graph part, $(W, R_\Box)$.

The *chromatic number* $\chi(F)$ of a frame $F = (W, R_\Box, R_A)$ is defined to be $\chi(W, R_\Box)$.

For $m < \omega$ let $$\eta_m = E \bigwedge_{i<m} (\Box q_i \to q_i).$$

Lemma 2 *η_m is valid in a frame F iff $\chi(F) > m$.*

So we can construct modal logics Λ whose frames have large chromatic number, by including many η_m as axioms.

faint idea?

The Erdős graphs G_n have

- large chromatic number $(> n)$
- a 'limit' G with chromatic number 2.

Try to find a modal logic Λ whose frames have large chromatic number. Hopes:

1. the G_n will validate Λ
2. their limit G won't
3. 'so' Λ is not elementary
4. Λ will be canonical(!)

— a tall order!

some attempts

A. Just let Λ include η_2.
A frame validates Λ iff it has chromatic number > 2.

So G_n validates Λ (all $n \geq 2$).
But G doesn't.

But Λ isn't canonical. Can be shown using generalised Erdős graphs.

B. Add all η_m $(m < \omega)$ to Λ.
Now a frame validates Λ iff it has chromatic number ∞.

Λ is now canonical (Hughes 1990. Idea: canonical frame has a reflexive point x — with $R_\Box(x, x)$. So its chromatic number is ∞.)

But the G_n do not validate Λ.

trick

Try to find Λ such that *the chromatic number of frames validating Λ increases with the size of the frame.*

- infinite frames validating Λ should have chromatic number ∞
- large finite frames validating Λ should have large chromatic number
- canonicity provable by Hughes's argument
- The G_n will validate Λ if we set Λ up right
- G (limit) won't: it's infinite but $\chi(G) = 2$.

details

For $n < \omega$ let
$$\sigma_n = \bigwedge_{i<n} E\big(p_i \wedge \bigwedge_{j<i} \neg p_j\big).$$

For a frame $F = (W, R_\Box, R_A)$, write $|F|$ for $|W|$.

Lemma 3 *σ_n is satisfiable over a frame F iff $|F| \geq n$.*

Corollary 4 *For $n, m < \omega$, the formula $\sigma_n \to \eta_m$ is valid in a frame F iff $(|F| \geq n \Rightarrow \chi(F) > m)$.*

Definition 5 Let $\mathcal{E}$ be the logic axiomatised by
$$\{\sigma_{|G_m|} \to \eta_m \quad : \quad m < \omega\},$$
where G_m is the mth Erdős graph.

A frame F validates $\mathcal{E}$ if *for all m, if $|F| \geq |G_m|$ then $\chi(F) > m$.*
So every G_n validates $\mathcal{E}$.

canonicity

Lemma 6 *$\mathcal{E}$ is canonical.*

Proof. In the canonical frame $\mathcal{F}$ for $\mathcal{E}$, R_A is an equivalence relation. (This uses the extra axioms $A\varphi \to \Box\varphi$, $A\varphi \to \varphi$, $E\varphi \to AE\varphi$.)

$\mathcal{F}$ breaks up into subframes based on the equivalence classes of R_A.

Let F be any such subframe. Fact: *it's enough to show F validates $\mathcal{E}$.*

1. If F is finite, it *will* validate $\mathcal{E}$ on general modal principles.
2. If F is infinite, Hughes's argument shows that it contains a *reflexive point.*

   So $\chi(F) = \infty$.

   So F validates η_m, so it certainly validates $\sigma_n \to \eta_m$, for any n, m.

■

non-elementarity

Lemma 7 *$\mathcal{E}$ is not elementary.*

Proof. Assume for contradiction that $\mathcal{E}$ is the logic of some elementary class $\mathcal{C}$ of frames.

Fact (van Benthem, Goldblatt): can assume wlog. that every *finite* frame of the form $(W, R_\Box, W \times W)$ validating $\mathcal{E}$ is in $\mathcal{C}$.

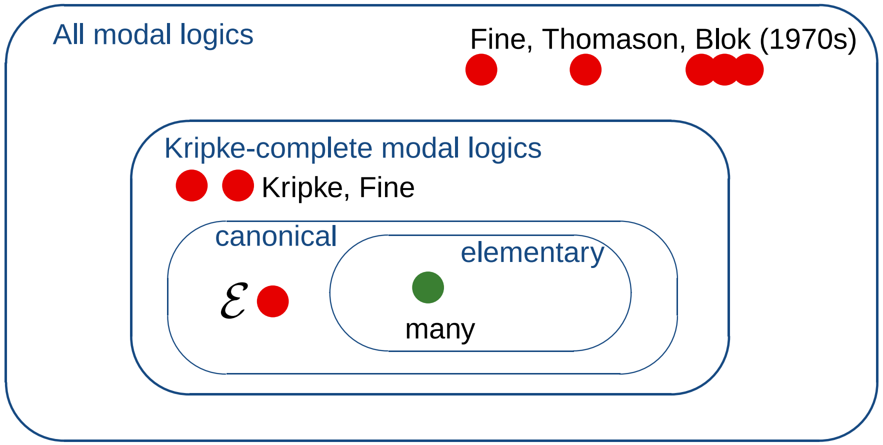
Regarded as a frame, the nth Erdős graph G_n is a finite frame validating $\mathcal{E}$.
So $G_n \in \mathcal{C}$ (for all n).

Now consider G, the 'limit' of the G_n.

- $G \in \mathcal{C}$, as G is a limit of frames in $\mathcal{C}$, and $\mathcal{C}$ is elementary,
- $G \notin \mathcal{C}$, as G does not validate $\mathcal{E}$ (because G is infinite but $\chi(G) = 2$).

Contradiction. ■

summary



Can eliminate use of A in $\mathcal{E}$ by a coding argument of Thomason.

conclusion

- The converse of Fine's theorem is false in general: not every canonical modal logic is the logic of an elementary class of frames.
- Can refine the example in various ways.
- A later example (with transitive $R_\Box$) was constructed directly, avoiding Erdős graphs and the A box. But probabilistic constructions still seem needed elsewhere in canonicity.

problems

1. Characterise canonical and elementary logics in some independent way, to clarify differences between them.
2. (van Benthem): Elementary = canonical + ?
   Canonical = elementary – ??

some references (some at `www.doc.ic.ac.uk/~imh`)

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