# Monodic packed fragment with equality is decidable

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**Abstract**

We prove decidability of satisfiability of sentences of the monodic packed fragment of first-order temporal logic with equality and connectives Until and Since, in models with various flows of time and domains of arbitrary cardinality. We also prove decidability over models with finite domains, over flows of time including the real order.

## 1 Introduction

In [6], a new 'monodic' fragment of predicate temporal logic was introduced. It consists of those formulas whose subformulas beginning with a temporal connective have at most one free variable. It was shown that if the first-order part of the language was also restricted to (roughly speaking) a decidable fragment of first-order logic, then satisfiability of monodic sentences was decidable over a wide range of (strict linear) flows of time.

The proof in [6] of decidability of a monodic sentence $\varphi$ relied on representing a temporal model of $\varphi$ by a 'quasimodel'. In a quasimodel, each element of the domain at each time is represented only by its *type* — the set of subformulas $\psi(x)$ of $\varphi$ that are true at the element at that time. The quasimodel records which types are realized at each time. (The integrity of domain elements over time is represented in the quasimodel by functions called runs that select a type at each time; monodicity ensures that this is sufficient.) A quasimodel is simpler than a model because there are only finitely many types; assuming decidability of the underlying first-order logic, it followed that the existence of a quasimodel was decidable over a wide range of flows of time.

The crucial lemma in the proof was of course that $\varphi$ has a temporal model iff it has a quasimodel, and the crucial part of this was to construct a model of $\varphi$ from a quasimodel. To show that the runs in a quasimodel could be turned into the points of the domain of a temporal model involved replacing a first-order structure $M$ by a larger one, $M^{(\kappa)}$, for some infinite cardinal $\kappa$. $M^{(\kappa)}$ should realize the same types as $M$, but each of them should be realized by $\kappa$ elements of $M^{(\kappa)}$. This was done by excluding equality from the logic. Then, every element of $M$ could simply be 'duplicated' $\kappa$ times to form $M^{(\kappa)}$. Lacking equality, the logic could not see what had happened.

In the presence of equality, this trick no longer works. As a trivial example, $\forall y(y = x)$ is not preserved by duplicating elements. Indeed, it was shown in [9] that with equality, the monodic validities over natural numbers time are not even recursively enumerable, and undecidability results for the monadic monodic fragment with only two variables but with equality are proved in [2].

In this paper, we show that the decidability argument can be resurrected if we limit the first-order part of monodic formulas to come from the *packed fragment* of first-order logic, defined by

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Marx in [7]. The packed fragment is more expressive than the guarded fragment and (for non-binary signatures) the loosely guarded fragment; the clique-guarded fragment of [3] is a syntactic variant of it. In spite of this rather good expressive power, it is still amenable to duplication of elements. For example, in a relational signature, the *disjoint union* of $\kappa$ copies of a structure $M$ realizes the same types as $M$, if these types consist of packed formulas. But simply duplicating (some) elements of $M$ as before (see proposition 7 below) works in the presence of constants, and also serves to prove decidability over temporal models with finite domains.

In section 3.3 we will prove:

**Theorem 1 (decidability)** *Let $\mathcal{C}$ be any of $\{(\mathbb{N},<)\}$, $\{(\mathbb{Z},<)\}$, $\{(\mathbb{Q},<)\}$, the class of all finite strict (i.e., irreflexive) linear orders, or any first-order definable class of strict linear orders. It is decidable whether a sentence of the monodic packed fragment with Until, Since, and equality in a signature without function symbols has a model with flow of time in $\mathcal{C}$.*

**Theorem 2 (finite domain decidability)** *Let $\mathcal{C}$ be any of the above, or $\{(\mathbb{R},<)\}$. It is decidable whether a sentence of the monodic packed fragment with Until, Since, and equality in a signature without function symbols has a model with finite domain and flow of time in $\mathcal{C}$.*

Section 3 contains the proofs, and section 2 the preliminary definitions.

**Notation 3** We will usually identify (notationally) a structure $M$ with its domain. We will write $c^M$ for the interpretation in $M$ of a constant $c$ of the signature of $M$.

## 2 Monodic packed fragment with equality

The following combines the definition of monodic formulas from [6] with a minor modification of Marx's definition of the packed fragment.

**Definition 4 (monodic packed fragment)** *Let $L$ be a signature with at most constants and relation symbols (no function symbols). An $L$-formula $\gamma$ is said to be a* packing guard *if $\gamma$ is a conjunction of atomic and existentially-quantified atomic formulas (possibly equalities) such that any two distinct free variables of $\gamma$ co-occur free in some conjunct of $\gamma$. The* monodic packed fragment *of predicate temporal logic in signature $L$ consists of the following formulas:*

- *Any atomic $L$-formula (which can be an equality, $\top$, or $\bot$) is monodic packed.*
- *Boolean combinations of monodic packed formulas are monodic packed.*
- *If $\gamma$ is a packing guard, $\varphi$ is a monodic packed formula, every free variable of $\varphi$ is free in $\gamma$, and $\bar{y}$ is a tuple of variables, then $\exists\bar{y}(\gamma \wedge \varphi)$ is a monodic packed formula.*
- *If $x$ is a variable and $\varphi, \psi$ are monodic packed formulas with free variable at most $x$, then $\varphi U \psi$ and $\varphi S \psi$ are monodic packed formulas. We read them as '$\varphi$ until $\psi$', '$\varphi$ since $\psi$', respectively.*

*The* packed fragment $\mathsf{PF}(L)$ *of the first-order language over $L$ consists of those monodic packed $L$-formulas that do not involve $U$ or $S$.*

Note that $\mathsf{PF}(L)$ is closed under subformulas, including subformulas of packing guards.

**Definition 5 (temporal structures and semantics)** *Let $L$ be as above and let $T$ be a strict linear order. A* temporal $L$-structure with flow of time $T$ *is a pair $\mathfrak{M} = (D, (M_t)_{t \in T})$, where $D$ is a non-empty set (the* domain of $\mathfrak{M}$*), each $M_t$ is an $L$-structure with domain $D$, and $c^{M_t} = c^{M_u}$ for all constants $c \in L$ and all $t, u \in T$.*

*Given $\mathfrak{M}$, $t \in T$, and an assignment $h$ of variables into $D$, we define the semantics of monodic packed formulas written with variables from the domain of $h$ as follows:*

- *For atomic $\alpha$, we let $\mathfrak{M}, t, h \models \alpha$ iff $M_t, h \models \alpha$.*
- *$\mathfrak{M}, t, h \models \neg\varphi$ iff $\mathfrak{M}, t, h \not\models \varphi$, and $\mathfrak{M}, t, h \models \varphi \wedge \psi$ iff $\mathfrak{M}, t, h \models \varphi$ and $\mathfrak{M}, t, h \models \psi$.*
- *$\mathfrak{M}, t, h \models \exists x\varphi$ iff $\mathfrak{M}, t, g \models \varphi$ for some assignment $g$ that agrees with $h$ except perhaps on $x$.*
- *$\mathfrak{M}, t, h \models \varphi U \psi$ iff there is $u \in T$ with $u > t$, $\mathfrak{M}, u, h \models \psi$, and $\mathfrak{M}, v, h \models \varphi$ for all $v \in T$ with $t < v < u$.*
- *$\mathfrak{M}, t, h \models \varphi S \psi$ iff there is $u \in T$ with $u < t$, $\mathfrak{M}, u, h \models \psi$, and $\mathfrak{M}, v, h \models \varphi$ for all $v \in T$ with $u < v < t$.*

Note that our temporal structures have constant domains and the interpretations of constants and assignments to variables are rigid. Also, we use the 'strict' semantics of Until and Since ($u > t$ rather than $u \geq t$, etc). They can express connectives such as the non-strict and weak Until and Since, Tomorrow, etc. Indeed, [6] proves expressive completeness results for monodic formulas written with them, over strict Dedekind-complete linear orders.

**Notation 6** The notation $\psi(\bar{x})$ will indicate that all free variables of formula $\psi$ are in the tuple $\bar{x}$; not all variables in $\bar{x}$ need actually occur free in $\psi$.

Let $h$ assign a tuple $\bar{a}$ of elements of $D$ (we write '$\bar{a} \in D$' to indicate this) to the variables $\bar{x}$. For a formula $\psi(\bar{x})$, we may write $\mathfrak{M}, t \models \psi(\bar{a})$ rather than $\mathfrak{M}, t, h \models \psi$. If $\psi$ is a sentence, we just write $\mathfrak{M}, t \models \psi$. Similar notation is used for first-order structures too.

# 3 Decidability of monodic packed fragment with equality

We now give a decision procedure for the monodic packed fragment.

## 3.1 Types in the packed fragment

Fix a sentence $\varphi$ of the monodic packed fragment written with at most constants and relation symbols (no function symbols). Let $L$ be the finite signature consisting of all the constants and relation symbols that occur in $\varphi$. For each subformula $\alpha U \beta$ of $\varphi$, which by monodicity of $\varphi$ has either one or no free variables, introduce a new unary (respectively, nullary) relation symbol $R_{\alpha U \beta}$. Do the same for subformulas $\alpha S \beta$ of $\varphi$. Let $\overline{L}$ denote the augmentation of $L$ by these new relation symbols.

For each subformula $\psi$ of $\varphi$, let $\overline{\psi}$ be the $\overline{L}$-formula obtained from $\psi$ by replacing all maximal subformulas $\alpha U \beta(y)$ of $\psi$ with one free variable, say $y$, by $R_{\alpha U \beta}(y)$, replacing all maximal subsentences $\alpha U \beta$ of $\psi$ by $R_{\alpha U \beta}$, and similarly for $S$. Note that $\overline{\psi}$ is in $\mathsf{PF}(\overline{L})$ and has the same free variables as $\psi$.

Fix a variable $x$. For a packed formula $\psi(y)$, choose a formula $\psi(x/y)$ obtained by renaming the variables in $\psi$ in a one-one way so that at most $x$ is free. $\psi(x/y)$ is packed.

Let $\mathcal{C}_x\varphi$ denote the set of all formulas $\overline{\psi}(x/y)$ and their negations, where $\psi(y)$ is a subformula of $\varphi$, together with all $\overline{L}$-formulas $x = c$ and $x \neq c$ for constants $c \in L$. Note that $\mathcal{C}_x\varphi \subseteq \mathsf{PF}(\overline{L})$.

A *type* is a maximal boolean-consistent subset of $\mathcal{C}_x\varphi$. There are finitely many types. A type $p$ is said to be a *constant type* if $x = c \in p$ for some constant $c \in L$. For an $\overline{L}$-structure $M$ and $a \in M$, we define $tp_M(a) = \{\psi(x) \in \mathcal{C}_x\varphi : M \models \psi(a)\}$. An element $a \in M$ *realizes* a type $p$ if $p = tp_M(a)$.

**Proposition 7** *Let $N$ be any $\overline{L}$-structure. Let $\xi$ be a map assigning a cardinal $\xi(p)$ to each type $p$, such that:*

- $\xi(p) = 0$ *if $p$ is not realized in $N$,*
- $\xi(p) = 1$ *if $p$ is a constant type that is realized in $N$,*
- $\xi(p) \geq |N|$ *otherwise.*

*Then there is an $\overline{L}$-structure $N^\xi$ such that each type $p$ is realized in $N^\xi$ by exactly $\xi(p)$ elements.*

**Proof** The idea is to make copies of the elements of the domain of $N$ that are not named by constants. For each $a \in N$ let $\kappa_a$ be a non-zero cardinal such that $\kappa_{c^N} = 1$ for all constants $c \in L$. (We will impose further conditions on the $\kappa_a$ later.) We use $i, j$ to denote ordinals. The domain of $N^\xi$ will be $\{(a,i) : a \in N,\ i < \kappa_a\}$. A subset $S$ of this domain is said to be *thin* if whenever $(a,i),(a,j) \in S$ then $i = j$. The $\overline{L}$-symbols are interpreted in $N^\xi$ as follows:

$$\begin{array}{rcl} c^{N^\xi} & = & (c^N, 0) \\ N^\xi \models R((a_1,i_1),\ldots,(a_n,i_n)) & \iff & \{(a_1,i_1),\ldots,(a_n,i_n)\} \text{ is thin} \\ & & \text{and } N \models R(a_1,\ldots,a_n), \end{array}$$

for all constants $c \in \overline{L}$, $n$-ary relation symbols $R \in \overline{L}$, and elements $(a_1,i_1),\ldots,(a_n,i_n)$ of the domain. We let $\pi : N^\xi \to N$ be the homomorphism taking each $(a,i) \in N^\xi$ to $a \in N$. Fix a set $V$ of variables. If $\eta : V \to N^\xi$ is an assignment, then $\eta' \stackrel{\text{def}}{=} \pi \circ h : V \to N$ is an assignment of the same variables into $N$.

**Lemma 8** *Let $\eta : V \to N^\xi$ be an assignment, and assume that the range $\eta(V)$ of $\eta$ is thin. Then $N^\xi, \eta \models \alpha$ iff $N, \eta' \models \alpha$ for any atomic $\overline{L}$-formula $\alpha$ written with variables from $V$.*

**Proof** Because $\kappa_{c^N} = 1$ for all constants $c$, the set $\eta(V) \cup \{c^{N^\xi} : c \in L \text{ a constant}\}$ is thin. The lemma now follows because the restriction of $\pi$ to any thin set is a partial isomorphism from $N^\xi$ to $N$. ❑

Let $\eta : V \to N^\xi$ be an assignment with thin range. Let $\bar{y}$ be a tuple of variables in $V$, and suppose that $h : V \to N$ agrees with $\eta'$ except perhaps on $\bar{y}$. Lift $h$ to an assignment $\eta_h : V \to N^\xi$ by:

$$\eta_h(v) = \begin{cases} \eta(w) & \text{if } h(v) = \eta'(w) \text{ for some } w \in V; \\ (h(v), 0) & \text{otherwise.} \end{cases}$$

Then it is easily checked that $\eta_h$ is well-defined, $\eta_h(V)$ is thin, $(\eta_h)' = h$, and $\eta_h$ agrees with $\eta$ except perhaps on $\bar{y}$.

**Lemma 9** *For any packed $\overline{L}$-formula $\psi$ written with variables from $V$, and any assignment $\eta : V \to N^\xi$ such that $\eta(V)$ is thin, we have $N^\xi, \eta \models \psi$ iff $N, \eta' \models \psi$. Hence, for all $(a,i) \in N^\xi$ we have $tp_{N^\xi}((a,i)) = tp_N(a)$.*

**Proof** We give a syntactic proof, by induction on $\psi$; the lemma can also be proved by bisimulations, using [7, fact 4.4]. The result for atomic $\psi$ follows from lemma 8. The boolean cases are easy and we omit them. So consider a packed formula $\theta$ of the form $\exists\bar{y}(\gamma \wedge \psi)$, where $\gamma$ is a packing guard and $\psi$ is a packed formula for which the claim is assumed inductively.

Assume that $N, \eta' \models \theta$. So there is an assignment $h : V \to N$ agreeing with $\eta'$ except perhaps on $\bar{y}$, and with $N, h \models \gamma \wedge \psi$. By the inductive hypothesis, $N^\xi, \eta_h \models \psi$. To see that $N^\xi, \eta_h \models \gamma$ as well, take any conjunct $\exists\bar{z}\alpha$ of $\gamma$ ($\bar{z}$ could be empty). Since $N, h \models \gamma$, there is $g : V \to N$ agreeing with $h$ except perhaps on $\bar{z}$ and with $N, g \models \alpha$. By lemma 8, $N^\xi, (\eta_h)_g \models \alpha$ also, and so $N^\xi, \eta_h \models \exists\bar{z}\alpha$. Hence, $N^\xi, \eta_h \models \gamma \wedge \psi$, whence certainly $N^\xi, \eta_h \models \theta$. Since $\eta$ and $\eta_h$ agree on the free variables of $\theta$, we obtain $N^\xi, \eta \models \theta$.

For the converse, assume that $N^\xi, \eta \models \theta$. So there is some $\zeta : V \to N^\xi$ agreeing with $\eta$ on the free variables of $\theta$ and with $N^\xi, \zeta \models \gamma \wedge \psi$. We may assume that $\bar{y}$ is a non-empty tuple of free variables of $\gamma$, and that for some $y$ in $\bar{y}$ we have $\zeta(v) = \zeta(y)$ for every $v \in V$ that does not occur free in $\gamma$.

It now follows that $\zeta(V)$ is thin. For, let $v, w \in V$ be distinct variables such that $\zeta(v) = (a,i)$ and $\zeta(w) = (a,j)$. We require $i = j$. By the assumption just made, we can suppose that $v, w$ occur free in $\gamma$. So there is a conjunct $\exists\bar{z}\alpha$ of $\gamma$ in which $v, w$ both occur free. If $\alpha$ is an equality, then since $v, w$ are free in $\exists\bar{z}\alpha$, we have $\zeta(v) = \zeta(w)$, so $i = j$. So suppose that $\alpha$ is of the form $R(x_1, \ldots, x_n)$, where $R \in \overline{L}$. We have $N^\xi, \zeta \models \exists\bar{z}R(x_1, \ldots, x_n)$, so choose an assignment $\varepsilon : V \to N^\xi$ agreeing with $\zeta$ except perhaps on $\bar{z}$ and with $N^\xi, \varepsilon \models R(x_1, \ldots, x_n)$. By definition of $N^\xi$, $\{\varepsilon(x_1), \ldots, \varepsilon(x_n)\}$ must be thin. Since this set contains $(a,i), (a,j)$, it follows again that $i = j$. We are done.

So by the inductive hypothesis, $N, \zeta' \models \psi$. Since $\pi$ is a homomorphism, $N, \zeta' \models \gamma$. So certainly $N, \zeta' \models \theta$. Since $\zeta'$ and $\eta'$ agree on the free variables of $\theta$, we have $N, \eta' \models \theta$ as required.

That $tp_{N^\xi}((a,i)) = tp_N(a)$ for all $(a,i) \in N^\xi$ is now immediate, since types consist of packed $\overline{L}$-formulas with at most one free variable, and $\{(a,i)\}$ is thin. ❑

For each type $p$ realized in $N$, let $R_p$ be the set of elements of $N$ that realize $p$. Choose an arbitrary element $a_p \in R_p$, let $\kappa_{a_p} > 0$ be such that $|R_p \setminus \{a_p\}| + \kappa_{a_p} = \xi(p)$, and for all other elements $b \in N$ realizing $p$ let $\kappa_b = 1$. Note that $\kappa_{c^N} = 1$ for each constant $c \in L$. This specifies $N^\xi$ completely.

Let $p$ be a type. If $p$ is not realized in $N$ then $\xi(p) = 0$; and $p$ is not realized in $N^\xi$, since by lemma 9, $tp_{N^\xi}((a,i)) = tp_N(a)$ for all $a \in N$ and $i < \kappa_a$. If $p$ is realized in $N$, then by lemma 9, the elements realizing $p$ in $N^\xi$ are precisely $\{(a_p, i) : i < \kappa_{a_p}\} \cup \{(b,0) : b \in R_p \setminus \{a_p\}\}$. This set has cardinality $\xi(p)$, as required. ❑

### 3.2 Quasimodels

The proof of decidability from now on is exceedingly similar to that of [6], using proposition 7 to replace lemma 9 of [6]. We include the proof to make the presentation self-contained.

**Definition 10 (model)** *A temporal $L$-structure $\mathfrak{M}$ with flow of time $T$ (say) is said to be a* model *of $\varphi$ if $\mathfrak{M}, t \models \varphi$ for some $t \in T$.*

A *state candidate* (for $\varphi$) is a set of types. A *realizable state candidate* is a state candidate of the form $\Sigma = \{tp_M(a) : a \in M\}$, for some $\overline{L}$-structure $M$; $M$ is said to *realize* $\Sigma$. A *finitely realizable state candidate* is a state candidate that is realized by a finite $\overline{L}$-structure.

Now fix a strict (irreflexive) linear order $T$. Given realizable state candidates $\Sigma_t$ ($t \in T$), a *run in* $(\Sigma_t)_{t \in T}$ is a choice function $\rho$ on $(\Sigma_t)_{t \in T}$ (so $\rho(t) \in \Sigma_t$ for all $t \in T$), satisfying:

1. for all constants $c \in L$ and all $t, u \in T$, we have $x = c \in \rho(t)$ iff $x = c \in \rho(u)$,

2. for all $t \in T$ and all subformulas $\alpha U \beta(y)$ of $\varphi$, $\overline{\alpha U \beta}(x/y) \in \rho(t)$ iff there is $u > t$ with $\overline{\beta}(x/y) \in \rho(u)$ and with $\overline{\alpha}(x/y) \in \rho(v)$ for all $v \in T$ with $t < v < u$,

3. a similar clause for $S$.

**Definition 11 (quasimodel)** *A* quasimodel *for $\varphi$ (over $T$) is a pair $((\Sigma_t)_{t\in T}, \mathcal{R})$ where $(\Sigma_t)_{t\in T}$ is a sequence of realizable state candidates and $\mathcal{R}$ is a set of runs in $(\Sigma_t)_{t\in T}$ such that $\Sigma_t = \{\rho(t) : \rho \in \mathcal{R}\}$ for all $t \in T$, and $\overline{\varphi} \in \bigcup \Sigma_t$ for some $t \in T$. The quasimodel is* finitary *if each $\Sigma_t$ is a finitely realizable state candidate and $\mathcal{R}$ is finite.*

**Theorem 12** *$\varphi$ has a model with flow of time $T$ (and finite domain) iff there is a (respectively, finitary) quasimodel for $\varphi$ over $T$.*

**Proof** Let $\mathfrak{M} = (D, (M_t)_{t\in T})$ be a model of $\varphi$ with flow of time $T$. Expand each $M_t$ to an $\overline{L}$-structure $\overline{M}_t$ by letting $\overline{M}_t \models R_\psi(d)$ iff $\mathfrak{M}, t \models \psi(d)$, for each unary $R_\psi \in \overline{L} \setminus L$ and each $d \in D$, and similarly letting $\overline{M}_t \models R_\psi$ iff $\mathfrak{M}, t \models \psi$, for each nullary $R_\psi \in \overline{L} \setminus L$. Let $\Sigma_t = \{tp_{\overline{M}_t}(d) : d \in D\}$. For each $d \in D$ define $\rho_d : T \to \bigcup_{t\in T} \Sigma_t$ by $\rho_d(t) = tp_{\overline{M}_t}(d)$. Then $((\Sigma_t)_{t\in T}, \{\rho_d : d \in D\})$ can be checked to be a quasimodel for $\varphi$ over $T$. If $D$ is finite then clearly the $\overline{M}_t$ and $\{\rho_d : d \in D\}$ are finite, so the quasimodel is finitary.

For the converse, assume that $((\Sigma_t)_{t\in T}, \mathcal{R})$ is a quasimodel for $\varphi$ over $T$. For each $t \in T$, choose an $\overline{L}$-structure $N_t$, finite if possible, that realizes $\Sigma_t$. Since there are only finitely many realizable state candidates, we may assume that $\{N_t : t \in T\}$ is finite. Let $\kappa = \max\{|N_t| : t \in T\}$. Define a multiset $\mathcal{R}^{(\kappa)}$ as follows. For each $\rho \in \mathcal{R}$, if some $\rho(t)$ $(t \in T)$ is a constant type then put one copy of $\rho$ into $\mathcal{R}^{(\kappa)}$; otherwise, put $\kappa$ copies of $\rho$ into $\mathcal{R}^{(\kappa)}$. For each type $p$ and each $t \in T$, let

$$\xi_t(p) = |\{\rho \in \mathcal{R}^{(\kappa)} : \rho(t) = p\}|$$

(the right-hand side denotes multiset cardinality, counting multiplicities).

**Lemma 13** *For each type $p$ and each $t \in T$,*

- $\xi_t(p) = 0$ *if $p$ is not realized in $N_t$,*
- $\xi_t(p) = 1$ *if $p$ is a constant type that is realized in $N_t$,*
- $\xi_t(p) \geq |N_t|$ *otherwise.*

**Proof** If $p$ is not realized in $N_t$ then $p \notin \Sigma_t$ so there is no $\rho \in \mathcal{R}$ with $\rho(t) = p$. Hence $\xi_t(p) = 0$.

If $p$ is a constant type realized in $N_t$, then certainly $p \in \Sigma_t$, so there is at least one $\rho \in \mathcal{R}$ with $\rho(t) = p$. Suppose that $\rho, \rho' \in \mathcal{R}$ satisfy $\rho(t) = \rho'(t) = p$. We claim that $\rho = \rho'$. Let $u \in T$; we show that $\rho(u) = \rho'(u)$. Choose a constant $c \in L$ with $x = c \in p$. So $x = c \in \rho(t) \cap \rho'(t)$. As $\rho, \rho'$ are runs, $x = c \in \rho(u) \cap \rho'(u)$. Pick $a, a' \in N_u$ with $tp_{N_u}(a) = \rho(u)$ and $tp_{N_u}(a') = \rho'(u)$. Then $N_u \models a = c \wedge a' = c$, so $a = a'$ and $\rho(u) = \rho'(u)$. As $u$ was arbitrary, $\rho = \rho'$ as claimed.

Hence, there is exactly one $\rho \in \mathcal{R}$ with $\rho(t) = p$. Since $p$ is a constant type, by construction of $\mathcal{R}^{(\kappa)}$ there is exactly one $\rho \in \mathcal{R}^{(\kappa)}$ with $\rho(t) = p$. So $\xi_t(p) = 1$.

Suppose now that $p$ is realized in $N_t$ and is not a constant type. As before, $p \in \Sigma_t$. Choose $\rho \in \mathcal{R}$ with $\rho(t) = p$. Because $\rho$ is a run, no $\rho(u)$ $(u \in T)$ is a constant type. So by construction of $\mathcal{R}^{(\kappa)}$, there are $\kappa$ copies of $\rho$ in $\mathcal{R}^{(\kappa)}$. So $\xi_t(p) = |\{\rho \in \mathcal{R}^{(\kappa)} : \rho(t) = p\}| \geq \kappa \geq |N_t|$. ❑

By the lemma, proposition 7 applies and we can form the $\overline{L}$-structure $N_t^{\xi_t}$ for each $t \in T$. For each $t$, the number of realizations of any type $p$ in $N_t^{\xi_t}$ is the same as the number of $\rho \in \mathcal{R}^{(\kappa)}$ such that $\rho(t) = p$. So we may choose a bijection $\iota_t : \mathcal{R}^{(\kappa)} \to N_t^{\xi_t}$ such that $tp_{N_t^{\xi_t}}(\iota_t(\rho)) = \rho(t)$ for all $\rho \in \mathcal{R}^{(\kappa)}$. Let $\overline{M}_t$ be the unique $\overline{L}$-structure with domain $\mathcal{R}^{(\kappa)}$ such that $\iota_t : \overline{M}_t \to N_t^{\xi_t}$ is an isomorphism. We get $\overline{M}_t$ by pulling back the $\overline{L}$-structure on $N_t^{\xi_t}$ via $\iota_t^{-1}$. Then $tp_{\overline{M}_t}(\rho) = \rho(t)$ for all $\rho \in \mathcal{R}^{(\kappa)}$ and $t \in T$. So

$$\overline{M}_t \models \psi(\rho) \iff \psi(x) \in \rho(t), \text{ for all } \psi(x) \in C_x\varphi,\ \rho \in \mathcal{R}^{(\kappa)},\ t \in T. \tag{1}$$

Let $M_t$ denote the $L$-reduct of $\overline{M}_t$. Note that for each constant $c \in L$, $t,u \in T$, and $\rho \in \mathcal{R}^{(\kappa)}$, we have $c^{M_t} = \rho$ iff $\overline{M}_t \models \rho = c$, iff $x = c \in \rho(t)$ (by (1)), iff $x = c \in \rho(u)$ (as $\rho$ is a run), iff $\overline{M}_u \models \rho = c$ (by (1) again), iff $c^{M_u} = \rho$. Hence, $c^{M_t} = c^{M_u}$. So we can form the temporal $L$-structure $\mathfrak{M} = (\mathcal{R}^{(\kappa)}, (M_t)_{t\in T})$, since the interpretation of constants is rigid.

**Lemma 14** *For each $t \in T$, each subformula $\psi(\bar{x})$ of $\varphi$, and each tuple $\bar{\rho} \in \mathcal{R}^{(\kappa)}$, we have*

$$\mathfrak{M}, t \models \psi(\bar{\rho}) \iff \overline{M}_t \models \overline{\psi}(\bar{\rho}).$$

**Proof** By induction on $\psi$. For atomic $\psi$ the result is simple, because $\mathfrak{M}, t \models \psi(\bar{\rho})$ iff $M_t \models \psi(\bar{\rho})$; as $\overline{\psi} = \psi$ and $\overline{M}_t$ expands $M_t$, this is iff $\overline{M}_t \models \overline{\psi}(\bar{\rho})$. The boolean cases are straightforward, noting that $\overline{\psi \wedge \chi} = \overline{\psi} \wedge \overline{\chi}$ and $\overline{\neg\psi} = \neg\overline{\psi}$. We consider the case of quantifiers. Assuming the result for $\psi(\bar{x}, y)$, we have $\mathfrak{M}, t \models \exists y \psi(\bar{\rho}, y)$ iff $\mathfrak{M}, t \models \psi(\bar{\rho}, \sigma)$ for some $\sigma \in \mathcal{R}^{(\kappa)}$. Inductively, this is iff $\overline{M}_t \models \overline{\psi}(\bar{\rho}, \sigma)$ for some $\sigma \in \mathcal{R}^{(\kappa)}$. This is iff $\overline{M}_t \models \exists y \overline{\psi}(\bar{\rho}, y)$, iff $\overline{M}_t \models \overline{\exists y \psi}(\bar{\rho}, y)$.

The interesting case is of course the temporal operators. We only do $U$; $S$ is similar. Consider a subformula $\alpha U \beta(y)$ of $\varphi$. Let $\rho \in \mathcal{R}^{(\kappa)}$. We have $\mathfrak{M}, t \models \alpha U \beta(\rho)$ iff there is $u > t$ in $T$ with $\mathfrak{M}, u \models \beta(\rho)$ and $\mathfrak{M}, v \models \alpha(\rho)$ for all $v \in T$ with $t < v < u$. Inductively, this is iff there is $u > t$ in $T$ with $\overline{M}_u \models \overline{\beta}(\rho)$ and $\overline{M}_v \models \overline{\alpha}(\rho)$ for all $v \in T$ with $t < v < u$. It follows from (1) that this holds iff there is $u > t$ in $T$ with $\overline{\beta}(x/y) \in \rho(u)$ and $\overline{\alpha}(x/y) \in \rho(v)$ for all $v \in T$ with $t < v < u$. As $\rho$ is a run, this is iff $\overline{\alpha U \beta}(x/y) \in \rho(t)$. By (1) again, this is iff $\overline{M}_t \models \overline{\alpha U \beta}(\rho)$, as required. ❑

Since $((\Sigma_t)_{t\in T}, \mathcal{R})$ is a quasimodel for $\varphi$, there is $t \in T$ and $p \in \Sigma_t$ with $\overline{\varphi} \in p$. Choose $\rho \in \mathcal{R}$ with $\rho(t) = p$. There is at least one copy of $\rho$ in $\mathcal{R}^{(\kappa)}$; choose one, and denote it by $\rho$ too. Then $\overline{\varphi} \in \rho(t)$, so by (1), $\overline{M}_t \models \overline{\varphi}$. By lemma 14, we obtain $\mathfrak{M}, t \models \varphi$. Thus, $\varphi$ has a model. We note that if the given quasimodel is finitary then $\mathcal{R}$ is finite, and we chose the $N_t$ to be finite. Hence, $\kappa$ is finite and the domain $\mathcal{R}^{(\kappa)}$ of the resulting model $\mathfrak{M}$ is also finite. This completes the proof of theorem 12. ❑

### 3.3 Decidability

**Lemma 15** *It is decidable whether a state candidate $\Sigma$ for a sentence $\varphi$ of the monodic packed fragment is realizable.*

**Proof** Each type $p \in \Sigma$ satisfies $p \subseteq C_x\varphi \subseteq \mathsf{PF}(\overline{L})$. Therefore, $\bigwedge p(x) \in \mathsf{PF}(\overline{L})$. Hence, up to logical equivalence, the $\overline{L}$-sentence

$$\alpha_\Sigma = \Big(\bigwedge_{p\in\Sigma} \exists x \bigwedge p(x)\Big) \wedge \Big(\forall x \bigvee_{p\in\Sigma} \bigwedge p(x)\Big)$$

is in $\mathsf{PF}(\overline{L})$ too. (The guards for $\exists x, \forall x$ here can be taken to be $x = x$.) Evidently, $\Sigma$ is a realizable state candidate iff $\alpha_\Sigma$ has a model, and this question is decidable [7, 3]. ❑

We can now prove theorem 1.

**Proof [of theorem 1]** Let $\varphi$ be a sentence of the monodic packed fragment. We will express the existence of a quasimodel for $\varphi$ by a sentence of monadic second-order logic that can be constructed effectively and uniformly from $\varphi$.

Form $L, \mathcal{C}_x\varphi$, etc., as before; this can be done effectively. Introduce a unary predicate variable $Q_\psi$ for each $\psi \in \mathcal{C}_x\varphi$. If $p$ is a type, let

$$\chi_p(t) \quad = \quad \bigwedge_{\psi \in p} Q_\psi(t) \quad \wedge \bigwedge_{\psi \in (\mathcal{C}_x\varphi)\setminus p} \neg Q_\psi(t),$$

saying that the $Q_\psi(t)$ define the type $p$ at $t$. Let $\rho$ denote the conjunction of the following three formulas, saying that the $Q_\psi$ define a run. For brevity, we write $\overline{\alpha U \beta}$ instead of $\overline{\alpha U \beta}(x/y)$, and similarly for $\alpha S \beta, \alpha, \beta$.

$$\begin{aligned}
&\forall tu \bigwedge_{x=c \in \mathcal{C}_x\varphi} (Q_{x=c}(t) \leftrightarrow Q_{x=c}(u)),\\
&\forall t \bigwedge_{\overline{\alpha U \beta} \in \mathcal{C}_x\varphi} \Big(Q_{\overline{\alpha U \beta}}(t) \leftrightarrow \exists u (t < u \wedge Q_{\overline{\beta}}(u) \wedge \forall v (t < v < u \to Q_{\overline{\alpha}}(v)))\Big),\\
&\forall t \bigwedge_{\overline{\alpha S \beta} \in \mathcal{C}_x\varphi} \Big(Q_{\overline{\alpha S \beta}}(t) \leftrightarrow \exists u (u < t \wedge Q_{\overline{\beta}}(u) \wedge \forall v (u < v < t \to Q_{\overline{\alpha}}(v)))\Big).
\end{aligned}$$

By lemma 15, we may effectively construct from $\varphi$ the finite set $\mathfrak{S}$ of all realizable state candidates for $\varphi$. For each $\Sigma \in \mathfrak{S}$ let $P_\Sigma$ be a unary predicate variable. We now define the monadic second-order sentence $\sigma_\varphi$ as follows:

$$\begin{aligned}
\exists_{\Sigma \in \mathfrak{S}} P_\Sigma \Bigg( \forall t \bigvee_{\Sigma \in \mathfrak{S}} \Big[P_\Sigma(t) \wedge \bigwedge_{\Sigma \neq \Sigma' \in \mathfrak{S}} \neg P_{\Sigma'}(t)\Big] \quad \wedge \quad \bigvee \{\exists t\, P_\Sigma(t) : \Sigma \in \mathfrak{S},\ \overline{\varphi} \in \bigcup \Sigma\} \\
\wedge\ \forall t \bigwedge_{p \in \Sigma \in \mathfrak{S}} \Big[P_\Sigma(t) \to \exists_{\psi \in \mathcal{C}_x\varphi} Q_\psi \Big(\rho \wedge \chi_p(t) \wedge \forall u \bigwedge_{\Sigma' \in \mathfrak{S}} \Big[P_{\Sigma'}(u) \to \bigvee_{q \in \Sigma'} \chi_q(u)\Big]\Big)\Big]\Bigg).
\end{aligned}$$

Evaluated in a flow of time $T$, $\sigma_\varphi$ says that the $P_\Sigma$ ($\Sigma \in \mathfrak{S}$) partition $T$. Thus, a unique realizable state candidate $\Sigma_t$ is allotted to every $t \in T$. This, $\sigma_\varphi$ says, is done so that some type in some $\Sigma_t$ contains $\overline{\varphi}$, and for every $t \in T$ and every type $p \in \Sigma_t$, there exists a run (described by the $Q_\psi$) in $(\Sigma_t)_{t \in T}$ that hits $p$ at $t$. Thus, $T \models \sigma_\varphi$ iff there exists a quasimodel for $\varphi$ over $T$ — for '$\Rightarrow$', let the quasimodel be $((\Sigma_t)_{t \in T}, \mathcal{R})$, where $\mathcal{R}$ is the set of all runs in $(\Sigma_t)_{t \in T}$. By theorem 12, this is iff $\varphi$ has a model with flow of time $T$.

Clearly, $\sigma_\varphi$ is effectively constructible from $\varphi$. So to decide whether $\varphi$ has a model with flow of time $T$, we only need decide whether $T \models \sigma_\varphi$, and this can be done by applying standard results on decidability of monadic second-order logic.

We use this idea to establish theorem 1. We will be brief; see [6] for more details. The statements '$(\mathbb{N}, <) \models \sigma_\varphi$' and '$(\mathbb{Z}, <) \models \sigma_\varphi$' are decidable, by a result of Büchi [1]. It also follows from Büchi's theorem that it is decidable whether $\sigma_\varphi$ is true in some finite linear order. The decidability of '$(\mathbb{Q}, <) \models \sigma_\varphi$' follows from Rabin's theorem on the decidability of *S2S* [8].

Finally, let $\mathcal{F}$ be a class of strict linear orders definable by a first-order sentence $\chi$ of linear order. By considering the 'standard translation' of $\varphi$ into two-sorted first-order logic and applying the downward Löwenheim–Skolem–Tarski theorem, it can be seen that $\varphi$ has a model with flow of time in $\mathcal{F}$ iff it has a model with *countable* flow of time in $\mathcal{F}$. This is iff $\sigma_\varphi$ is true in some countable order in $\mathcal{F}$. By relativising $\sigma_\varphi$ to a unary predicate variable which is stated to satisfy $\chi$, we can obtain a new monadic second-order sentence $\sigma_\varphi^{(\chi)}$, effectively constructible from $\varphi$, and such that $\sigma_\varphi$ is true in some countable order in $\mathcal{F}$ iff $(\mathbb{Q},<) \models \sigma_\varphi^{(\chi)}$. By Rabin's theorem, this last statement is decidable. ❑

[6] gives a decision procedure over $(\mathbb{N},<)$ of elementary complexity.

**Proof [of theorem 2]** Using theorem 12 and the arguments of [6, theorem 36, corollary 37][1], it suffices to show that it is decidable whether a state candidate $\Sigma$ is finitely realizable. But $\Sigma$ is finitely realizable iff $\alpha_\Sigma$ (as in lemma 15) has a finite model. The packed fragment has the finite model property [4, 5], so this is iff $\alpha_\Sigma$ has a model. The result now follows from the decidability of the packed fragment. ❑

**Remark 16** Since the packed fragment has the finite model property, any realizable state candidate is finitely realizable. However, not every satisfiable monodic packed sentence is true in some model with finite domain. It is readily checked that the monodic packed sentence

$$\varphi = \Box\exists x(P(x) \wedge \neg(\top S P(x)))$$

is satisfied in the model $\mathfrak{M} = (\mathbb{N},(M_n)_{n\in\mathbb{N}})$ with flow of time $(\mathbb{N},<)$, domain $\mathbb{N}$, and with $P^{M_n} = \{n\}$, but is false in all models over $(\mathbb{N},<)$ with finite domains. Here, $\Box\psi$ abbreviates $\psi \wedge \neg(\top U \neg\psi \vee \top S \neg\psi)$, stating that $\psi$ is always true. Cf. [6, theorem 25]. The set of runs in any quasimodel for $\varphi$ over an infinite flow of time must be infinite.

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[1] Minor alterations are needed because the definitions have changed to accommodate equality; this has allowed a simpler definition of 'state candidate' than in [6].