Axiomatizing hybrid logic using modal logic

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Abstract

We study hybrid logics with nominals and 'actuality' operators $@_i$.
We recall the method of ten Cate, Marx, and Viana to simulate hybrid logic using modalities and 'nice' frames, and we show that the hybrid logic of a class of frames is the modal logic of the class of its corresponding nice frames.
Using these results, we show how to axiomatize the hybrid logic of any elementary class of frames.
Then we study *quasimodal logics*, which are hybrid logics axiomatized by modal axioms together with basic hybrid axioms common to any hybrid logic, using only orthodox inference rules. We show that the hybrid logic of any elementary modally definable class of frames, or of any elementary class of frames closed under disjoint unions, bounded morphic images, ultraproducts and generated subframes, is quasimodal. We also show that the hybrid analogues of modal logics studied by McKinsey–Lemmon and Hughes are quasimodal.

Keywords: Axiomatization, quasimodal logic, nice frame, McKinsey, Lemmon, Hughes.

1 Introduction

The broad aim of this paper is to study the process of axiomatising hybrid logics, using modal methods. There are two main strands. First, we generalise to hybrid logic the method of [6] for axiomatising the modal logic of any elementary class of Kripke frames. Second, we begin a study of how we can axiomatize the hybrid logic of a class K of frames *if we are given the modal logic of K as 'free' axioms.*

Different approaches have been taken in the literature to try to axiomatize the hybrid logic of a class of frames. For example, we can find in [3] axiomatizations for the hybrid logic with operator @ alone, @ and ↓, @ and ∀, etc. The axiomatizations for @ use 'non-orthodox' Burgess–Gabbay-style inference rules, such as Name and BG, and cover only classes of frames definable by pure formulas (i.e., hybrid formulas containing no propositional variables, but only nominals). Further classes of frames can be covered by adding more non-orthodox rules.

In [2], the authors give a slightly different way to axiomatize the hybrid logic of a class of frames characterised by a set of pure formulas. However, the axiomatization is done from scratch: even though hybrid logic is based on modal logic, this axiomatization does not directly take advantage of modal completeness theorems. Moreover, extra 'non-orthodox' inference rules are still needed to make the axiomatization complete.

An interesting result can be found in [10]. In this article, a theorem states that any class of frames defined by Sahlqvist formulas can be axiomatized in a *smart* way, in the sense that the axiomatization reflects the fact that hybrid logic is based on modal logic, and uses modal completeness theorems. No non-orthodox rules are needed in this case. [10] also shows that pure and Sahlqvist axioms cannot be readily combined. So there are limits to the classes of frames that are covered.

As we can see, there are two gaps in these results. First, we know that we can axiomatize the class of all frames, any class of frames axiomatized by modal Sahlqvist formulas, and any class of frames defined by pure sentences (if we add extra rules). We would like to be able to axiomatize more classes of frames, preferably using only orthodox inference rules. To this end, we use *nice frames*, which have been proposed by ten Cate et al. (in [10]) as a way to simulate hybrid operators using modal logic. Using classes of nice frames, we show how to axiomatize, with orthodox rules, the hybrid logic of any elementary class of frames.

Second, we would like to separate, in an axiomatization, the hybrid part from the modal part. We would like to know when it is possible to make the axiomatization reflect the fact that hybrid logic is based on modal logic, and when we cannot. So we study the following question: given a class of frames, and its modal axiomatization, what further axioms do we need to axiomatize the hybrid logic of this class? A first step towards an answer will be presented as *quasimodal classes*, in section 5. These are classes of frames whose hybrid logic is axiomatized by their modal logic together with only basic hybrid axioms, expressing the meaning of the hybrid operators and valid in all frames. We give some criteria for a class to be quasimodal, and consider some examples, including interesting classes of frames whose modal logics were introduced and studied by McKinsey–Lemmon and Hughes.

Scope of this paper

In this paper, the modal signature is a single box with accessibility relation noted R. The hybrid logic we deal with is $\mathcal{H}(@)$, i.e., the hybrid logic with nominals and 'actuality' operators @. We do not study operators such as $\downarrow$, $\exists$, etc.

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For convenience, and to fix notation, in section 2 we recall some standard definitions. In section 3, we present and extend *nice frames*, which have been proposed by ten Cate et al. (in [10]) as a way to simulate hybrid operators using modal logic. Section 4 gives a method of axiomatizing the hybrid logic of any elementary class of frames. We study in section 5 the hybrid axiomatization of classes of frames, when a known modal axiomatization is given. At last, in section 6, we apply results of this latter section to modal logics studied by McKinsey–Lemmon and Hughes.

2 Basic definitions

We assume familiarity with modal logic ([2] has ample information). Here, we recall the basics of hybrid logic, and set up some notation. Throughout, we fix a countably infinite set $\mathcal{P}$ of propositional letters, and a countable non-empty set $\mathcal{N}$ of nominals. Any element of $\mathcal{P} \cup \mathcal{N}$ is a hybrid $\mathcal{H}(@)$-formula, as is $\top$; if ϕ, ψ are $\mathcal{H}(@)$-formulas, so are $\neg\phi$, $\phi \wedge \psi$, and $\Box\phi$; and if $i \in \mathcal{N}$ then $@_i\phi$ is also an $\mathcal{H}(@)$-formula. No other things are hybrid $\mathcal{H}(@)$-formulas.

Semantics of $\mathcal{H}(@)$-formulas is given by models of the form $\mathcal{M} = \langle W, R, V \rangle$, where $\langle W, R \rangle$ is a Kripke frame and $V : \mathcal{P} \cup \mathcal{N} \to \wp(W)$ (here, $\wp$ denotes power set) is a valuation such that $|V(i)| = 1$ for each $i \in \mathcal{N}$. We write the hybrid satisfaction relation as $\models_h$ in order to distinguish it from ordinary modal evaluation which we write as $\models_m$. For $w \in W$ and a $\mathcal{H}(@)$-formula ϕ, we define $\mathcal{M}, w \models_h \phi$ by induction on ϕ: for $x \in \mathcal{P} \cup \mathcal{N}$, we let $\mathcal{M}, w \models_h x$ iff $w \in V(x)$; the boolean connectives are handled as usual; $\mathcal{M}, w \models_h \Box\phi$ iff $\mathcal{M}, v \models_h \phi$ for every v with $(w, v) \in R$; and $\mathcal{M}, w \models_h @_i\phi$ iff $\mathcal{M}, v \models_h \phi$, where $V(i) = \{v\}$. A $\mathcal{H}(@)$-formula ϕ is valid in $\mathcal{M}$ if $\mathcal{M}, w \models_h \phi$ for all worlds w of $\mathcal{M}$, valid in a frame $\mathfrak{F}$ if it is valid in all models with frame $\mathfrak{F}$, and valid in a class K of frames if it is valid in all frames in K. We use the notations $\mathcal{M} \models_h \phi$, $\mathfrak{F} \models_h \phi$, $K \models_h \phi$, respectively, for these notions.

Notation: for a set S of modal formulas and a modal formula ϕ, we write $S \vdash_m \phi$ if ϕ is provable using as axioms all propositional tautologies, normality of $\Box$ (i.e., $\Box(p \to q) \to (\Box p \to \Box q)$), and formulas in S, and using the standard inference rules of modus ponens, substitution, and box-generalization (from ϕ, derive $\Box\phi$). Formally, any set of modal formulas containing the axioms and closed under the rules contains ϕ.[1] We will define a hybrid variant $\vdash_h$ later (definition 3.8).

3 Hybrid Extension of Modal Classes

As hybrid logic is close to modal logic, it would be interesting if one could apply modal theorems to hybrid logic. Ten Cate et al. have, in [10], defined nice frames, which give a way to simulate hybrid operators using modal logic, and so to apply results from modal logic to hybrid logic. In this section, we extend this method.

3.1 Nice Frames

We recall here how [10] proposed to treat hybrid operators as modalities.

Definition 3.1 (from [10, §3]) A *non-standard* frame is a frame of the form $\mathfrak{F} = \langle W, R, (R_i)_{i \in \mathcal{N}}, (S_i)_{i \in \mathcal{N}} \rangle$, where R and the R_i are binary relations on W, and S_i are unary relations on W.

This is in contrast with what we will call *standard frames,* which are ordinary Kripke frames of the form $\langle W, R \rangle$. The idea behind non-standard frames is that for

[1] Warning: this notation is not standard. We are *not* using it to mean that $\vdash_m \sigma_1 \wedge \ldots \wedge \sigma_n \to \phi$ for some $\sigma_1, \ldots, \sigma_n \in S$. For example, for distinct $p, q \in \mathcal{P}$, we have $p \vdash_m q$ in our notation but not in this one.

any nominal i, $@_i$ is treated as a unary modality with accessibility relation R_i, and i is treated as a nullary modality with accessibility relation S_i. Thus, the semantics associated to non-standard frames is:

$$\begin{aligned} \mathcal{M}, w \models_m i \quad &\text{iff } w \in S_i \\ \mathcal{M}, w \models_m @_i\phi \ &\text{iff } \forall w'(wR_iw' \implies \mathcal{M}, w' \models_m \phi), \end{aligned}$$

where w is a world of a model $\mathcal{M}$ based on a non-standard frame.

A non-standard frame is said to be *nice* if for each $i \in \mathcal{N}$, S_i is a singleton and $\forall xy(R_ixy \leftrightarrow S_iy)$ is true. A model is said to be *non-standard* (respectively *nice*) if its frame is non-standard (resp. nice).

The following lemma shows that evaluating a formula in a nice frame is, in a certain sense, equivalent to evaluating it in the hybrid valuation.

Lemma 3.2 *(from [10, lemma 3.2])*
Let $\mathcal{M}_m = \langle W, R, (R_i)_{i\in\mathcal{N}}, (S_i)_{i\in\mathcal{N}}, V_m\rangle$ be a nice model, where $V_m : \mathcal{P} \to \wp(W)$, and let $\mathcal{M}_h = \langle W, R, V_h\rangle$ (where $V_h = V_m \cup \{(i, S_i) \mid i \in \mathcal{N}\}$) be its corresponding hybrid model. Then, for any world $w \in W$ and hybrid formula ϕ, we have:

$$\mathcal{M}_h, w \models_h \phi \text{ iff } \mathcal{M}_m, w \models_m \phi.$$

Definition 3.3 (from [10, §2]) We call Δ the following set of axioms, for all $i, j \in \mathcal{N}$ and $p \in \mathcal{P}$:

(agree)	$@_ip \to @_j@_ip$
(propagation)	$@_ip \to \Box@_ip$
(elimination)	$(@_ip \wedge i) \to p$
(ref)	$@_ii$
(self-dual)	$@_ip \leftrightarrow \neg@_i\neg p$

We now extend [10, lemma 3.3]. The proof is straightforward.[2]

Lemma 3.4 *For any non-standard frame $\mathfrak{F}$, $\mathfrak{F}$ is nice iff $\mathfrak{F}$ is weakly connected (i.e. there exists a path between any two worlds using any relation or reverse relation of $\mathfrak{F}$) and $\mathfrak{F}$ validates Δ.*

3.2 Hybrid Extension

An important limitation of nice frames is that evaluation of nominals is bound to the frame, whereas in hybrid logic, it is bound to the model. So, an hybrid formula valid, in the non-standard semantics, in a frame, is only valid, in the hybrid semantics, in some of the models that can be built over this frame: those that agree on the valuation of the nominals. The idea we develop here to solve this problem is

[2] You can find in [9] the full proofs of some propositions and theorems for which we only give sketches or omit proofs in this paper.

to consider, in the same class of nice frames, all the possible evaluations of nominals of a given modal frame. This will be done in *hybrid extensions*.

Before defining this extension, we give the following notations. Given a class K of standard frames, we call ML(K) the modal logic of K: that is, the set of all modal formulas valid in K. The logic HL(K) is the hybrid logic of K. At last, L being a class of non-standard frames, NS(L) (for Non-Standard) is the hybrid logic that is valid in L in the non-standard semantics, i.e., NS(L) is the set of hybrid formulas valid in L when $@_i$ and i (for all $i \in \mathcal{N}$) are interpreted as modalities.

In other words:

$$\begin{aligned} \mathrm{ML}(K) &= \{\text{modal formulas } \phi : K \models_m \phi\} \\ \mathrm{HL}(K) &= \{\text{hybrid } \mathcal{H}(@)\text{-formulas } \phi : K \models_h \phi\} \\ \mathrm{NS}(L) &= \{\text{hybrid } \mathcal{H}(@)\text{-formulas } \phi : L \models_m \phi\} \end{aligned}$$

Definition 3.5 Let K be a class of standard frames. The *hybrid extension* of K is the class $\mathcal{H}(K)$ of nice frames that can be built over K:

$$\mathcal{H}(K) = \{\text{nice frames } \langle W, R, (R_i)_{i\in\mathcal{N}}, (S_i)_{i\in\mathcal{N}}\rangle \text{ such that } \langle W, R\rangle \in K\}.$$

Let $\mathfrak{F} = \langle W, R, (R_i)_{i\in\mathcal{N}}, (S_i)_{i\in\mathcal{N}}\rangle$ be a non-standard frame. The *modal reduct* of $\mathfrak{F}$ is the standard frame $\mathfrak{F}^{\mathsf{M}} = \langle W, R\rangle$.

Given this definition, lemma 3.2, which described in which sense we can consider that non-standard and modal semantics are equivalent, can be extended to classes.

Theorem 3.6 *For all classes K of standard frames, the hybrid logic of K is the modal logic of $\mathcal{H}(K)$:*

$$\mathrm{NS}(\mathcal{H}(K)) = \mathrm{HL}(K).$$

Proof (sketch) We know by lemma 3.2 that evaluating a formula in a non-standard model or in the corresponding hybrid model are equivalent. From this, we can show a similar result with class of frames, that is: K being a class of standard frames, and ϕ an hybrid formula, $K \models_h \phi$ iff $\mathcal{H}(K) \models_m \phi$. Then, it is straightforward to extend this result to a whole logic instead of a single formula, and we get $\mathrm{NS}(\mathcal{H}(K)) = \mathrm{HL}(K)$. □

We show in the next theorem that hybrid extensions also preserve the fact that a class is elementary. We introduce notation $\models_f$ which describes first-order evaluation.

Theorem 3.7 *K being an elementary class of standard frames, its hybrid extension $\mathcal{H}(K)$ is elementary.*

Proof. Suppose $K = \{\mathfrak{F} : \mathfrak{F} \models_f T\}$, where T is a first-order theory in the signature $\{R\}$, and let K' be the class of non-standard models of $T' = T \cup H$, where:

$$H = \bigcup_{i\in\mathcal{N}} \{\forall xy(R_i(x,y) \leftrightarrow S_i(y)), \forall xy((S_i(x) \land S_i(y)) \rightarrow x = y), \exists x S_i(x)\}$$

Let us show that $K' = \mathcal{H}(K)$.
($\supseteq$) By definition, frames of $\mathcal{H}(K)$ are nice, so they validate H. By definition, K validates T, so, $\mathcal{H}(K)$ validates T as well ($\mathcal{H}(K)$ only adds relations to K, that do not occur in T). So $\mathcal{H}(K) \subseteq K'$.
($\subseteq$) Frames of K' validate H, so they are nice. They validate T by definition.
Let $\mathfrak{F} = \langle W, R, (R_i), (S_i) \rangle$ be a frame of K'. $K' \models_f T$, so $\mathfrak{F} \models_f T$. But formulas of T are first-order formulas that do not use R_i and S_i, so $\mathfrak{F}^{\mathsf{M}} \models_f T$ as well. Thus, $\mathfrak{F}^{\mathsf{M}} \in K$, by definition, and, as $\mathfrak{F}$ is nice, $\mathfrak{F} \in \mathcal{H}(K)$. □

The converse theorem is also true, but we will not use it in this article, and the proof is longer. It can be found in [9].

3.3 *Hybrid and non-standard modal provability*

These notions will be needed in the following sections.

Definition 3.8 For a set S of $\mathcal{H}(@)$-formulas and an $\mathcal{H}(@)$-formula ϕ, we write $S \vdash_h \phi$ if ϕ is provable using as axioms all propositional tautologies, normality of $\Box$ and $@_i$ ($i \in \mathcal{N}$), and formulas in $S \cup \Delta$, using the rules modus ponens, sorted substitution (if ϕ is derivable then so is any formula built from ϕ by uniformly replacing proposition letters by arbitrary formulas and nominals by nominals), box-generalization, and @-generalization (from ϕ, derive $@_i\phi$ for any $i \in \mathcal{N}$). This system is as in [10]. (See footnote 1.)

We write $S \vdash_m^{ns} \phi$ if $S \vdash_m \phi$ when nominals and $@_i$s are regarded as nullary (respectively, unary) modalities. (So the axioms and rules of $\vdash_m$ of section 2 are augmented by normality and generalization for each $@_i$.)

To remind ourselves that Δ is included as basic axioms of $\vdash_h$, we will often say that a hybrid logic of the form $\{\phi \in \mathcal{H}(@) : \Lambda \vdash_h \phi\}$ is *axiomatized by* $\Lambda \cup \Delta$.

Lemma 3.9 *Let S be a set of $\mathcal{H}(@)$-formulas that is closed under substitution of nominals in its formulas (that is, if $\phi \in S$ then $\phi' \in S$ for any ϕ' obtained from ϕ by substitution of nominals for nominals). Let ϕ an $\mathcal{H}(@)$-formula. The following are equivalent:*

- (i) $S \vdash_h \phi$,
- (ii) $S \cup \Delta \vdash_m^{ns} \phi$.

Proof. The only difference between $S \cup \Delta \vdash_m^{ns}$ and $S \vdash_h$ is that substitution of nominals for nominals is an inference rule of only the latter. Still, an easy induction on length of a formal proof will show that if $S \cup \Delta \vdash_m^{ns} \phi$ then $S \vdash_h \phi$, and if $S \vdash_h \phi$ then $S \cup \Delta \vdash_m^{ns} \phi'$ for any ϕ' obtained from ϕ by substitution of nominals for nominals. This is enough to prove the lemma. □

4 Elementarily Generated Hybrid Logics

[6] gave a method of constructing an axiomatization of the modal logic of any elementary class of frames. We will use this result, together with the hybrid extension we just defined, to axiomatize the hybrid logic of any elementary class of frames.

Theorem 4.1 *For any elementary class of frames K, one can construct a set Λ of axioms for $HL(K)$ from first-order sentences defining K.*

Proof. Let K be an elementary class of frames. By theorem 3.7, we know that $\mathcal{H}(K)$ is elementary. Thus, [6, theorem 5.16] says that there exist a modal axiomatization Λ of $NS(\mathcal{H}(K))$, constructed from first-order sentences defining $\mathcal{H}(K)$, which can themselves be constructed from first-order sentences defining K (see proof of theorem 3.7). Formally, $NS(\mathcal{H}(K)) = \{\phi \in \mathcal{H}(@) : \Lambda \vdash_m^{ns} \phi\}$. This remains true if we close Λ under substitution of nominals for nominals, since by definition of $\mathcal{H}(K)$, such substitutions do not affect validity in $\mathcal{H}(K)$.

By theorem 3.6, $NS(\mathcal{H}(K))$ is the hybrid logic of K. Moreover, $NS(\mathcal{H}(K))$ is an *hybrid* logic, in the sense that it contains the axioms Δ (for frames of $\mathcal{H}(K)$ are nice, so they validate Δ). So $\Lambda \vdash_m^{ns} \delta$ for every $\delta \in \Delta$, which means that '$\Lambda \vdash_m^{ns}$' is the same as '$\Lambda \cup \Delta \vdash_m^{ns}$'. Therefore, $HL(K) = NS(\mathcal{H}(K)) = \{\phi \in \mathcal{H}(@) : \Lambda \vdash_m^{ns} \phi\} = \{\phi \in \mathcal{H}(@) : \Lambda \cup \Delta \vdash_m^{ns} \phi\}$. By lemma 3.9, this is $\{\phi \in \mathcal{H}(@) : \Lambda \vdash_h \phi\}$, as required. □

Of course, the axiomatization we obtain is always infinite, and not always transparent. See [6] for further discussion.

A remark about the result of [6] we use here is necessary. Although in [6], the case of nullary modalities is unfortunately not explicitly considered, we do need it here, as the S_i have arity 0. However, [6] should work for this case as well: we do not see what would go wrong with nullary modalities.

Otherwise, we can also simulate i in a different way: we define the unary modality i with the following semantics:

$$\mathfrak{F}, V, x \models_m i(\phi) \text{ iff } \forall y \text{ if } S'_i(x, y) \text{ then } \mathfrak{F}, V, y \models_m \phi$$

where S'_i is defined by: $S'_i(x, y)$ iff $x = y$ and $S_i(x)$.

Then, the nominal i would we written $i(\top)$, and we would eventually get: $\mathfrak{F}, V, x \models_m i(\top)$ iff $S_i(x)$.

5 Quasimodal Classes

Hybrid logic is defined as a layer over modal logic. However, axiomatization of the hybrid logic of classes of frames is often done from scratch, without separating modal axioms from hybrid ones.

In this section, we study the hybrid logic of classes of frames whose modal logic is already known. That is, given a class of frames and its modal axiomatization, which hybrid axioms do we need to add to get the hybrid logic of the same class of frames? Some classes will only need Δ, which describes the hybrid operators, but some others may need more axioms, which means that some properties of the frames could not be expressed using modal logic only. So, we obtain a measure of how much more information than the modal logic is carried by the hybrid logic of the class.

As a first attempt, we propose here to study cases where no hybrid axioms are needed, except Δ, which defines hybrid logic. In such logics, which will be called *quasimodal logics*, hybrid semantics does not bring anything more, concerning axiomatization, than its own definition.

Definition 5.1

(i) An hybrid logic Λ is *quasimodal*[3] if it is axiomatized by modal axioms (that is, without nominals or @-operators), together with Δ (which, recall, is included as a set of basic axioms in $\vdash_h$). Formally, Λ is quasimodal if there exists a set Θ of modal formulas such that $\Lambda = \{\phi \in \mathcal{H}(@) : \Theta \vdash_h \phi\}$.

(ii) We say that a class of standard frames is *quasimodal* if its hybrid logic is quasimodal.

As an attempt to characterise quasimodal classes of frames, we will show in theorem 5.4 that a class of frames closed under disjoint union, bounded morphism, generated subframes and ultraproducts is quasimodal. First we prove a useful lemma.

Lemma 5.2 *Let K be a class of standard frames that contains the canonical frame of* ML(K)*. Then* HL(K) *is quasimodal.*

Sketch of proof Let $\Lambda = \{\phi \in \mathcal{H}(@) : ML(K) \vdash_h \phi\}$ be the hybrid logic axiomatized by ML$(K) \cup \Delta$. We show that $\Lambda =$ HL(K).

First, as ML$(K) \subseteq$ HL(K), by soundness of $\vdash_h$ we get: $\Lambda \subseteq$ HL(K).

For the converse inclusion, consider an hybrid formula ϕ not in Λ. By lemma 3.9, $\Lambda \cup \{\neg\phi\}$ is modally consistent in the non-standard sense. Let $\mathcal{C}$ be the canonical model of Λ in the language $\mathcal{H}(@)$, in the non-standard sense. Let $\mathcal{M}$ be a submodel of $\mathcal{C}$, generated by a world in which $\neg\phi$ is true. Let $\mathfrak{F}$ be the (non-standard) frame of $\mathcal{M}$. Since Δ is Sahlqvist and valid in $\mathcal{M}$, it is valid in $\mathfrak{F}$. As $\mathfrak{F}$ is weakly connected, by lemma 3.4 we see that $\mathfrak{F}$ is nice. We know that $\mathcal{M}$ does not validate ϕ in the non-standard modal sense. So by lemma 3.2, ϕ is not valid in the modal reduct $\mathfrak{F}^{\mathsf{M}}$ of $\mathfrak{F}$ in the hybrid sense (i.e., $\mathfrak{F}^{\mathsf{M}} \not\models_h \phi$).

For each $i \in \mathcal{N}$, introduce a new propositional letter q_i, and for each formula $@_i\psi$ of $\mathcal{H}(@)$, introduce a new propositional letter $q_{@_i\psi}$. Let $\mathcal{Q}$ be $\mathcal{P}$ together with these new letters; it is countable. Define, for each $\psi \in \mathcal{H}(@)$, a modal formula $\psi{\downarrow}$, by induction: $i{\downarrow} = q_i$ for $i \in \mathcal{N}$, $p{\downarrow} = p$ for $p \in \mathcal{P}$, $\top{\downarrow} = \top$, $\downarrow$ commutes with the boolean operators and $\Box$, and $(@_i\psi){\downarrow} = q_{@_i\psi}$. It can be checked that for each world w of $\mathcal{M}$, the set $w{\downarrow} = \{\psi{\downarrow} : \psi \in w\}$ is a maximal consistent set in the modal logic $ML(K)$, and that the map $(w \mapsto w{\downarrow})$ is an injective bounded morphism from $\mathfrak{F}^{\mathsf{M}}$ into the canonical frame $\mathfrak{G}$ (say) of $ML(K)$ based on $\mathcal{Q}$. So $\mathfrak{F}^{\mathsf{M}}$ is isomorphic to a generated subframe of $\mathfrak{G}$. As hybrid $\mathcal{H}(@)$-validity is preserved under generated subframes, $\mathfrak{G} \not\models_h \phi$. But we are given that $\mathfrak{G} \in K$. So $\phi \notin HL(K)$. □

Remark 5.3 Ten Cate et al. proved the following statement in [10, theorem 3.4]. *Let Σ be a set of modal Sahlqvist formulas not containing nominals or satisfaction operators. Then, the hybrid logic axiomatized by $\Sigma \cup \Delta$ is sound and strongly complete for the class of frames defined by Σ.* We show that this theorem is a particular case of the previous lemma.

[3] "Quasimodal" logic (and classes) were called "stable" in [9]

Let K be the class of frames defined by Σ. We show that we can apply lemma 5.2 on K. By Sahlqvist's completeness theorem, the canonical frame for $\mathrm{ML}(K)$ validates Σ, thus this canonical frame is in K. Therefore, by lemma 5.2, the hybrid logic of K is axiomatized by $\Delta \cup \Sigma$.

From now on, *nearly all frames mentioned are standard modal frames,* so we drop the adjective 'standard'. We can now give some properties that make a class of frames quasimodal. They show that quasimodal frame classes and hybrid logics are rather common. We call *the three fundamental frame constructions* the following operations: generated subframes, bounded morphic images, and disjoint unions (these operations are known to preserve modal validity).

Theorem 5.4

(i) If Λ is a canonical modal logic, then the class of all frames that validate Λ is quasimodal.

(ii) A class of frames closed under the three fundamental frame constructions and ultraproducts is quasimodal.

(iii) A modally definable elementary class of frames is quasimodal.

Proof. Part (i) is immediate from lemma 5.2.

Let K be a frame matching conditions of (ii). By a result of Goldblatt [4, (4.9), p. 580], the canonical frame of $\mathrm{ML}(K)$ is in $\mathbb{SHU}d\mathbb{P}uK = K$ (for a class L of frames, $\mathbb{S}L$, $\mathbb{H}L$, $\mathbb{U}dL$ and $\mathbb{P}uL$ represent, respectively, the class of generated subframes, bounded morphic images, disjoint unions, and ultraproducts of frames of L). By lemma 5.2, K is quasimodal.

Part (iii) follows, since any class of frames satisfying the hypotheses of (ii) also satisfies the hypotheses of (ii). □

Remark 5.5 The theorem is not a characterisation, as one can easily find a quasimodal class that is not modally definable, and a quasimodal class that is not closed under the three fundamental operations.

Example 5.6 This theorem can be applied to prove that the following classes of frames are quasimodal: reflexive frames, euclidean frames, transitive frames, dense frames, symmetric frames, frames validating K4.1, the class of all frames, and the empty class.

The previous theorem can be used to show the quasi-modality of some classes, as we did in the previous example. Here is a theorem that will be used to prove that some classes are not quasimodal.

Theorem 5.7 *Let Λ be the modal logic of the quasimodal class K of frames, and let L be a class of frames containing K and having the same modal logic Λ.*

(i) K and L have the same hybrid logic.

(ii) L is quasimodal.

Proof.

(i) $(\supseteq)$ $K \subseteq L$, so $\mathrm{HL}(K) \supseteq \mathrm{HL}(L)$.

($\subseteq$) As K is quasimodal, its hybrid logic is axiomatized by $\Lambda \cup \Delta$: that is, $\Lambda \vdash_h \mathrm{HL}(K)$. But L, by hypothesis, validates Λ. And by definition of the hybrid semantics, it validates Δ as well. Thus, L validates $\Lambda \cup \Delta$, which entails $\mathrm{HL}(K)$. So $\mathrm{HL}(K) \subseteq \mathrm{HL}(L)$.

(ii) We just proved that K and L have the same hybrid logic. As K is quasimodal, L is quasimodal as well.

□

Two straightforward applications of this theorem will be used to give some examples of non-quasimodal classes.

Corollary 5.8

(i) *K being a class of frames, if there exists a frame that validates* $\mathrm{ML}(K)$ *and invalidates* $\mathrm{HL}(K)$*, then K is not quasimodal.*

(ii) *Let K be a class of frames, and ϕ an hybrid formula valid in K. If ϕ is not valid in the closure of K under the three fundamental operations and ultraproducts, then K is not quasimodal.*

Proof. (i) Suppose that there exists a frame $\mathfrak{F}$ that validates $ML(K)$ but not $\mathrm{HL}(K)$. Then $K \cup \{\mathfrak{F}\}$ has the same modal logic as K, but not the same hybrid logic. So by theorem 5.7, K cannot be quasimodal.

(ii) The closure of K under the operations contains K, and these classes have the same modal logic $\mathrm{ML}(K)$, since these operations preserve modal validity. If K were quasimodal, theorem 5.7 would be applicable, and K and its closure would have the same hybrid logic, which is false because of ϕ. □

We show an application of this corollary in the following example, in which we can use either part (i) or (ii) of the corollary.

Example 5.9 As $i \to \Box\neg i$ characterises irreflexivity and is not preserved by bounded morphic images, the class of irreflexive frames is not quasimodal.

By a *tree,* we mean an acyclic frame that contains an unique root from which each world is accessible, and in which every world which is not the root has a unique predecessor. The class of all trees is not quasimodal. The formula $(@_i\Diamond k \wedge @_j\Diamond k) \to @_i j$ (which states that a world cannot have two ancestors) is valid in every tree. However, using the three fundamental frame constructions applied to trees, one can build a frame in which this formula can be falsified.

The class of worlds having at most n worlds (for each $n \geq 1$) is characterised by $\bigvee_{0 \leq x < y \leq n} @_{i_x} i_y$, which is not preserved by disjoint union. Thus, this class is not quasimodal.

Lemma 5.2 and especially theorem 5.7 rather suggest that among the frame classes that validate a given modal logic, any quasimodal classes tend to congregate among the larger ones. This is supported by an interesting behaviour that can be observed with the two latter classes of frames in the example just given. Indeed, any non-empty class of frames included in one of them is not quasimodal.

Let us prove this result for the class of trees. The proof is very similar for the class of frames of at most n worlds. Let K be a non-empty class of frames included in the class of all trees. Thus, this class contains only trees, and formula $(@_i\Diamond k \wedge @_j\Diamond k) \rightarrow @_i j$ is valid in it. However, one can, out of K, and with the three fundamental frame constructions, build a frame that is not a tree. Thus, K is not quasimodal.

We might therefore conjecture that *a class included in a non-quasimodal class of frames is not quasimodal.* But this is wrong: for instance, the class of frames with an empty relation is quasimodal (it is elementary and modally definable) and included in the class of irreflexive frames, which is not quasimodal. A question that arises is: which conditions on a non-quasimodal class are needed to make every subclass of it non-quasimodal? By theorem 5.7, 'any subclass of it has the same modal logic' is a sufficient condition, but are there more enlightening ones?

6 McKinsey–Lemmon and Hughes's Logics

The McKinsey–Lemmon and Hughes logics are remarkable modal logics that have been studied for a long time. They are not finitely axiomatizable, each one is the logic of an elementary class of frames, and for each of these logics, the class of frames validating it is not elementary.

We first detail a little more of what is known about these logics, and then we give a few new results about their hybrid extensions.

6.1 Background

Hughes's Logic

KMT, or *Hughes's logic*, introduced in [7], is the logic axiomatized by the following axioms:

$$\Diamond\left((\Box p_1 \rightarrow p_1) \wedge \ldots \wedge (\Box p_n \rightarrow p_n)\right) \qquad \text{(for each } n \geq 1\text{).}$$

It is the logic of both $\mathcal{C}_T$, the class of frames in which each world has a reflexive successor, and $\mathcal{C}_T^*$, the class of frames validating KMT. $\mathcal{C}_T$ is elementary and defined by the first-order sentence $\forall x \exists y (Rxy \wedge Ryy)$. Hughes proved that KMT is not finitely axiomatizable [7, theorem 10] and that $\mathcal{C}_T^*$ is not elementary [7, theorem 11].

McKinsey–Lemmon Logic

KM^∞, or the *McKinsey–Lemmon logic*, is axiomatized by the following axioms:

$$\Diamond\left((\Diamond p_1 \rightarrow \Box p_1) \wedge \ldots \wedge (\Diamond p_n \rightarrow \Box p_n)\right) \qquad \text{(for each } n \geq 1\text{).}$$

An early study of this logic is in [8, p. 74]. It is the logic of $\mathcal{C}_\infty$, the class of frames in which each world has a successor with at most one successor, and $\mathcal{C}_\infty^*$, the class of frames validating KM^∞.

$\mathcal{C}_\infty$ is defined by the first-order sentence $\forall x \exists y (Rxy \wedge \forall zt (Ryz \wedge Ryt \rightarrow z = t))$. On the other hand, $\mathcal{C}_\infty^*$ is not elementary. This has been proved independently, in [1, theorem 21] and [5, theorem 2.2], by the same argument, based on one of van Benthem [11] for McKinsey's formula $\Box\Diamond p \to \Diamond\Box p$ (which is equivalent to the instance for $n = 1$ of the above axiom schema).

[5, corollary 4.5] shows that KM^∞ is not finitely axiomatizable.

6.2 Hybrid extensions of the McKinsey–Lemmon and Hughes Logics

In this section, we shall prove that $\mathcal{C}_T$ and $\mathcal{C}_T^*$ on one hand, and $\mathcal{C}_\infty$ and $\mathcal{C}_\infty^*$ on the other hand, have the same hybrid logic, and that these hybrid logics are quasimodal. We also prove that the hybrid logics of these classes are not finitely axiomatizable.

Proposition 6.1

(i) $\mathcal{C}_T$ and $\mathcal{C}_T^*$ have the same hybrid logic, and are quasimodal.

(ii) $\mathcal{C}_\infty$ and $\mathcal{C}_\infty^*$ have the same hybrid logic, and are quasimodal.

Sketch of proof (i) As $\mathcal{C}_T$ is defined as the class of frames in which each world has a reflexive successor, it is elementary, and one can easily check that it is closed under bounded morphic images, disjoint unions and generated subframes. So we can apply theorem 5.4: $\mathcal{C}_T$ is quasimodal. By definition, $\mathcal{C}_T$ is included in $\mathcal{C}_T^*$, and they have the same modal logic, so we can apply theorem 5.7 to prove the first part of this proposition.

(ii) The same proof applies for $\mathcal{C}_\infty$, as it is defined as the class of frames in which each world has a successor with at most one successor. □

We know that KMT and KM^∞ are not finitely axiomatizable. We can wonder whether the hybrid operators we introduce are sufficient to build a finite set of axioms that entails KMT and KM^∞. We show in corollary 6.3 that this is not the case: the hybrid logic of $\mathcal{C}_T$ and $\mathcal{C}_T^*$ (resp. $\mathcal{C}_\infty$ and $\mathcal{C}_\infty^*$), which is axiomatized by $KMT \cup \Delta$ (resp. $KM^\infty \cup \Delta$), is not finitely axiomatizable.

Lemma 6.2 *Let $\alpha_1, \alpha_2, \ldots$ be modal formulas and suppose that for each $n \geq 1$,*

(i) $\alpha_{n+1} \vdash_m \alpha_n$,

(ii) *there exists a frame $\mathfrak{F}_n$ validating α_n but not α_{n+1}.*

Then the hybrid logic Θ axiomatized by $\{\alpha_1, \alpha_2, \ldots\} \cup \Delta$ is not finitely axiomatizable.

Proof. Assume for contradiction that θ is a $\mathcal{H}(@)$-formula that axiomatizes Θ. As θ is a theorem of Θ, there is n such that $\{\alpha_n\} \vdash_h \theta$. As $\mathfrak{F}_n \models_h \alpha_n$, we have $\mathfrak{F}_n \models_h \theta$. But $\{\theta\} \vdash_h \alpha_{n+1}$, so $\mathfrak{F}_n \models_h \alpha_{n+1}$, which is a contradiction. □

Corollary 6.3 *$HL(\mathcal{C}_\infty)$ and $HL(\mathcal{C}_T)$ are not finitely axiomatizable.*

Proof. Since both these logics are quasimodal, by [5, lemmas 3.7–3.8] and [7, theorem 7], $HL(\mathcal{C}_\infty)$ and $HL(\mathcal{C}_T)$ are axiomatized by Δ together with formulas α_n with the above properties. So we can apply lemma 6.2. □

7 Conclusion

In this paper, we axiomatized the hybrid logic of two kind of frames: elementary classes of frames, and quasimodal classes of frames. Both these results used hybrid extensions, which we defined and studied, building on the nice frames of [10].

We axiomatized hybrid logics of elementary classes of frames, by applying [6] to the hybrid extension; this shows the usefulness of these nice frames and of the hybrid extensions.

Quasimodal logics have then been studied for the purpose of separating the hybrid and modal parts of hybrid axiomatization. The general question we would like to answer is: given an hybrid logic, which is the modal part of this logic, and which is the hybrid part? Quasimodal logics (and quasimodal classes) are the elements of the answer to this question in the simplest case, where the hybrid component is reduced to the five axiom schemas defining hybrid logic. However, it would be interesting to continue this work (see further work below).

Finally, we studied the Hughes and McKinsey–Lemmon logics. Each of these modal logics is the logic of (at least) two classes of frames, one elementary, and one non-elementary. We showed that the hybrid logics of these two classes are the same: this means that hybrid logic does not seem to be powerful enough to differentiate these two classes of frames. We also proved that these hybrid logics are not finitely axiomatizable. These two results, analogous to known modal results, show that these logics might be as interesting in the hybrid case as they were in the modal case.

Further Work

We give here some directions that we think may be interesting to investigate.

Can the hybrid operator $\downarrow$ be simulated by modalities, in a similar way to the modal simulation of @ in [10] and in section 3?

An idea is to consider, for a standard frame $\mathfrak{F}$, the disjoint union of the frames of the hybrid extension of $\{\mathfrak{F}\}$ (the class having only one frame $\mathfrak{F}$). We get a frame containing the disjoint union of all the possible valuations of $\mathfrak{F}$. Then, the operator $\downarrow$ is simply a move from one subframe to another one having a different valuation.

We only dealt with frames having one unary modality $\Box$. Can we extend the results of this paper to arbitrary frames with many polyadic modalities? Is it interesting?

We began to work on characterising quasimodal classes in section 5. Although we gave some properties about quasimodal classes, we could not solve this problem.

We showed that one might have expected the statement *"A class included in a class that is not quasimodal is not quasimodal either"* to be true. However, we showed that it is false. Can we find a weaker, and true, version of this statement, such as *"A class included in a non-quasimodal class matching* [some conditions] *is not quasimodal"*?

This is a problem totally different from the previous one we stated, but, given a finite set Γ of hybrid axioms, is the problem *"Is the hybrid logic axiomatized by* Γ *quasimodal?"* decidable? If it is so, what is its complexity?

Quasimodal logics were what seemed the easiest way to separate hybrid axioms from modal axioms. Indeed, in quasimodal logics, we surely cannot remove the hybrid axioms, since they are the very definition of nominals and hybrid operators.

It is possible to have a similar separation for a wider range of logics? Blackburn et al., in [2], studied the hybrid logic of frames axiomatized by pure formulas (formulas not containing propositional variables, but only nominals). Although they give an interesting result (the class of frames validating a set Π of pure formulas is axiomatized by the hybrid logic $\Delta \cup \Pi$), first, they need to add extra rules to the logic, and second, we do not know if fewer hybrid axioms would be needed if we added the entire modal logic as axioms. Can we find a theorem without these extra rules? Can we prove something like: *The hybrid logic of a class of frames defined by a set Π of pure formulas is axiomatized by its modal logic together with $\Delta \cup \Pi$?*

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