Axiomatising modal logics of elementary classes of Kripke frames

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Thanks to Hilary for inviting me

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introduction

In this talk, *elementary modal logic* means the modal logic of an *elementary* class of Kripke frames (defined by a first-order theory).

Question: how to axiomatise?

- partial results
- general method
    1. convert first-order axioms to *positive bounded form*
    2. convert positive bounded sentences to *hybrid logic*
    3. compute *modal approximants* of hybrid formulas
- remarks

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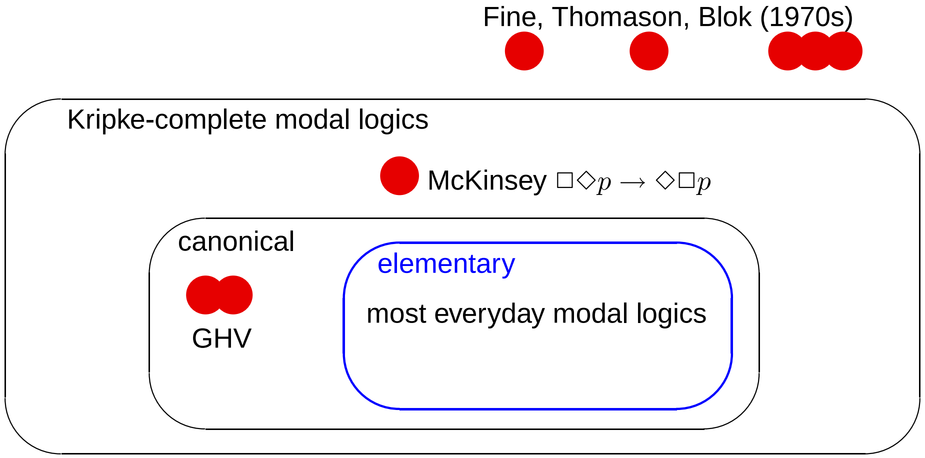
the world of modal logics



Fine, Thomason, Blok (1970s)

Kripke-complete modal logics

McKinsey $\Box\Diamond p \to \Diamond\Box p$

canonical

GHV

elementary

most everyday modal logics

elementary ⇒ canonical (Fine, 1970s)

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elementary modal logics

They are numerous. Most everyday modal logics are elementary.

Two live problems:

1. When does a given set of modal axioms axiomatise an elementary modal logic?
2. Given (the defining theory of) an elementary class of frames, how can we find axioms for its modal logic?

We will start by looking at four attacks on problem 1.

elementary modal logics 1: Sahlqvist-axiomatisable logics

Definition 1 (Sahlqvist formula)

- any positive modal formula (built from atoms, $\top, \bot$ using $\wedge, \vee, \Box, \Diamond$) is a Sahlqvist formula
- Any negated boxed atom $\neg\Box^n p$ is a Sahlqvist formula
- If φ, ψ are Sahlqvist formulas then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\Box\varphi$.

Examples

Many common logics are Sahlqvist-axiomatisable.
$\Box p \to p$ — equivalent to $[\neg\Box p] \vee [p]$
$\Diamond\Box p \to \Box\Diamond p$ — equivalent to $\Box[\neg\Box p] \vee [\Box\Diamond p]$

facts

- Every Sahlqvist formula σ has a *local first-order correspondent* $\chi(x)$ (say), in the frame language. χ can be computed from σ.

  For any Kripke frame $\mathcal{F} = (W, R)$ and $w \in W$:
  σ is valid in $\mathcal{F}$ at w iff $\mathcal{F} \models \chi(w)$.

  E.g., local correspondent of $\Box p \to p$ is $R(x, x)$.

  So σ is valid in $\mathcal{F}$ iff $\mathcal{F} \models \forall x \chi(x)$ — *global correspondent* of σ.
- Any Sahlqvist formula axiomatises the modal logic of the class of frames satisfying its global correspondent.
  So *all Sahlqvist-axiomatisable modal logics are elementary.*
- Generalised by (e.g.,) Goranko, Vakarelov, Kikot.

McKinsey–Lemmon logic

This is an interesting logic from the Lemmon notes (1966):

$$KM^\infty : \qquad \Diamond\big((\Box p_1 \vee \Box\neg p_1) \wedge \ldots \wedge (\Box p_k \vee \Box\neg p_k)\big) \qquad (\text{all } k \geq 1).$$

The axiom for $k = 1$ is the McKinsey formula.

Facts (Goldblatt–IH 2006):

- class of all frames for KM^∞ is non-elementary (shown independently by Balbiani–Shapirovsky–Shehtman 2006)
- any axiomatisation has infinitely many non-canonical formulas

So KM^∞ *is not Sahlqvist-axiomatisable.*

Still, KM^∞ *is elementary:* Lemmon showed it is the logic of the class of frames satisfying

$$\forall x \exists y (R(x, y) \wedge \forall zt(R(y, z) \wedge R(y, t) \to z = t))$$

elementary modal logics 2: Balbiani–Shapirovsky–Shehtman

BSS showed how to generalise Sahlqvist's approach to cover KM^∞.

Theorem 2 (BSS, AiML-06) *Let $\sigma(p_1, \ldots, p_k)$ be a Sahlqvist formula with local correspondent $\chi(x)$. Then the logic axiomatised by*

$$\{\Diamond\big(\sigma(p_1^1, \ldots, p_k^1) \wedge \ldots \wedge \sigma(p_1^n, \ldots, p_k^n)\big) \quad : \quad n \geq 1\}$$

where the p_i^j are distinct atoms, axiomatises the logic of the class of frames satisfying $\forall x \exists y (R(x, y) \wedge \chi(y))$.

So all such logics are elementary.

BSS and KM^∞

The Balbiani–Shapirovsky–Shehtman theorem covers KM^∞ nicely:

- by definition, KM^∞ is axiomatised by $\{\Diamond((\Diamond p_1 \to \Box p_1) \wedge \ldots \wedge (\Diamond p_n \to \Box p_n)) : n \geq 1\}$
- $\Diamond p \to \Box p$ is Sahlqvist with local correspondent

$$\chi(y) = \forall z \forall t (R(y,z) \wedge R(y,t) \to z = t)$$

- so by BSS, KM^∞ is the logic of the class of frames satisfying $\forall x \exists y (R(x,y) \wedge \chi(y))$ — as was already known (Lemmon 1966).

Similar logics (e.g., one of Hughes 1990) are also covered.

The full scope of this approach is not clear.

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elementary modal logics 3 and 4: Venema

3: Sahlqvist by expansions:

KM^∞ is finitely axiomatisable in an expanded modal signature, by:

$$\delta \to (\Diamond p \to \Box p), \quad \Diamond\delta.$$

δ is a new 0-ary modality.

The full scope of this approach is not clear.

4: Additivity axioms:

$\pi(p \vee q) \to \pi(p) \vee \pi(q)$ for any positive modal formula $\pi(x)$.

This always axiomatises an elementary modal logic. (Venema showed how to obtain the first-order definition.)

Sometimes it is Sahlqvist-axiomatisable (eg $\pi(x) = \Box\Diamond x$), and sometimes not (eg $\pi(x) = \Diamond\Box x$ — follows from [Fine 1975]).

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taking stock

problem 1 When does a given set of modal axioms axiomatise an elementary modal logic?

problem 2 Given (the defining theory of) an elementary class of frames, how can we find axioms for its modal logic?

All four approaches aim to solve problem 1. But Venema said his original motivation for the additivity approach was to 'axiomatise [the modal logic of] an elementary frame class'. *This is problem 2!*

To attack problem 2, we might try to:

1. Take a potshot at a nice general form of modal axiom. Show these axioms axiomatise all and only the elementary modal logics. (This is problem 1 taken to the extreme!)
2. Start with the defining theory of an elementary class of frames. Try to obtain modal axioms from it.

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general method of axiomatising any elementary logic

We follow the second route. Price: the axioms are not always nice.

Start with a class $\mathcal{C}$ of Kripke frames defined by a first-order theory T.

Step 1: replace T by $U = \{\varepsilon : \varepsilon$ a *pseudo-equation,* $T \vdash \varepsilon\}$.

Step 2: convert sentences in U to a set Φ of equivalent *quasipositive hybrid sentences.*

Step 3: from each $\varphi \in \Phi$, synthesise an infinite set φ^* of *modal* axioms by *approximating* the nominals in φ by modal formulas.

Then $\bigcup\{\varphi^* : \varphi \in \Phi\}$ axiomatises the modal logic of $\mathcal{C}$.

step 1: replace T by pseudo-equations

Positive bounded formulas:

$$\begin{aligned}\Pi := \quad & R(x,y) \mid x = y \mid \top \mid \bot \mid \Pi \wedge \Pi \mid \Pi \vee \Pi \mid \\ & \forall y(R(x,y) \to \Pi) \mid \exists y(R(x,y) \wedge \Pi) \qquad \text{where } x \neq y.\end{aligned}$$

Pseudo-equations: $\forall x\, \pi(x)$ where $\pi(x)$ is positive bounded.

Recall $\mathcal{C}$ defined by T. Put $U = \{\varepsilon : \varepsilon \text{ a pseudo-equation}, T \vdash \varepsilon\}$.

Theorem 3 (Goldblatt 1995) *Mod*(U) *is the closure* $\overline{\mathcal{C}}$ *of* $\mathcal{C}$ *under disjoint unions, bounded morphic images, generated subframes, and ultraroots.*

$\overline{\mathcal{C}}$ has the same modal logic as $\mathcal{C}$.

So we can and do replace $\mathcal{C}, T$ *by* $\overline{\mathcal{C}}, U$.

Step 1 is not needed if T is already pseudo-equational.

step 2: hybrid logic

Hybrid logic = modal logic + nominals (like first-order variables)

$$H := i \mid \top \mid \bot \mid \neg H \mid H \wedge H \mid H \vee H \mid \Diamond H \mid \Box H \mid \forall i H \mid \exists i H$$

$i, j, \ldots$ are the nominals. *Sentence* — no free nominals.

semantics

Fix a Kripke frame $\mathcal{F} = (W, R)$.
An *assignment/valuation* into $\mathcal{F}$ is a map $h : \{\text{nominals}\} \to W$.

- $\mathcal{F}, h, w \models i$ iff $w = h(i)$
- Boolean and modal operators as usual
- $\mathcal{F}, h, w \models \forall i \varphi$ iff $\mathcal{F}, g, w \models \varphi$ for all assignments g with $g(j) = h(j)$ for all nominals $j \neq i$.
- $\exists$ similar

validity

A hybrid sentence φ is *valid* in a frame $\mathcal{F} = (W, R)$ (written $\mathcal{F} \models \varphi$) if $\mathcal{F}, h, w \models \varphi$ for every $w \in W$ and every hybrid assignment h.

Example

Recall $\quad \Diamond\big((\Box p_1 \vee \Box\neg p_1) \wedge \ldots \wedge (\Box p_k \vee \Box\neg p_k)\big) \quad (k \geq 1)$

axiomatises the logic KM^∞ of the class of frames satisfying

$$\forall x \exists y (R(x,y) \wedge \forall z t (R(y,z) \wedge R(y,t) \to z = t))$$

These frames are the ones validating the hybrid sentence

$$\Diamond \exists i \Box i.$$

quasipositive hybrid formulas and pseudo-equations

Actually we only need the *quasipositive fragment* of hybrid logic.

$$\begin{aligned}\Phi := \quad & i \mid \top \mid \bot \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Diamond\Phi \mid \Box\Phi \mid \exists i \Phi \mid \\ & \forall i(\Diamond i \to \Phi) \mid \Diamond j \wedge \forall i(\Box(j \to \Diamond i) \to \Phi) \mid \\ & \Diamond(j \wedge \Diamond j') \wedge \forall i(\Box(j \to \Box(j' \to \Diamond i)) \to \Phi) \mid \cdots\end{aligned}$$

Proposition 4 *Any pseudo-equation* ε *can be easily translated into a quasipositive hybrid sentence* φ *that is valid in precisely the frames satisfying* ε*.*

Example:

$$\varepsilon = \forall x \exists y \big(R(x,y) \wedge \forall z (R(y,z) \to R(x,z) \vee R(z,x) \vee x = z)\big)$$

translates to

$$\varphi = \exists x \big[x \wedge \exists y \big(\Diamond y \wedge \forall z \big([\Box(y \to \Diamond z)] \to \Diamond z \vee [\Diamond\Diamond(z \wedge \Diamond x)] \vee z\big)\big)\big].$$

step 2 conclusion

Using proposition 4, we *replace the pseudo-equational theory U defining $\overline{\mathcal{C}}$ by a set Φ of quasipositive hybrid sentences valid in precisely the frames in $\overline{\mathcal{C}}$.*

Steps 1 and 2 are not needed if we can define our elementary frame class by quasipositive hybrid sentences in the first place.

E.g., the frames for KM^∞ are defined by $\Diamond\exists i\Box i$.

step 3: modal approximants of quasipositive hybrid formulas

Idea: approximate nominals by modally-definable (clopen) sets in Kripke model.

Let S be a finite set of modal formulas. For any $X \subseteq S$, we can approximate the nominal i by

$$i_{S,X} \quad = \quad \bigwedge\{\alpha : \alpha \in X\} \wedge \bigwedge\{\neg\beta : \beta \in S \setminus X\}.$$

Simulate $\forall i\varphi$ by *conjunction* over all $X \subseteq S$.

Simulate $\exists i\varphi$ by *disjunction* over all $X \subseteq S$.

We get a *modal 'approximant'* of a hybrid formula φ with respect to S.

example: approximants of $\Diamond\exists i\Box i$ axiomatise KM^∞

Recall $\quad \Diamond\big((\Box p_1 \vee \Box\neg p_1) \wedge \ldots \wedge (\Box p_k \vee \Box\neg p_k)\big) \quad (k \geq 1)$

axiomatises logic KM^∞ of class of frames validating

$$\varphi \quad = \quad \Diamond\exists i\Box i.$$

Approximate φ w.r.t. finite set $S = \{p_1, \ldots, p_k\}$ of atoms:

$$\varphi_S \quad = \quad \overbrace{\Diamond}^{\Diamond}\ \overbrace{\bigvee_{X\subseteq S}}^{\exists i}\ \overbrace{\Box}^{\Box}\ \overbrace{\Big(\bigwedge_{p\in X} p \wedge \bigwedge_{p\in S\setminus X} \neg p\Big)}^{i}.$$

φ_S is equivalent to the kth axiom of KM^∞.
Conclude $\{\varphi_S : S \text{ finite}\}$ axiomatises KM^∞!

In general, if we define 'approximant' properly, then:

For any quasipositive hybrid sentence φ, the set of its approximants axiomatises the modal logic of $\{\mathcal{F} : \mathcal{F} \models \varphi\}$.

soundness...

Lemma 5 *If a quasipositive sentence φ is valid in a frame $\mathcal{F}$, then all approximants of φ are valid in $\mathcal{F}$.*

This is essentially a monotonicity principle. Clopens are coarser than nominals, so should be OK for *positive* hybrid φ (monotonic).

Problem: $\forall$. For some models $\mathcal{M}$ on $\mathcal{F}$, $\bigwedge_{X\subseteq S}$ may include X such that the approximant $i_{S,X}$ is true at no world of $\mathcal{M}$.

Solution: *relativise* $\forall$ to exclude such 'inconsistent' X.

$\forall i\ i$ — bad. $\quad \forall i(\Diamond i \to i)$ — OK.

General (quasipositive) form:

$$\underbrace{\Diamond(j \wedge \Diamond j')}_{\text{any length}} \wedge \forall i(\Box(j \to \Box(j' \to \Diamond i)) \to \varphi)$$

But quasipositive is not positive! This causes technical problems.

. . . and completeness

Definition 6 Kripke model $\mathcal{M} = (W, R, m)$ is said to be *descriptive* if:

1. Whenever $t \neq u$ in W, there is a modal formula α such that $\mathcal{M}, t \models \alpha$ and $\mathcal{M}, u \models \neg\alpha$.
2. If $w, v \in W$, then $R(w, v)$ iff ($\mathcal{M}, w \models \Diamond\alpha$ whenever $\mathcal{M}, v \models \alpha$).
3. Any set of modal formulas is satisfied in $\mathcal{M}$ if all finite subsets of it are.

Example: the canonical model of any modal logic.

Lemma 7 *If all approximants of a quasipositive sentence φ are valid in a* descriptive *model $\mathcal{M}$, then φ is valid in $\mathcal{M}$'s Kripke frame.*

Can prove by extending Sahlqvist's completeness theorem.

example: $\Diamond\exists i\Box i$

Fix descriptive model $\mathcal{M} = (\underbrace{W, R}_{\mathcal{F}}, m)$. Recall $i_{S,X} = \bigwedge_{\alpha\in X} \alpha \wedge \bigwedge_{\beta\in S\setminus X} \neg\beta$.

We show $\mathcal{M}, w \models (\Diamond\exists i\Box i)_S$ (all S) $\Rightarrow \mathcal{F}, w \models \Diamond\exists i\Box i$.

1\) Let $\mathcal{X}$ be a set of modal formulas. For each finite set S of modal formulas, put $X_S := S \cap \mathcal{X}$. Let h be a hybrid assignment.
Assume '$\mathcal{X} \Vdash h$': i.e., $(\forall w \in W)\big(\mathcal{M}, w \models i_{S,X_S}$ (all S) $\Rightarrow \mathcal{F}, h, w \models i\big)$.
Then (easy) $(\forall w \in W)\big(\mathcal{M}, w \models (\Box i)_{S,X_S}$ (all S) $\Rightarrow \mathcal{F}, h, w \models \Box i\big)$.

2\) Fix w. Suppose that $\mathcal{M}, w \models (\exists i\Box i)_S$ (all S).
Then $\mathcal{M}, w \models \bigvee_{X\subseteq S}(\Box i)_{S,X}$ (all S). So $\forall S \exists X \subseteq S(\mathcal{M}, w \models (\Box i)_{S,X})$.

By König's tree lemma and descriptiveness, can find $\mathcal{X}, h$ such that $\mathcal{X} \Vdash h$ and $\mathcal{M}, w \models (\Box i)_{S,X_S}$ (all S).

$\mathcal{X} \Vdash h$, so by (1), $\mathcal{F}, h, w \models \Box i$.
So clearly, $\mathcal{F}, w \models \exists i\Box i$.

example ctd

3\) Suppose $\mathcal{M}, w \models (\Diamond\exists i\Box i)_S$ (all S) — i.e., $\mathcal{M}, w \models \bigwedge_S \Diamond(\exists i\Box i)_S$.
Esakia's lemma converts $\bigwedge\Diamond$ to $\Diamond\bigwedge$:
there is $v \in W$ with $R(w, v)$ and $\mathcal{M}, v \models (\exists i\Box i)_S$ (all S).
By (2), $\mathcal{F}, v \models \exists i\Box i$. So $\mathcal{F}, w \models \Diamond\exists i\Box i$.

So if all approximants $(\Diamond\exists i\Box i)_S$ are valid in $\mathcal{M}$, then $\Diamond\exists i\Box i$ is valid in its frame $\mathcal{F}$.

tech points

Cases $\wedge, \vee$ are 'similar'. $\forall$ needs care: quasipositive $\not\Rightarrow$ positive.

We require throughout a monotonicity lemma.

It seems necessary to allow different nominals $i, j, k, \ldots$ to be approximated with different sets S.

step 3 conclusion

Corollary 8 *The approximants of a set Φ of quasipositive sentences axiomatise the logic of the class of frames validating all $\varphi \in \Phi$.*

Proof. Let $\mathcal{K}$ be the class of frames validating every $\varphi \in \Phi$.
Let Λ_Φ be the logic axiomatised by approximants of sentences in Φ.

1. By soundness (lemma 5), Λ_Φ *is valid over* $\mathcal{K}$.
2. Suppose $\alpha \notin \Lambda_\Phi$. We show α *is not valid over* $\mathcal{K}$.

   Let $\mathcal{M}$ be the canonical model of Λ_Φ. Let $\mathcal{F}$ be its Kripke frame.

   Let $\varphi \in \Phi$. Each approximant of φ is an *axiom,* so is valid in $\mathcal{M}$.

   $\mathcal{M}$ is descriptive. By completeness (lemma 7), φ is valid in $\mathcal{F}$.

   This holds for all $\varphi \in \Phi$. So $\mathcal{F} \in \mathcal{K}$.

   By general theory, $\mathcal{M}$ satisfies $\neg\alpha$. So α *is not valid in* $\mathcal{F}$. ■

conclusion

Corollary 9 *The elementary modal logics are precisely those axiomatised by the approximants of sets Φ of quasipositive sentences.*

Proof. $\Leftarrow$: Any such Φ defines an elementary class of frames. By corollary 8, the logic Λ_Φ is elementary too.

$\Rightarrow$: Given first-order theory T defining an elementary frame class $\mathcal{C}$,

step 1 turns T into a pseudo-equational theory U defining $\overline{\mathcal{C}}$. The modal logics of $\mathcal{C}, \overline{\mathcal{C}}$ are the same.

step 2 turns U into a set Φ of quasipositive hybrid sentences valid in precisely the frames in $\overline{\mathcal{C}}$

step 3 turns Φ into approximants. By corollary 8, they axiomatise the logic of $\overline{\mathcal{C}}$. ■

remarks

Syntactic characterisation of elementary modal logics, by approximants. New way to study them.

1. proof works for multiple polyadic modalities
2. ‘explains’ elementarity of KM^∞ and other non-Sahlqvist logics by Sahlqvist-like means. But BSS do it better?
3. axioms can be ‘natural’ — eg KM^∞
4. some logics need infinitely many quasipositive sentences
5. *open problem* to find finite axiomatisation where one exists

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