Aspects of relation algebras

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Much of the material is joint work with Robin Hirsch

What are we going to do?

We have 4 tutorials. The plan is:

1. Introduction to relation algebras.
   Representations.
   Atom structures.
   Examples.

2. Using games to build representations.
   Axiomatising the finite representable relation algebras.

3. 'Rainbow' construction (Hirsch).
   Monk's theorem that the representable relation algebras are not finitely axiomatisable.

4. Weakly associative algebras (WA, Maddux). Relativised representations.
   Finite base property for WA by Herwig's theorem.

There are *exercises* for you to try.

Why relation algebras?

- They have a long history—De Morgan, Peirce, Schröder, Tarski, Lyndon, Monk, Andréka, Németi, Maddux, . . .

- They provide a simple introduction to algebraic logic.

- Algebraic logic has strong connections to modal and dynamic logics. Recent Dutch–Hungarian work. Techniques cross over, both sides gain. *Arrow logic* is the modal analogue of relation algebras.

- Relation algebras have practical applications in computing: eg. databases, artificial planning, specification theory (eg. fork algebras). RelMiCS group.

- Can develop all of mathematics by relation algebras. See book of Tarski–Givant, 1987.

General references

Good introductions to relation algebras and more can be found in:

1. R Maddux. Introductory course on relation algebras, finite-dimensional cylindric algebras, and their interconnections.
In H Andréka, J D Monk, and I Németi, editors, *Algebraic Logic*, number 54 in Colloq. Math. Soc. J. Bolyai, pages 361–392, Amsterdam, 1991. North-Holland.

2. I Németi. Algebraisations of quantifier logics, an introductory overview.
Studia Logica, 50(3/4):485 – 570, 1991.

The standard reference on algebraic logic is

3. L Henkin, J D Monk, and A Tarski.
Cylindric Algebras.
Part I, 1971. Part II, 1985. North-Holland.

Some papers relevant to the tutorials:

4. H Andréka, I Hodkinson, and I Németi. Finite algebras of relations are representable on finite sets. *J. Symbolic Logic*, 64:243–267, 1999.

5. R Hirsch and I Hodkinson. Axiomatising various classes of relation and cylindric algebras. *Logic J. IGPL,* 5:209–229, 1997.

6. R Hirsch and I Hodkinson. Step by step—building representations in algebraic logic. *J. Symbolic Logic*, 62:225–279, 1997.

7. R Hirsch and I Hodkinson. Complete representations in algebraic logic. *J. Symbolic Logic*, 62:816–847, 1997.

8. R Lyndon. The representation of relational algebras. *Ann. Math,* 51:707–729, 1950.

9. J D Monk. On representable relation algebras. *Michigan Math. J.*, 11:207–210, 1964.

Revision: boolean algebras

Definition 1 A boolean algebra is an algebra $\mathcal{B} = \langle B, +, \cdot, -, 0, 1\rangle$ satisfying, for all $x, y \in B$:

- $+$ is associative and commutative: $(x+y)+z = x+(y+z)$ and $x+y = y+x$
- complement: $-(-x) = x$
- form of distributivity: $x = x \cdot y + x \cdot -y$
- connections:
  $x \cdot y = -(-x + -y)$
  $x + (-x) = 1$
  $-1 = 0$.

Abbreviation:
$x \leq y$ means $x \cdot y = x$ or equivalently $x+y = y$.

Representations of boolean algebras

The motivation for this definition comes from *fields of sets.*

If U is a set, $\wp U$ denotes the power set (set of all subsets) of U.

Suppose that $B \subseteq \wp U$ contains $\emptyset, U$ and is closed under union, intersection, complement. For example, $B = \wp U$ itself.

Then $\langle B, \cup, \cap, U \setminus -, \emptyset, U\rangle$ is a boolean algebra, called a *field of sets.*

In fact, every boolean algebra is isomorphic to a field of sets (Stone's theorem, 1936).

Unary and binary relations

A *unary relation* on a set U is just a subset of U.

So boolean algebras are to do with unary relations. In fact, they embody all truths about unary relations (by Stone's theorem).

Can we do the same for *binary relations* (subsets of $U \times U$)?

We can consider *fields of binary relations on* U. These are subsets A of $\wp(U \times U)$.
We want A to be closed under the boolean operations, as before.
But there are natural operations on *binary relations* that we want to consider too.

Operations on binary relations

Which operations would we like?

Consider the binary relations *son, daughter.*

son(x,y) = y is a son of x, etc.

$$x \xrightarrow{\text{son}} y$$

From these, we know we can derive many other relations, such as:

- child
- parent
- father, mother, brother, sister
- grandchild, grandson
- grandfather, grandmother, grandparent
- aunt, uncle, niece, nephew.

Can we think of basic operations that will let us form these complex relations from *son* and *daughter*?

The relation algebra operations

Clearly, child = son + daughter.
This is still boolean; we use + rather than $\cup$.

If we can form the *converse* of a binary relation:

$$R^{\smile}(x,y) \iff R(y,x)$$

then we can do parent:

- parent = child$^{\smile}$, or
- parent = (son + daughter)$^{\smile}$, or
- parent = son$^{\smile}$ + daughter$^{\smile}$.

If we can form the *composition* of two relations:

$$R\,;S(x,y) \iff \exists z(R(x,z) \wedge S(z,y))$$

then we can do more:

- grandchild = child ; child
- grandparent = parent ; parent

Need for equality

Can we say 'brother'?

How about parent ; son ?

Trouble is, parent(me,mum)$\wedge$son(mum,me).
So $\exists z$(parent(me,z) $\wedge$ son(z,me)).
So by definition, [parent ; son](me,me).
But I am not a brother of me.

Suppose we can express equality, by adding a constant 1' for it. Then

- brother = (parent ; son) $\cdot -$ 1'

With 1' we can express 'mother' too:

- mother = ((parent ; daughter)$\cdot 1'$) ; parent
- father = parent $\cdot$ −mother

The non-boolean operations we usually take are indeed ; , $^{\smile}$, and 1'.

Exercise 1 Can you express *sister* with these operations? How about *nephew*? And *aunt*?

Proper relation algebras

If we use these operations on binary relations, we get a new kind of algebra.

Definition 2 A *proper relation algebra* is an algebra

$$\mathcal{A} = \langle A, \cup, \cap, E \setminus -, \emptyset, E, =, ^{\smile}, ; \rangle$$

where $A \subseteq \wp(U \times U)$ for some non-empty set U, E is an equivalence relation on U, and the operations are as explained already. (A must be closed under them—so, e.g., $\emptyset, E, = \in A$).

Eg: $\mathcal{A} = \langle \wp(U \times U), \cup, \cap, \setminus, \emptyset, U \times U, =, ^{\smile}, ; \rangle$: called the *full power set algebra on U.*

Usually E will be $U \times U$ itself. But for technical reasons we don't insist on this.

Q: Can we axiomatise these algebras, like we did for boolean algebras?

Relation algebras

Definition 3 (Tarski, 1940s)
A *relation algebra* is an algebra

$$\mathcal{A} = \langle A, +, \cdot, -, 0, 1, 1', ^{\smile}, ; \rangle$$

where $\langle A, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra, and the following hold for all $x, y, z \in A$:

(R1) $(x ; y) ; z = x ;(y ; z)$ (associativity)

(R2) $(x + y) ; z = x ; z + y ; z$

(R3) $x ; 1' = x$

(R4) $x^{\smile\smile} = x$

(R5) $(x + y)^{\smile} = x^{\smile} + y^{\smile}$

(R6) $(x ; y)^{\smile} = y^{\smile} ; x^{\smile}$

(R7) $x^{\smile} ; -(x ; y) \leq -y$.

Representations

These axioms are an attempted analogue, for binary relations, of the BA axioms. Do they work?

Definition 4 A *representation* of a relation-type algebra $\mathcal{A}$ is an isomorphism from $\mathcal{A}$ to a proper relation algebra. $\mathcal{A}$ is *representable* if it has a representation.

Definition 5 RA is the class of all relation algebras. RRA is the class of representable relation-type algebras.

It's easily seen that the RA axioms are sound:

Exercise 2 RRA $\subseteq$ RA.

Equivalently, a proper relation algebra is a relation algebra.
Can we prove an analogue of Stone's theorem, showing completeness?
That is, is *any* relation algebra representable?

The answer

A: Sadly, no [Lyndon, 1950].

$$\text{RRA} \subset \text{RA}.$$

We (you) prove this on Tues.

Moreover, RRA is not finitely axiomatisable in first-order logic [Monk, 1964].

We prove this on Wed.

In fact, RRA is not finitely axiomatisable in 2nd-order logic, 3rd-order logic, ... It can't be axiomatised by equations using a finite set of variables, nor by Sahlqvist equations. It's really nasty.

At least it is recursively axiomatisable. See Tues.

Atoms

We will mostly consider finite algebras. For these, it's easier to work with their atoms.

Definition 6 An *atom* of a (boolean algebra or) relation algebra is a $\leq$-minimal non-zero element. A relation algebra is *atomic* if every non-zero element is $\geq$ an atom.

Exercise 3 Any finite relation algebra is atomic.

It can be shown that in an atomic relation algebra $\mathcal{A}$:

- 1' is determined by {atoms a : a $\leq$ 1'}.
- if a is an atom then so is $a^{\smile}$. And $^{\smile}$ is determined by its values on atoms.
- ; in $\mathcal{A}$ is determined by the set of triples of atoms (x,y,z) such that x ; y $\geq$ z$^{\smile}$. We call these *consistent triangles.*

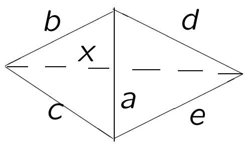
Atom structures

So if we specify
- a finite set X of atoms,
- the set Id of atoms under 1',
- how the function $\smile$ behaves on the atoms,
- the set C of consistent triangles,

is there a relation algebra with these atoms?

Not quite:

Definition 7 A *'relation algebra atom structure'* is a structure $(X, Id, \smile, C)$ satisfying:
- $x^{\smile\smile} = x$ (for all $x \in X$)
- $x = y^{\smile}$ iff $\exists z \in Id((x, y, z)$ consistent)
- if (x, y, z) is consistent then so are (y, z, x) and $(z^{\smile}, y^{\smile}, x^{\smile})$ ('Peircean transforms')
- diamond completion (for associativity):



If $(a, b, c), (a, d, e)$ consistent then there is x with $(b, x, d^{\smile}), (c^{\smile}, x, e)$ consistent.

Exercise 4 Show that if $a \in Id$ then $a^{\smile} = a$.

Atom structures and relation algebras

Definition 8 For an atomic relation algebra $\mathcal{A}$, write $At\mathcal{A}$ for its *atom structure*:

$$At\mathcal{A} = (\{\text{atoms of } \mathcal{A}\}, Id, \smile, C),$$

where $Id = \{a : a \leq 1'\}$, $\smile$ is as in $\mathcal{A}$ (restricted to atoms), $C = \{(a, b, c) : c^{\smile} \leq a ; b\}$.

Theorem 9 (Lyndon, 1950)
1. If $\mathcal{A}$ is an atomic relation algebra then $At\mathcal{A}$ is a r.a. atom structure as in definition 7.
2. For any relation algebra atom structure $\mathcal{S} = (X, Id, \smile, C)$, there is an atomic relation algebra $\mathcal{A}$ with atom structure $\mathcal{S}$.
If X is finite, $\mathcal{A}$ is unique.

Proof. 1. Exercise!
2. $\mathcal{A} = \langle \wp X, \cup, \cap, \setminus, \emptyset, \wp X, Id, \smile, ; \rangle$
is a relation algebra with atom structure $\mathcal{S}$, where for $a, b \subseteq X$:
- $a^{\smile} = \{x^{\smile} : x \in a\}$,
- $a ; b = \{z : \exists x \in a \, \exists y \in b \, (x, y, z^{\smile}) \in C\}$.

If $\mathcal{S}$ is finite and $At\mathcal{B} = \mathcal{S}$, then $b \mapsto \{x \in X : x \leq b\}$ is an isomorphism : $\mathcal{B} \to \mathcal{A}$. ∎

So we can specify a finite relation algebra by specifying its atom structure.

Examples of relation algebras

Example 10 The smallest non-trivial relation algebra, $\mathcal{I}$, has atoms $1', \sharp$, both self-converse. The consistent triangles are the Peircean transforms of $(1', 1', 1')$ and $(\sharp, \sharp, 1')$.

It is representable: take $U = \{0, 1\}$ and interpret 1' as $=$ and $\sharp$ as $\neq$.

Example 11 The *point algebra* $\mathcal{P}$ has 3 atoms, $1', a, a^{\smile}$.

The consistent triangles are all Peircean transforms of:
- $(x, x^{\smile}, 1')$ for all atoms x (of course)
- $(a, a, a^{\smile})$.

The point algebra is representable: take U to be the rational numbers, and interpret 1' as $=$ and a as $<$.

McKenzie's algebra (1970)

Example 12 This is the smallest non-representable relation algebra. We call it $\mathcal{K}$. It has 4 atoms:

$$1', \quad a, \quad a^{\smile}, \quad \sharp = \sharp^{\smile}.$$

The consistent triangles are all Peircean transforms of:
- $(x, x^{\smile}, 1')$ for all x
- $(a, a, a^{\smile})$ (like point algebra so far)
- $(a, a^{\smile}, \sharp)$, $(a^{\smile}, a, \sharp)$, and $(a, \sharp, \sharp)$.

This defines a relation algebra (exercise).

We'll see on Tues (exercise 12) that it's not representable.

Another example (from Jipsen)

Example 13 This algebra (say $\mathcal{J}$) has 4 atoms: 1', r, b, g. They are all self-converse.

The *inconsistent* triangles are the Peircean transforms of

- identity ones (x,y,1') if $x \neq y$, as usual
- (b,b,b)
- (b,b,g)
- (b,g,g)

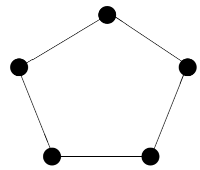
This is representable (see exercise 11, Tues).

As of 1996, it's not known whether it has a finite representation.

Exercises

Exercise 5 *If in $\mathcal{I}$ we say that $(\sharp, \sharp, \sharp)$ is consistent too, do we get a (representable) relation algebra?*

Exercise 6 In the pentagon,



let $e(x, y)$ hold iff (x, y) is an edge, and $n(x, y)$ iff (x, y) is a non-edge (and $x \neq y$). Show that $\{1', e, n\}$ form the atoms of a proper relation algebra. What are the consistent triangles?

Exercise 7 *Does $\mathcal{P}$ have a finite representation?*

Exercise 8 Show that the atom structure for $\mathcal{K}$ in example 12 is a relation algebra atom structure.

Tutorial 2: Games

Today's topic: How can we tell whether a (finite) relation algebra is representable?

We use networks and games. This view developed by Hirsch-IH based on Lyndon [8]. We make some assumptions, for simplicity:

- Most relation algebras here will be finite. So we can work with (relation algebra) atom structures instead of algebras.
- We will always assume 1' is an atom of relation algebras here. So $Id = \{1'\}$ in atom structures.

It follows that in a representation of a relation algebra, we can always assume that 1 is represented as $U \times U$ (not just an equivalence relation on U:

Exercise 9 Let $\mathcal{A}$ be a representable relation algebra and suppose that 1' is an atom of $\mathcal{A}$. Show that $\mathcal{A}$ has a representation h such that $h(1)$ is of the form $U \times U$.

Networks

A network approximates a representation.

Definition 14 (network) Let $\mathcal{S} = (A, Id, \smile, C)$ be an atom structure. An *(atomic) pre-network* over $\mathcal{S}$ is a pair $N = (N_1, N_2)$ where N_1 is a non-empty set and $N_2 : N_1 \times N_1 \to A$ is a labelling function.

N is a *network* if:

- $N_2(x, x) = 1'$ for all $x \in N_1$
- $(N_2(x, y), N_2(y, z), N_2(z, x))$ is consistent for all $x, y, z \in N_1$.

Some networks satisfy $N(x, y) = 1' \Rightarrow x = y$. We call these *strict networks.*

Some basic things about pre-networks

Let $N = (N_1, N_2)$ and $N' = (N'_1, N'_2)$ be pre-networks (over some atom structure).

Definition 15 N, N' are *isomorphic*, written $N \cong N'$, if there is a bijection $\theta : N_1 \to N'_1$ with $N_2(x, y) = N'_2(\theta(x), \theta(y))$ for $x, y \in N_1$.

Definition 16 We write $\boldsymbol{N \subseteq N'}$ if $N_1 \subseteq N'_1$ and $N'_2\lceil(N_1 \times N_1) = N_2$ (that is, $N_2 \subseteq N'_2$).

Definition 17 If $N_0 \subseteq N_1 \subseteq \cdots$ are pre-networks, define $\bigcup_{t<\omega} N_t$ to be the pre-network

$$\Big(\bigcup_{t<\omega}(N_t)_1, \bigcup_{t<\omega}(N_t)_2\Big).$$

Clearly, if the N_t are all networks then so is $\bigcup N_t$.

Notation 18 We generally write N for any of N, N_1, N_2 (sometimes write N_1 as $dom(N)$, but usually distinguish by context). We let $|N|$ denote $|N_1|$ and say N is finite if N_1 is.

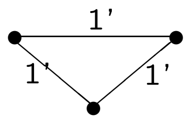
Examples of networks

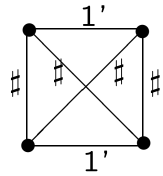
Lemma 19 *In a network N we have $N(x, y) = N(y, x)^{\smile}$ for all $x, y \in N$.*

Proof. $(N(x, y), N(y, x), N(x, x))$ is consistent. But $N(x, x) = 1'$. So $N(x, y) = N(y, x)^{\smile}$. ■

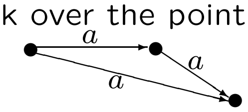
So in diagrams, we only need label arcs one way. And if atoms are self-converse, we don't need arrows.
Labels on reflexive arcs ($N(x, x)$) are 1', so we don't need to write this.

Here are 2 networks over (the atom structure of) the small algebra $\mathcal{I}$ (example 10):





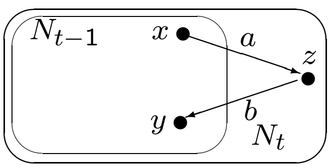
And a network over the point algebra $\mathcal{P}$:



Game on atomic networks

Definition 20 (game) Fix $n \le \omega$, an atom structure $\mathcal{S}$, and a pre-network N (over $\mathcal{S}$).
Players $\forall$ and $\exists$ play a game $\boldsymbol{G_n(N, \mathcal{S})}$ with n rounds: $0, 1, \ldots, t, \ldots$, for $t < n$, in which pre-networks $N_0, N_1, \ldots$ over $\mathcal{S}$ are built.

• In round 0, $\exists$ places a copy of N on the board (so $N_0 \cong N$).
• In round $t > 0$, if N_{t-1} has been built so far, $\forall$ chooses $x, y \in N_{t-1}$ and atoms a, b of $\mathcal{S}$ such that $(a, b, N_{t-1}(y, x))$ is consistent.
$\exists$ must respond with a pre-network $N_t \supseteq N_{t-1}$ with a node $z \in N_t$ such that $N_t(x, z) = a$ and $N_t(z, y) = b$.
We can assume $z \notin N_{t-1}$ if we want.



Winning: After n rounds, if all N_t ($t < n$) are *networks* then $\exists$ wins. Otherwise, $\forall$ wins.

Games and representations

Definition 21 A *winning strategy* for a player in $G_n(N, \mathcal{S})$ is a set of rules that, if followed, always lead to a win for that player.

Theorem 22 *Let $\mathcal{A}$ be a countable relation algebra.*
Let I denote the unique (up to isomorphism) 1-node network over $At\mathcal{A}$.

1. *If $\exists$ has a winning strategy in $G_\omega(I, At\mathcal{A})$, then $\mathcal{A}$ is representable.*

2. *The converse holds if $\mathcal{A}$ is finite.*

Proof of 1.

Let the game commence.
Let $\exists$ use her winning strategy, and let $\forall$ make every possible move at some stage of play (he can do so because $At\mathcal{A}$ is countable).
Play builds a chain of networks

$$I = N_0 \subseteq N_1 \subseteq \cdots$$

Let $N = \bigcup_{t<\omega} N_t$.
Define equivalence relation $\sim$ on $dom(N)$ by

$$x \sim y \iff N(x, y) = 1'.$$

For $r \in \mathcal{A}$, let

$$\widehat{r} = \{(x/{\sim}, y/{\sim}) : x, y \in N,\ N(x, y) \le r\}.$$

Then

$$\widehat{\mathcal{A}} = \langle \{\widehat{r} : r \in \mathcal{A}\}, \cup, \cap, \setminus, \emptyset, N/{\sim} \times N/{\sim}, =, ^{\smile}, ;\rangle$$

is a proper relation algebra, and $r \mapsto \widehat{r}$ is an isomorphism from $\mathcal{A}$ to $\widehat{\mathcal{A}}$.

So $\mathcal{A}$ is representable.

Proof of 2: preliminaries

Assume $\mathcal{A}$ is finite, and (without loss of generality) a proper relation algebra of the form

$$\mathcal{A} = \langle A, \cup, \cap, \setminus, \emptyset, U \times U, =, ^{\smile}, ;\rangle$$

on a set U (so $A \subseteq \wp(U \times U)$).

Lemma 23
1. If $x, y \in U$ then there is a unique atom $\alpha(x, y)$ of $\mathcal{A}$ with $(x, y) \in \alpha(x, y)$.
2. Let $X \subseteq U$. Then $Nwk(X) \stackrel{\text{def}}{=} (X, \alpha)$ is a network over $At\mathcal{A}$.

Proof.
1. Let $a \in A$ be $\le$-minimal with $(x, y) \in a$.
a exists as $\mathcal{A}$ is finite and $(x, y) \in U \times U = 1^{\mathcal{A}}$.
Check that a is an atom of $\mathcal{A}$.

If a, b are distinct atoms containing (x, y), then $(x, y) \in a \cdot b = \emptyset$, contradiction.

2. Easy. ■

Proof of 2 ($\exists$'s winning strategy)

$\exists$'s strategy in $G_\omega(I, At\mathcal{A})$ is to ensure that each N_t played satisfies:
- $dom(N_t) \subseteq U$
- $(x, y) \in N_t(x, y)$ for all $x, y \in U$.

That is, $N_t = Nwk(dom(N_t))$.
This is a network, by lemma 23.
So if she can do it, it's a winning strategy.

It's easy to do in round 0:
$\exists$ can take $N_0 = Nwk(x) \cong I$ for any $x \in U$.

If she has done it as far as N_{t-1}, let $\forall$ pick $x, y \in N_{t-1}$ and atoms a, b with $(a, b, N_{t-1}(y, x))$ consistent.
So in $\mathcal{A}$ we have $N_{t-1}(x, y) \le a\, ;b$.
By assumption, $(x, y) \in a\, ;b$.

So there is $z \in U$ with $(x, z) \in a$ and $(z, y) \in b$.
$\exists$ defines $N_t \stackrel{\text{def}}{=} Nwk(dom(N_{t-1}) \cup \{z\})$. ■

Finite vs. infinite games

Theorem 24 *Let $\mathcal{S}$ be a finite atom structure and N a finite pre-network over $\mathcal{S}$. The following are equivalent:*
1. $\exists$ has a winning strategy in $G_\omega(N, \mathcal{S})$.
2. $\exists$ has a winning strategy in $G_n(N, \mathcal{S})$ for all finite n.

Proof. $1 \Rightarrow 2$: trivial.
$2 \Rightarrow 1$: $\exists$'s strategy in $G_\omega(N, \mathcal{S})$ is:
$(*)$ 'in each round, t, ensure I have a winning strategy in $G_n(N_t, \mathcal{S})$ for infinitely many n'.

$(*)$ is true in round 0 by assumption.
If $(*)$ holds for N_{t-1}, let $\forall$ make his move.
For infinitely many n, $\exists$ has a winning strategy in $G_{n+1}(N_{t-1}, \mathcal{S})$.
Let its response to $\forall$'s move be N_t^n, say.
Note: $\exists$ has a winning strategy in $G_n(N_t^n, \mathcal{S})$.
As $\mathcal{S}$ is finite, there is an infinite set $X \subseteq \omega$ such that the N_t^n $(n \in X)$ are all isomorphic.
$\exists$ lets $N_t = N_t^n$ (any $n \in X$). This keeps $(*)$.

$(*)$ implies each N_t is a network. So this strategy is winning for $\exists$. ■

Expressing winning strategy in logic

Theorem 25 *Let $n < \omega$. There is a first-order sentence σ_n such that for any atom structure $\mathcal{S}$, $\exists$ has a winning strategy in $G_n(I, \mathcal{S})$ iff $\mathcal{S} \models \sigma_n$. And $\{\sigma_n : n < \omega\}$ is recursive.*

Proof. For each finite set X, we write a formula φ_n^X with free variables v_{xy} $(x, y \in X)$, such that for any pre-network N with domain X, $\exists$ has a winning strategy in $G_n(N, \mathcal{S})$ iff $\mathcal{S} \models \varphi_n^X(N(x,y)/v_{xy})_{x,y \in X}$.

We define φ_n^X by induction on n. We let

$$\varphi_0^X \stackrel{\text{def}}{=} \bigwedge_{x \in X} (v_{xx} = 1') \wedge \bigwedge_{x,y,z \in X} C(v_{xy}, v_{yz}, v_{zx}))$$

Given φ_n^X, we pick $z \notin X$, let $Z = X \cup \{z\}$, and let φ_{n+1}^X be

$$\bigwedge_{x,y \in X} \forall ab\Big(C(a, b, v_{yx}) \to \exists_{w \in X} v_{wz}, v_{zz}, v_{zw} \\ (v_{xz} = a \wedge v_{zy} = b \wedge \varphi_n^Z)\Big)$$

We then let $\sigma_n = \varphi_n^{\{x\}}(1'/v_{xx})$ (any x). ∎

Axioms for representability

We can now axiomatise the finite representable relation algebras.

Definition 26 For $n < \omega$, λ_n is the sentence obtained by translating σ_n into the language of relation algebras, by:
- relativising quantifiers to atoms ('x is an atom' is definable by $\forall y(y < x \leftrightarrow y = 0)$).
- replacing $C(x, y, z)$ by $z^\smile \leq x \,;\, y$.

The following is immediate:

Lemma 27 *For any atomic relation algebra $\mathcal{A}$, $\mathcal{A} \models \lambda_n$ iff $At\mathcal{A} \models \sigma_n$.*

Theorem 28 (ess. Lyndon, [8]) *Let $\mathcal{A}$ be a finite relation algebra (with $1' \in At\mathcal{A}$). Then $\mathcal{A}$ is representable iff $\mathcal{A} \models \lambda_n$ for all finite n.*

Proof. By theorem 22, $\mathcal{A}$ is representable iff $\exists$ has a winning strategy in $G_\omega(I, At\mathcal{A})$.
By theorem 24, this is iff $\exists$ has a winning strategy in $G_n(I, At\mathcal{A})$ for all finite n.
By theorem 25, this is iff $At\mathcal{A} \models \sigma_n$ for all n.
By lemma 27, this is iff $\mathcal{A} \models \lambda_n$ for all n. ∎

What about infinite relation algebras?

Theorem 29 *If $\mathcal{A}$ is any atomic relation algebra, and $\mathcal{A} \models \lambda_n$ for all finite n, then $\mathcal{A}$ is representable.*

Proof. By lemma 27, $At\mathcal{A} \models \sigma_n$ for all n.
By theorem 25, $\exists$ has a winning strategy in $G_n(I, At\mathcal{A})$ for all $n < \omega$.
By (eg) saturation, there is countable $\mathcal{B} \equiv \mathcal{A}$ such that $\exists$ has a winning strategy in $G_\omega(I, \mathcal{B})$.
By theorem 22, $\mathcal{B} \in$ RRA.

Fact 30 RRA is elementary (first-order axiomatisable). In fact, it is a variety—equationally axiomatised.

So $\mathcal{A} \in$ RRA, too. ∎

The converse fails, even for atomic relation algebras (have a look at Lemma 23).

But arbitrary representable relation algebras (RRA) can be axiomatised by games in a similar way. See [5,6].

Exercises

Exercise 10 Show that $\exists$ has a winning strategy in $G_\omega(I, At\mathcal{P})$ where $\mathcal{P}$ is the point algebra of example 11.
[Hint: if stuck, use the known representation of $\mathcal{P}$ and the proof of theorem 22(2).]

Exercise 11 Show that $\exists$ has a winning strategy in $G_\omega(I, At\mathcal{J})$ where $\mathcal{J}$ is the 4-atom algebra of example 13.
[Hint: use atom r to fill in the edges.]

Exercise 12 Show that $\forall$ has a winning strategy in $G_\omega(I, At\mathcal{K})$ where $\mathcal{K}$ is McKenzie's 4-atom algebra (example 12).

Tutorial 3: Rainbow, Monk's theorem

Today we are going to prove the most important negative result in algebraic logic:

Theorem 31 (Monk, [9]) RRA *is not finitely axiomatisable in first-order logic.*

The idea is to construct finite relation algebras $\mathcal{A}_n$ ($2 \leq n < \omega$) such that for all n:
- $\forall$ has a winning strategy in $G_\omega(I, At\mathcal{A}_n)$
- $\exists$ has a winning strategy in $G_n(I, At\mathcal{A}_n)$.

Lemma 32 *Monk's theorem follows.*

Proof. Assume RRA is finitely axiomatised, by σ, say. Let

$$T = \{\text{RA axioms}\} \cup \{\lambda_n : n < \omega\} \cup \{\alpha, \neg\sigma\},$$

where α expresses atomicity. Then for any finite $\Sigma \subseteq T$ there is $\mathcal{A}_n \models \Sigma$. By first-order compactness, T has a model, say $\mathcal{B}$. By theorem 29, $\mathcal{B} \in$ RRA, contradicting $\mathcal{B} \models \neg\sigma$. ■

Rainbow algebras

We build the $\mathcal{A}_n$ using a simplification of a recent idea due to R. Hirsch, called the *rainbow construction.*

The simplified version is good enough for Monk's theorem.

The atom structure $At\mathcal{A}_n$ ($n \geq 2$) is given by:

ATOMS:
- 1'
- g_i for $i \leq n$ (greens)
- w (white)
- y, b (yellow, black)
- r_{ij} for distinct $i, j < n$.

All self-converse except $\mathsf{r}_{ij}^{\smile} = \mathsf{r}_{ji}$.

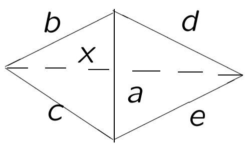
Inconsistent triangles: Peircean transfs of:
- $(x, y, 1')$ if $x \neq y^{\smile}$.
- green ones: $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{g}_k)$ for any $i, j, k \leq n$
- (g,g,w), (y,y,y), (y,y,b)
- any red triangle except $(\mathsf{r}_{ij}, \mathsf{r}_{jk}, \mathsf{r}_{ki})$.

This defines a finite relation algebra

There are finitely many atoms, so any relation algebra with this atom structure is finite.

We check that it's a relation algebra atom structure (definition 7):

- $x^{\smile\smile} = x$ for all atoms x: clear.
- $x = y^{\smile}$ iff $(x, y, 1')$ consistent: clear.
- if (x, y, z) is consistent then so are (y, z, x) and $(z^{\smile}, y^{\smile}, x^{\smile})$ ('Peircean transforms'): by definition.
- diamond completion (for associativity):

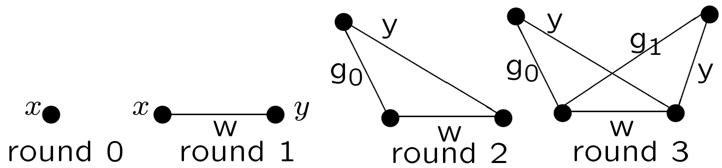


This will follow if we show $\exists$ has a winning strategy in $G_4(I, \mathcal{A}_n)$.
We do this in theorem 34.

$\forall$'s winning strategy

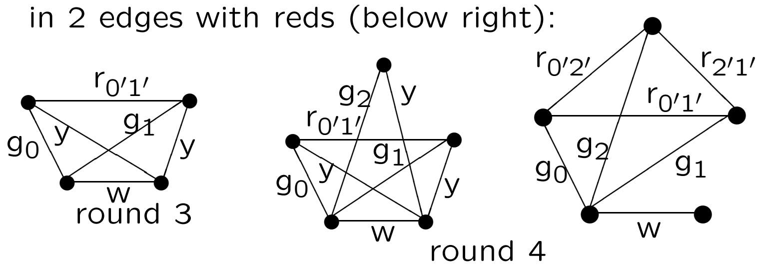
Theorem 33 $\forall$ *can win* $G_\omega(I, At\mathcal{A}_n)$ ($\forall n$).

Proof. His first 4 moves are:



Now $\exists$ has to fill in the missing edge. Which atom can she choose?
Only a red—say, $\mathsf{r}_{0'1'}$ (below left).

$\forall$ continues (below middle), and $\exists$ has to fill in 2 edges with reds (below right):



And so on: $\forall$ plays $x, y, \mathsf{g}_{i-2}, \mathsf{y}$ in round i.
$\exists$ will run out of reds in round $n+2$. ■

∃'s winning strategy

Theorem 34 *∃ can win $G_{n+2}(I, At\mathcal{A}_n)$ for each $n \geq 2$.*

Proof. At the start of some round t, $0 < t < n+2$, suppose the current network is $N = N_{t-1}$. Note that $|N| \leq t \leq n+1$.

Let ∀ move in round t by picking nodes x, y and atoms a, b with $(a, b, N(y, x))$ consistent.

We can assume ∀ never plays so that ∃ doesn't need to add a node. So (i) if $N(v, w) = 1'$ then $v = w$, and (ii) there is no $z \in N$ with $N(x, z) = a$ and $N(z, y) = b$.

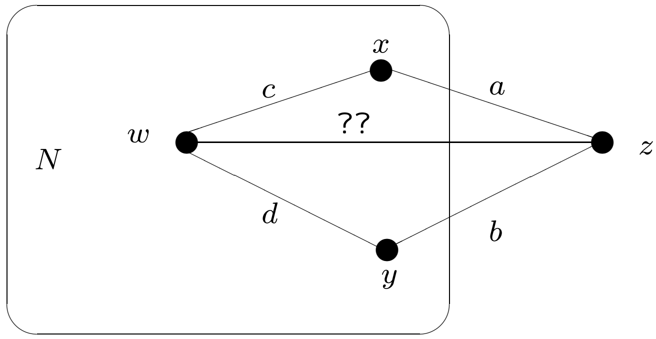
∃ makes N_t by adding a new node z, defining $N_t(x, z) = a$, $N_t(z, y) = b$, $N_t(z, z) = 1'$, and then choosing $N_t(w, z)$ for all $w \in N \setminus \{x, y\}$.

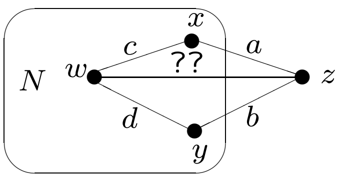
Of course, she defines $N_t(z, w) = N_t(w, z)^{\smile}$.

What does ∃ choose for the $N_t(w, z)$

- Use white if possible.
- If not, use black if possible.
- If not, use a suitable red.

Here, 'if possible' means 'if triangles w, x, z and w, y, z are rendered consistent'.





Let $N(w, x) = c$

and $N(w, y) = d$.

- If a, c not both green, and b, d not both green, she can pick ?? = w (white).
- Otherwise, if a, c not both yellow, and b, d not both yellow, she lets ?? = b (black).
- Otherwise, we can assume a is green, b yellow. ∃ must choose a red for $N_t(w, z)$ for each w in the set

$$R = \{w \in N : N(w, x) \text{ green, } N(w, y) = \text{y}\}.$$

If $v, w \in R$, $v \neq w$, then $N(v, w)$ is red (why?). Indeed, $\exists \rho : R \to n$ with

$$N(v, w) = \mathsf{r}_{\rho(v), \rho(w)} \quad \text{(all distinct } v, w \in R).$$

If $R = \{w\}$, we let $\rho(w) = 0$.
Note: $|rng(\rho)| \leq |N| - 2 \leq n - 1$.
So there is $i \in n \setminus rng(\rho)$.

∃ lets ?? $= N_t(w, z) = \mathsf{r}_{\rho(w), i}$ for all $w \in R$. This completes the labelling and she defines N_t to be the result.

Is N_t a network?

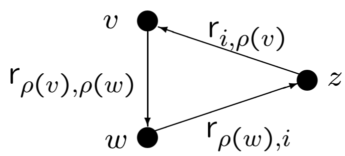
It's enough to check that all triangles (v, w, z) with $v, w \in N$, $v \neq w$, are consistently labelled.

Triangles of the form (x, y, z), (v, x, z), (v, y, z) are clearly consistent.

Consider a triangle (v, w, z) where $v, w \notin \{x, y\}$.

∃ labelled (v, z) and (w, z) with w, b, or a red. All triangles with 2 edges like this are consistent, except perhaps all-red triangles.

But ∃ only uses reds for (v, z) and (w, z) when $v, w \in R$. In this case, we have



and this is consistent. ■

Other results

The full rainbow construction and variations can be used to prove many more negative results, such as:

• The class of relation algebras with representations satisfying the condition of lemma 23(1) is not elementary [Hirsch; see [7]].
• RRA is not closed under 'completions' [IH]
• (Hence) RRA is not Sahlqvist-axiomatisable [Venema]
• * It is undecidable whether a finite relation algebra is representable [Hirsch–IH]

• Similar results for 'cylindric algebras' (higher-arity relations) can be proved too
(* : probably).

• For all $n \geq 5$ there is a first-order sentence that can be proved with n variables but not with $< n$ [Hirsch–IH–Maddux].
• For $n \geq 5$, the classes RA_n, SRaCA_n, and (for $m \geq 3, m+2 \leq n$) $\mathsf{SNr}_m\mathsf{CA}_n$ are not finitely axiomatisable [Hirsch–IH].

Tutorial 4: Finite base property

Relation algebras, even non-representable ones, are surprisingly complicated.

We mentioned that it is undecidable whether a finite relation algebra is representable.

Tarski proved that the equational theory of RA is undecidable.

One of the chief 'causes' is associativity.

So researchers like Maddux proposed weakening the associative law.

The semi-associative law is $(x\,;1)\,;1 = x\,;(1\,;1)$.

Even assuming only this, the equational theory of the resulting class, SA, is undecidable.

Weak associativity

Maddux (1982) proposed an even weaker law,

$$([x \cdot 1']\,;1)\,;1 = [x \cdot 1']\,;(1\,;1)$$

The class of all algebras satisfying (R2–R6) and this law is called WA.

The equational theory of WA is decidable.

Every WA is representable if we allow special 'relativised representations'.

And finite WAs have finite relativised representations [Andréka–IH–Németi].

This last is our topic today.

Relativised representations

Weak version of 'proper relation algebra': all operations are relativised to (intersected with) the unit.

Definition 35 A *relativised proper relation algebra* is an algebra

$$\mathcal{A} = \langle A, \cup, \cap, -, \emptyset, 1, 1', {}^{\smile}, ;\rangle$$

where $A \subseteq \wp(U \times U)$ for some non-empty set U, and for $r, s \in A$,
• $-r = 1 \setminus r$ (as before!)
• $1' = \{(x,y) \in 1 : x = y\}$
• $r^{\smile} = \{(x,y) \in 1 : (y,x) \in r\}$
• $r\,;s = \{(x,y) \in 1 : \exists z((x,z) \in r \wedge (z,y) \in s)\}$.

1 (the 'unit') need not be an equivalence relation on U.

Definition 36 A *relativised representation* of a relation-type algebra $\mathcal{A}$ is an isomorphism from $\mathcal{A}$ onto a relativised proper relation algebra.

WA and relativised representations

WAs are associated with relativised representations with reflexive symmetric unit.

Theorem 37 (Maddux, 1982) *Let $\mathcal{A}$ be a relation-type algebra. $\mathcal{A} \in$ WA iff $\mathcal{A}$ has a relativised representation in which the unit is reflexive and symmetric.*

We will prove a related result:

Theorem 38 ([4], 1999) *Let $\mathcal{A}$ be a* finite *relation-type algebra. $\mathcal{A} \in$ WA iff $\mathcal{A}$ has a relativised representation* with finite base *in which the unit is reflexive and symmetric.*

This is known as the finite base property for WA.

Exercise 13 Prove $\Leftarrow$.

Preliminaries: packed and live

Definition 39 Let M, N be structures for a finite relational language L.

1. A subset $X \subseteq M$ is *live* if $|X| = 1$ or there are $a_1, \ldots, a_n \in M$ and n-ary $R \in L$ such that $M \models R(a_1, \ldots, a_n)$ and $X \subseteq \{a_1, \ldots, a_n\}$.

2. M is *packed* if any two elements of M form a live set.

3. A *partial isomorphism* of M is a 1-1 partial map $f : M \to M$ such that for any $a_1, \ldots, a_n \in M$ and n-ary $R \in L$, $M \models R(a_1, \ldots, a_n) \leftrightarrow R(f(a_1), \ldots, f(a_n))$. f is an *automorphism* if it's bijective.

4. A *homomorphism* from M to N is a map $f : M \to N$ such that for any $a_1, \ldots, a_n \in M$ and n-ary $R \in L$, if $M \models R(a_1, \ldots, a_n)$ then $N \models R(f(a_1), \ldots, f(a_n))$.

Herwig's theorem

Theorem 40 (Herwig, 1998) *Let M be a finite structure in a finite relational language. There is a finite structure $M^+ \supseteq M$ such that*
- *any partial isomorphism of M extends to an automorphism of M^+.*
- *If $X \subseteq M^+$ is live then $g(X) \subseteq M$ for some automorphism g of M^+.*
- *If N is a packed structure, and $f : N \to M^+$ is a homomorphism, then there exists a homomorphism $f^- : N \to M$.*

For the proof see B. Herwig, Extending partial isomorphisms for the small index property of many ω-categorical structures. *Israel J. Math.*, 107:93–124, 1998.

Finite base property for WA

Proof of theorem 38 $\Rightarrow$. Let $\mathcal{A}$ be a finite WA. We assume for simplicity that 1' is an atom of $\mathcal{A}$.

Let M consist of a (disjoint) copy of each network over $At\mathcal{A}$ with at most 3 nodes.
Networks defined as for relation algebras, but require $N(x, y) = 1' \Rightarrow x = y$ (networks are strict).
Regard $At\mathcal{A}$ as a relational language. Let $M \models a(x, y)$ iff $\exists N \in M(x, y \in N \wedge N(x, y) = a)$.

Let M^+ be as in Herwig's theorem.

Let $h : \mathcal{A} \to \wp(M^+ \times M^+)$ be given by

$$h(r) = \{(x, y) : x, y \in M^+, M^+ \models a(x, y) \text{ for some atom } a \leq r\}.$$

We show h is a relativised representation of $\mathcal{A}$ (with range a relativised proper relation algebra with base M^+).

Checking h

Let $r, s \in \mathcal{A}$.

• h is 1–1: if $r \neq s$, take an atom $a \leq r \oplus s$ (symmetric difference).
There are $x, y \in M$ with $M \models a(x, y)$.
Then $(x, y) \in h(r) \oplus h(s)$.

• $h(r + s) = h(r) \cup h(s)$,
$h(r \cdot s) = h(r) \cap h(s)$: clear.

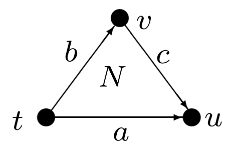
• $h(-r) = h(1) \setminus h(r)$,
$h(1') = \{(x, x) : (x, x) \in h(1)\}$,
$h(r^{\smile}) = \{(x, y) \in 1 : (y, x) \in h(r)\}$,
$h(1)$ is reflexive and symmetric: use liveness.

Exercise 14 Check the above.

• Finally we check $h(r\,;s) = \{(x, y) \in h(1){:}\exists z((x, z) \in h(r) \wedge (z, y) \in h(s))\}$.

$h(r\,;s) \subseteq \{(x, y) \in h(1) : \exists z(x, z) \in h(r), (z, y) \in h(s)\}$

If $(x, y) \in h(r\,;s)$, there's an atom $a \leq r\,;s$ with $M^{+} \models a(x, y)$.
So certainly $(x, y) \in h(1)$.
Also, $\{x, y\}$ is live. Let g be an automorphism of M^{+} with $g(x), g(y) \in M$.
Then $M \models a(g(x), g(y))$.
Now we may take atoms $b \leq r$, $c \leq s$ with $a \leq b\,;c$. So there's a network N in M with nodes t, u, v, say:

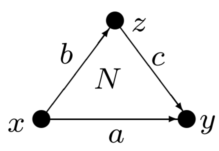


Clearly, $t \mapsto g(x), u \mapsto g(y)$ is a partial isomorphism of M.
So it extends to an automorphism f of M^{+}.
Let $z = g^{-1}f(v)$. Then $M^{+} \models b(x, z) \wedge c(z, y)$.
So $(x, z) \in h(r)$ and $(z, y) \in h(s)$, as required.

$h(r\,;s) \supseteq \{(x, y) \in h(1) : \exists z(x, z) \in h(r), (z, y) \in h(s)\}$

Let $x, y, z \in M^{+}$ with
$(x, y) \in h(1)$, $(x, z) \in h(r)$, $(z, y) \in h(s)$.
We require $(x, y) \in h(r\,;s)$.

Let $b \leq r, c \leq s$, and a be atoms with $M^{+} \models a(x, y) \wedge b(x, z) \wedge c(z, y)$. Then the substructure



is packed, and the inclusion map is a homomorphism: $N \to M^{+}$.

By Herwig's theorem there's a homomorphism $x \mapsto x', y \mapsto y', z \mapsto z'$ from N into M. So

$$M \models a(x', y') \wedge b(x', z') \wedge c(z', y').$$

So by definition of M, $(b, c, a^{\smile})$ is consistent.
So $a \leq b\,;c \leq r\,;s$. By definition of h,
$(x, y) \in h(r\,;s)$, as required. ■

Remarks

The size of the finite representation of $\mathcal{A}$ can be bounded recursively in $|\mathcal{A}|$.

The same idea proves that any universal sentence valid in finite WAs is valid in all WAs.

This gives the 'finite model property' for *arrow logic* in its relativised interpretation.

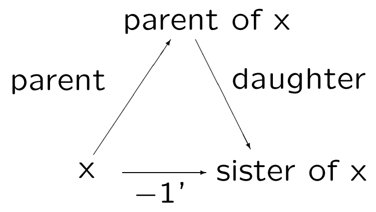
The method applies in several other situations. Eg: can find a finite relativised representation for any finite relation algebra, and associativity gives it more properties.

Exercise 15 Let h be a relativised representation of $\mathcal{A} \in$ WA. Show that the unit, $h(1)$, is a *reflexive and symmetric* binary relation on the base of h.

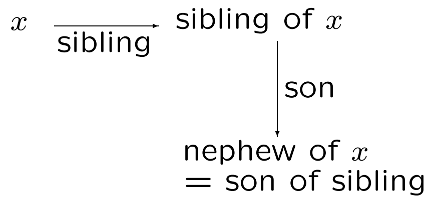
Answers to selected exercises

Exercise 1

1. sister = (parent ; daughter) $\cdot -1'$



2. nephew = sibling ; son



3. aunt = parent ; sister

Exercise 4

If $\mathcal{S} = (X, Id, ^{\smile}, C)$ is a relation algebra atom structure and $a \in Id$, show $a^{\smile} = a$.

We know $(a, a^{\smile}, b) \in C$ for some $b \in Id$.

As C is closed under Peircean transforms, $(a^{\smile}, b, a) \in C$.

But $a \in Id$, so $a^{\smile} = b^{\smile}$.

So $a = a^{\smile\smile} = b^{\smile\smile} = b$.
Thus, $(a, a^{\smile}, a) \in C$.

As C is closed under Peircean transforms, $(a^{\smile}, a^{\smile}, a) \in C$.

But $a \in Id$, so $a^{\smile} = a^{\smile\smile} = a$.

Exercise 5

The algebra with consistent triangles $(x, x^{\smile}, 1')$ and $(\sharp, \sharp, \sharp)$ is a representable relation algebra—one representation is on a set with three elements, interpreting 1' as = and $\sharp$ as $\neq$.

Exercise 6

The consistent triangles are
- $(x, x, 1')$ for $x \in \{1', e, n\}$,
- (e, e, n), (n, n, e).

The algebra with atoms $1', e, n$ is closed under the operations and the pentagon is a representation of it.

Exercise 7

In any representation of the point algebra, a is interpreted as an (irreflexive) dense partial order. ($a ; a = a$ gives transitivity and density, and $a \neq a^{\smile}$ gives antisymmetry.) The partial order has at least 2 elements, because a is interpreted as a non-empty relation.
But there is no such finite order.

Exercise 12

$\forall$ can win $G_5(I, \mathcal{K})$ as follows: