Interval temporal logics with chop-like operators

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Thanks to the TIME'10 organisers for inviting me!

Setting...

Most temporal logic is point-based.

Here we look at interval temporal logics.

Motivation: eating dinner takes a while.

Useful in databases, concurrency, AI: planning, event calculus.
Also in philosophy, linguistics.

We look at several connectives, but concentrate on 'chop' and its conjugates.

We consider expressiveness, decidability, axiomatisations.

Plan of talk

1. Interval temporal logics. Some possible connectives.
2. Expressiveness
3. Logics of classes of linear flows of time
   (a) Decidability
   (b) Axiomatisation
4. Summary of results
5. Conclusion
6. Some references

Interval semantics

Linear flow of time (irreflexive linear order) $(T, <)$
Eg $(\{0, 1, \ldots, n\}, <), (\mathbb{N}, <), (\mathbb{Z}, <), (\mathbb{Q}, <), \ldots$

Interval $[x, y] = \{z \in T : x \leq z \leq y\}$ (for $x, y \in T$, $x \leq y$)

Propositional atoms evaluated at intervals: so each atom stands for a binary relation on T.

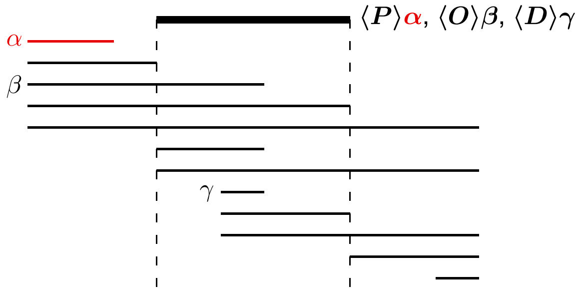
We add various *connectives* to express temporal properties.
Their semantics should be first-order definable (unless we use μ-calculus etc)

The atoms, connectives, and booleans make *temporal formulas.*

All formulas are evaluated at intervals.

Some possible interval connectives

Can include $\Diamond$s for some/all *Allen interval relations*:

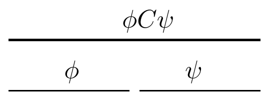


2^{12} possible choices of connectives.
Goranko, Montanari, Sciavicco are classifying their properties.
Goranko 2010: "$> 80\%$ of currently investigated fragments are undecidable over most linear orders".

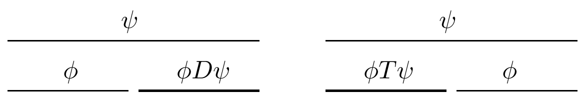
Chop logic 'CDT'

Three binary connectives: C, D, T.
These operators occur in Venema (1991).

C stands for *chop*:



Its conjugates D and T (*done* and *to come*?) are similar, but the 'decomposition' lies *before* or *after* the current interval:



We also have a constant π to pick out one-point intervals.

Formal syntax and semantics of CDT

Syntax of CDT-formulas:

$$p \text{ (atom)} \mid \top \mid \pi \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi C \psi \mid \varphi D \psi \mid \varphi T \psi$$

Can include or exclude any of C, D, T — get 7 different languages.
Can also omit π — so 14 in all.

Semantics: $(T, <)$ irreflexive linear order

$\mathcal{I}(T) = \{[x, y] : x, y \in T,\ x \leq y\}$
— set of all intervals of $(T, <)$ (with endpoints)

Assignment h : {atoms} $\to \wp(\mathcal{I}(T, <))$ — gives truth values of atoms

Model $\mathcal{M} = (T, <, h)$

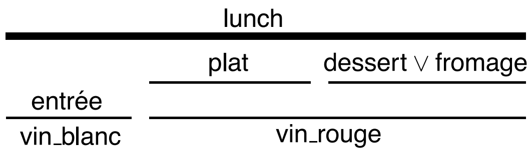
Evaluation in $\mathcal{M} = (T, <, h)$

- $\mathcal{M}, [x, y] \models p \iff [x, y] \in h(p)$ for atoms p
- $\mathcal{M}, [x, y] \models \pi \iff x = y$
- booleans $\top, \neg, \wedge$ evaluated as usual
- $\mathcal{M}, [x, y] \models \varphi C \psi \iff$ there is $z \in [x, y]$ with $\mathcal{M}, [x, z] \models \varphi$ and $\mathcal{M}, [z, y] \models \psi$
- $\mathcal{M}, [x, y] \models \varphi D \psi \iff$ there is $z \leq x$ with $\mathcal{M}, [z, x] \models \varphi$ and $\mathcal{M}, [z, y] \models \psi$
- $\mathcal{M}, [x, y] \models \varphi T \psi \iff$ there is $z \geq y$ with $\mathcal{M}, [y, z] \models \varphi$ and $\mathcal{M}, [x, z] \models \psi$.

Example

lunch $\rightarrow$

(entrée $\wedge$ vin_blanc) C [(plat C (dessert $\vee$ fromage)) $\wedge$ vin_rouge]



Expressiveness of CDT

Benchmark: L^n ($n \geq 3$): n-variable first-order logic with $<$ and binary relation symbols $P, Q, \ldots$ tied to atoms $p, q, \ldots$ and interpreted as *symmetric* relations (because can't distinguish $P(x,y), P(y,x)$ in CDT).

Venema (1991): *CDT and L^3 are equally expressive* over linear time.

- Any CDT-formula translates into L^3 ('standard translation'). E.g., $(\neg\pi \wedge pTq)C\neg q$ translates to $\exists z(x < z \leq y \wedge \exists y(y \geq z \wedge P(y,z) \wedge Q(x,z))) \wedge \neg Q(z,y))$.
- Any L^3-formula $\varphi(x,y)$ can be translated into CDT (and it's straightforward: use induction, DNF, simulate $\exists z\psi(x,y,z)$ with C, D, T).

Limits of expressiveness

Venema (1990): over $\mathcal{I}(\mathbb{Q},<)$, $L^3 \subset L^4 \subset \cdots$ *in expressiveness.*

Proof uses equivalence relation $\sim_n$ with n classes, each dense in $\mathbb{Q}$. *Need* n variables to say "$\exists \geq n$ classes" (proved by games). So $\sim_{n+1}$ distinguishable from $\sim_n$ in L^{n+1} but not in L^n.

Conclude:

- CDT cannot express that an atom defines an equivalence relation with at least 4 equivalence classes.
- *no finite set of temporal connectives is expressively complete for* $\bigcup_n L^n$ (contrast Kamp's theorem for point-based temporal logic).

Logics of classes of linear flows of time

Let $\mathcal{K}$ be a class of linear flows of time (= irreflexive linear orders). Examples:

- Lin: all linear flows
- Fin: all finite linear flows
- Dens: all dense linear flows
- Disc: all discrete linear flows

Write $C(\mathcal{K})$ for the set of all C-formulas *valid* (true at all intervals) in every model $\mathcal{M} = (T,<,h)$ for every $(T,<) \in \mathcal{K}$.

Define $C^\pi(\mathcal{K})$ similarly, but using $(C+\pi)$-formulas.

Similarly define $D(\mathcal{K}), D^\pi(\mathcal{K}), \ldots,$
and also $\mathcal{S}(\mathcal{K}), \mathcal{S}^\pi(\mathcal{K})$ for $\emptyset \neq \mathcal{S} \subseteq \{C, D, T\}$.

Decidability (lack of)

$C(\mathcal{K}), C^\pi(\mathcal{K}), D(\mathcal{K}), D^\pi(\mathcal{K}), T(\mathcal{K}), T^\pi(\mathcal{K})$ are *all undecidable* for $\mathcal{K} = \text{Lin}, \text{Fin}, \text{Dens}, \text{Disc}, \{(\mathbb{N},<)\}, \{(\mathbb{Z},<)\}, \{(\mathbb{Q},<)\}$

Exceptions: $D(\mathbb{N})$, $D(\text{Fin}), T(\text{Fin})$ — these are *open.*

Work of Goranko, IH, Lodaya, Montanari, Moszkowski, Sciavicco, 1983–today.

Proofs can encode

- *undecidable tiling problems* for infinite flows,
- *Post's correspondence problem* or *intersection of CF grammars* for finite flows.

Axiomatisations (better news. . .)

Venema (1991): CDT can be *finitely axiomatised* over various classes of linear flows of time (eg: all, dense, discrete, $\{\mathbb{Q}\}$).

Axioms: quite a lot. Some of the more interesting ones:

- $(\neg(\alpha T\beta)C\beta) \to \neg\alpha$ (conjugacy of C, T)
- $\alpha T(\beta C\gamma) \leftrightarrow \beta C(\alpha T\gamma) \vee (\gamma T\alpha)T\beta$ (linearity)

Rules: MP, generalisation, and a *non-orthodox rule:*

- from $HOR(p) \to \alpha$ (p an atom *not occurring in* α), infer α.

$HOR(p)$ is a CDT-formula saying "p holds *precisely* on intervals with the same (right-hand) endpoint as current interval".

Completeness proof constructs model 'step by step'.

Non-orthodox rules

Mainly arose in temporal logic. Later generalised to modal logic. Originated with Gabbay ('irreflexivity rule', 1981), Burgess (1980).

Have a *side-condition:* p does not occur in the formula being proved.

Not always *robust:* sound over intended semantics but can be unsound in extensions.

Non-orthodox rules are used to *name points* in canonical model. (So hybrid logic often has such rules.)

They make completeness proofs much easier — even 'trivial'. So some people prefer to avoid non-orthodox rules if possible.

Can we axiomatise CDT without non-orthodox rules?

No we can't. (Not finitely anyway.)

Theorem (IH–Montanari–Sciavicco)
Let $\mathcal{K}$ be a class of linear flows of time containing $(\mathbb{Q},<)$.
Let $\emptyset \neq \mathcal{S} \subseteq \{C, D, T\}$.

Then $\mathcal{S}(\mathcal{K}), \mathcal{S}^\pi(\mathcal{K})$ are *not finitely axiomatisable* with only orthodox rules (modus ponens, generalisation, substitution).

E.g., $CDT^\pi(\text{Lin}), C(\text{Dens}), T^\pi(\{(\mathbb{Q},<)\})$ are not finitely axiomatisable.

Proof sketch (for $C(\mathcal{K}), C^\pi(\mathcal{K})$ only)

We build a sequence of finite *abstract frames* $\mathcal{F}_n$ ($n \in \mathbb{N}$, $n \geq 2$).
(Based on a construction of Monk (1969).)

1. Their *limit* $\mathcal{F}_\omega$ validates $C^\pi(\mathbb{Q}, <)$.
2. But no individual $\mathcal{F}_n$ even validates $C(\mathsf{Lin})$.

Fix $\mathcal{K}$ with $(\mathbb{Q}, <) \in \mathcal{K} \subseteq \mathsf{Lin}$. (Eg $\mathcal{K} = \mathsf{Dens}$.)

Suppose $\alpha_1, \ldots, \alpha_j$ were a finite axiomatisation of $C(\mathcal{K})$ or $C^\pi(\mathcal{K})$.

Then as $\mathcal{K} \subseteq \mathsf{Lin}$, we have $\alpha_1, \ldots, \alpha_j \vdash \varphi$ for all $\varphi \in C(\mathsf{Lin})$.
Let $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_j$. By (2), α is not valid on any $\mathcal{F}_n$.
By properties of 'limit', α is not valid on $\mathcal{F}_\omega$.
But $(\mathbb{Q}, <) \in \mathcal{K}$, so $\alpha \in C^\pi(\mathbb{Q}, <)$. Contradiction to (1)!

(Red part uses soundness of our rules over the $\mathcal{F}_n$.
Venema's non-orthodox rules are unsound over the $\mathcal{F}_n$.)

Some details of the proof...

We look at

A. Abstract frames and models
B. The abstract frames $\mathcal{F}_n$
C. Why the abstract frames don't validate $C(\mathsf{Lin})$
D. The limit frame $\mathcal{F}_\omega$
E. Why $\mathcal{F}_\omega$ validates $C^\pi(\mathbb{Q}, <)$

A. Abstract frames

An *abstract frame* $\mathcal{F}$ is a set F of 'abstract intervals', plus information stating:

- which abstract intervals are supposed to be 'one-point intervals' (like $[x, x]$).

  We can represent this information by a subset $\Pi \subseteq F$.
  $a \in \Pi$ says that $\mathcal{F}$ thinks a *is a one-point interval.*

- which abstract intervals a are supposed to '*chop*' into abstract intervals b, c (like $[x, y]$ chops into $[x, z], [z, y]$).

  We can represent this information by a *3-ary relation* R on $\mathcal{F}$.
  $R(a, b, c)$ says that $\mathcal{F}$ thinks a *chops into* b, c.

Abstract models

Let $\mathcal{F} = (F, \Pi, R)$ be an abstract frame.

Take an assignment $h : \{\text{atoms}\} \to \wp(F)$.
So atoms are still interpreted as sets of (abstract) intervals.

Let $\mathcal{M} = (\mathcal{F}, h)$. *We can evaluate C^π-formulas in $\mathcal{M}$.*

For $a \in F$ define

- $\mathcal{M}, a \models p \iff a \in h(p)$ for atoms p
- $\mathcal{M}, a \models \pi \iff a \in \Pi$
- booleans $\top, \neg, \wedge$ evaluated as usual
- $\mathcal{M}, a \models \varphi C \psi \iff$ there are $b, c \in F$ with $R(a, b, c)$, $\mathcal{M}, b \models \varphi$, and $\mathcal{M}, c \models \psi$.

B. The abstract frames $\mathcal{F}_n$ $(n \geq 2)$

$\mathcal{F}_n = (F_n, \{1'\}, R)$, where

- $F_n = \{1'\} \cup (\{1, \ldots, m\} \times \{1, \ldots, n\})$, where $m = \lceil e \cdot n! \rceil$.
  The *colour* of $(p, q) \in F_n$ is q.
  Here is $\mathcal{F}_4$ $(m = \lceil 65.2387 \ldots \rceil = 66)$:

- $1'$ is the only 'one-point interval'. (It's abstract!)
- $R = \{(1', 1', 1'), (a, 1', a), (a, a, 1') : a \in F_n\} \cup$
  $\{(a, b, c) : a, b, c \neq 1',\ a, b, c$ *not all same colour*$\}$

Idea: if $a, b, c \neq 1'$, then a can be chopped into b, c iff a, b, c are not all the same colour.

C. $\mathcal{F}_n$ does not validate $C(\mathsf{Lin})$

Regard elements of F_n as propositional atoms.
Let a, b, c range over F_n.
Define

$$\begin{aligned} \theta &= \bigvee_a a \wedge \bigwedge_{a \neq b} \neg(a \wedge b) \wedge \bigwedge_{\neg R(a,b,c)} \neg(a \wedge (bCc)) \\ \sigma &= \textstyle\bigwedge_a (aC\top) \wedge \neg(\top C([\neg\theta]C\top)) \end{aligned}$$

Then σ *is satisfiable in* $\mathcal{F}_n$*:*
Let $h(a) = \{a\}$ (all $a \in F_n$).
An easy check shows $(\mathcal{F}_n, h), a \models \sigma$ for any $a \in F_n \setminus \{1'\}$.

But $\neg\sigma$ *is valid over linear interval models.*
This is because $|F_n| \gg$ number of colours.

D. The limit $\mathcal{F}_\omega$

$\mathcal{F}_\omega = (\{1'\} \cup (\mathbb{N} \times \mathbb{N}), 1', R)$.

The *colour* of $(x, y) \in \mathbb{N} \times \mathbb{N}$ is y.

R defined as before:
$R = \{(1', 1', 1'), (a, 1', a), (a, a, 1') : a \in F_\omega\} \cup$
$\{(a, b, c) : a, b, c \neq 1',\ a, b, c$ not all same colour$\}$

$\mathcal{F}_\omega$ is a *limit* of the $\mathcal{F}_n$:

Any C^π*-formula valid in* $\mathcal{F}_\omega$ *is valid in infinitely many* $\mathcal{F}_n$
(proof uses model theory).

E. $\mathcal{F}_\omega$ validates the logic $C^\pi(\mathbb{Q}, <)$

We *label* intervals of $\mathbb{Q}$ by elements of $\mathcal{F}_\omega$, 'step by step'.
We aim to achieve:

1. all elements of $\mathcal{F}_\omega$ occur as labels
2. $1'$ labels all and only the one-point intervals $[x, x]$
3. if $[x, y], [x, z], [z, y]$ are labeled by $a, b, c \in \mathcal{F}_\omega$, then $R(a, b, c)$
4. if a labels $[x, y]$, and $R(a, b, c)$, then there is $z \in [x, y]$ such that b labels $[x, z]$ and c labels $[z, y]$.

We can achieve 3–4 because there are as many colours in $\mathcal{F}_\omega$ as there are abstract intervals (labels): countably infinitely many of each.

$\mathcal{F}_\omega$ validates $C^\pi(\mathbb{Q},<)$

For modal logicians:
$[x,y] \mapsto \text{label}[x,y]$ is a surjective p-morphism : $\mathcal{I}(\mathbb{Q},<) \to \mathcal{F}_\omega$, so preserves validity.

In detail:
Let $h : \{\text{atoms}\} \to \mathcal{F}_\omega$ be any assignment.
Pull back h to an assignment $g : \{\text{atoms}\} \to \wp(\mathcal{I}(\mathbb{Q},<))$ by

$$g(p) = \{[x,y] : \text{label}[x,y] \in h(p)\}.$$

Check by induction on φ (written with C, π) that for all $[x,y]$,

$$(\mathbb{Q},<,g),[x,y] \models \varphi \iff (\mathcal{F}_\omega,h),\text{label}[x,y] \models \varphi.$$

Summary of results (rather gloomy)

$\mathcal{K}$	$C(\mathcal{K})$	$C^\pi(\mathcal{K})$	$D(\mathcal{K})$	$D^\pi(\mathcal{K})$	$T(\mathcal{K})$	$T^\pi(\mathcal{K})$
Lin	U,N	U [2],N	U,N	U,N	U,N	U,N
Fin	U	U [1]	?	U	?	U
Dens	U,N	U [3],N	U,N	U,N	U,N	U,N
Disc	U	U	U	U	U	U
$\mathbb{N}$	U	U [1]	?	U	U	U
$\mathbb{Z}$	U	U [1]	U	U	U	U
$\mathbb{Q}$	U,N	U [3],N	U,N	U,N	U,N	U,N

U: undecidable. N: not finitely axiomatisable with only orthodox rules.
[1] Moszkowski, [2] Goranko–Montanari–Sciavicco, [3] Lodaya.
Others: IH–Montanari–Sciavicco.
Note: finite axiomatisability *open* for discrete orders.

Conclusion

Interval logics with Chop and the like are

- very expressive, quite natural to use
- finitely axiomatisable with a non-orthodox rule
- some natural deductive systems exist (Venema 1991) (arbitrarily many variables may occur in proofs)
- usually undecidable
- often not finitely axiomatisable with only orthodox rules. So Venema's non-orthodox rule is *really needed.*

The methods we saw are borrowed from relation algebras.
They may give further results?
Eg: no axiomatisation by Sahlqvist/canonical axioms?

Some references

- Goranko, V., Montanari, A., and Sciavicco, G. A road map of interval temporal logics and duration calculi. *Journal of Applied Non-Classical Logics 14*, 1-2 (2004), 9–54.
- Hodkinson, I., Montanari, A., and Sciavicco, G. Non-finite axiomatizability and undecidability of interval temporal logics with C, D, and T. In *Proc. CSL* (2008), M. Kaminski and S. Martini, Eds., *LNCS 5213*, Springer-Verlag, pp. 308–322.
- Monk, J. D. Nonfinitizability of classes of representable cylindric algebras. *J. Symbolic Logic 34* (1969), 331–343.
- Venema, Y. Expressiveness and Completeness of an Interval Tense Logic. *Notre Dame J. Formal Logic 31* (1990) 529–547.
- Venema, Y. A logic with the chop operator. *J. Logic Computat. 1* (1991), 453–476.