On non-local propositional and weak monodic quantified CTL*

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Abstract

In this paper we prove decidability of two kinds of branching time temporal logics. First we show that the non-local version of propositional PCTL*, in which truth values of atoms may depend on the branch of evaluation, is decidable. Then we use this result to establish decidability of various fragments of quantified PCTL*, where the next-time operator can be applied only to formulas with at most one free variable, all other temporal operators and path quantifiers are applicable only to sentences, and the first-order constructs follow the pattern of any of several decidable fragments of first-order logic.

Key words: Branching time temporal logic, CTL*, predicate temporal logic, decidability, non-local semantics.

1 Introduction

This paper continues the investigation of the computational behaviour of first-order branching time temporal logics started in [13]. A 'negative' result obtained in [13] is the undecidability of the one-variable fragment of quantified computational tree logic CTL* (both bundled [3, 18] and 'unbundled' versions, and even with sole temporal operator 'now or some time in the future'). On the other hand, it was shown that by restricting applications of first-order quantifiers to *state* (i.e., path-independent) formulas, and applications of temporal operators and path quantifiers to formulas with at most one free variable (similarly to the linear time *monodic* logic of [11]), decidable fragments can be obtained. This is so even when we include the 'past' operator 'Since'.

Here we prove decidability of various expressive fragments of another kind of fragment of first-order CTL* with Since. This fragment is called the *weak monodic fragment* and consists of formulas in which

- the next-time operator is monodic, i.e., it only may be applied to formulas with at most one free variable,
- all other temporal operators and path quantifiers are 'nullodic:' they are applicable only to sentences, and

- there are no restrictions on first-order quantification.

The weak monodic fragment itself is clearly undecidable because it contains full first-order logic. We obtain decidable fragments of it by restricting its first-order part to some decidable fragment of first-order logic — for example, the two-variable or the guarded fragment.

To prove decidability, we first show decidability of the 'non-local' version of propositional CTL^*, where truth values of atoms may depend on the branch of evaluation.[1] We then reduce the weak monodic fragment to this logic. The main technical instrument in both proofs is the method of quasi-models [11, 13].

For possible applications of decidable fragments of first-order temporal logics, the reader may consult, e.g., [6].

2 Decidability of non-local $\mathcal{PCTL}^*$

The propositional language $\mathcal{PCTL}^*$ [5, 14] extends propositional logic with temporal connectives U, S ('until,' 'since') and a path quantifier E ('there exists a branch (or history)'). The dual path quantifier A ('for all branches (or histories)') is defined as an abbreviation: $\mathsf{A}\varphi = \neg\mathsf{E}\neg\varphi$. Other standard abbreviations we need are: $\Diamond_F\varphi = \top\,\mathsf{U}\,\varphi$ ('some time in the future'), $\Box_F\varphi = \neg\Diamond_F\neg\varphi$ ('always in the future'), $\Diamond_P\varphi = \top\,\mathsf{S}\,\varphi$ ('some time in the past'), $\bigcirc\varphi = \bot\,\mathsf{U}\,\varphi$ ('at the next moment'), and $\bigcirc_P\varphi = \bot\,\mathsf{S}\,\varphi$ ('at the previous moment').

Trees. This language is interpreted in (bundled and unbundled) models based on ω-trees. A *tree* is a strict partial order $\mathfrak{T} = \langle T, < \rangle$ such that for all $t \in T$, the set $\{u \in T : u < t\}$ is linearly ordered by $<$. When we write $\mathfrak{T}$ for a tree, it will be implicit that $\mathfrak{T} = \langle T, < \rangle$. We say that $\mathfrak{T}$ is countable if T is countable. A *full branch* of $\mathfrak{T}$ is a maximal linearly-ordered subset of T. An ω-*tree* is a rooted tree (i.e., it has a unique minimum element), each of whose full branches, ordered by $<$, is order-isomorphic to the natural numbers $\langle \mathbb{N}, < \rangle$. For an ω-tree $\mathfrak{T}$, and $t \in T$, we let $ht(t) = |\{u \in T : u < t\}|$.

We shall need to use the following important kind of ω-tree. For a non-empty set Λ, we write ${}^{<\omega}\Lambda$ for the set of all finite sequences of elements of Λ. For $\xi, \eta \in {}^{<\omega}\Lambda$, define $\xi \lhd \eta$ if η is the concatenation $\xi\hat{}\zeta$ for some non-empty $\zeta \in {}^{<\omega}\Lambda$ — that is, ξ is a proper initial segment of η. Then $\langle {}^{<\omega}\Lambda, \lhd \rangle$ is an ω-tree of cardinality $|\Lambda| + \aleph_0$.

We will also need to form 'products' of ω-trees. If $\mathfrak{T} = \langle T, <_T \rangle$ and $\mathfrak{U} = \langle U, <_U \rangle$ are ω-trees, we write $\mathfrak{T} \otimes \mathfrak{U}$ for

$$\langle\, \{(t,u) : t \in T,\ u \in U,\ ht(t) = ht(u)\},\ <\,\rangle,$$

where $(t,u) < (t',u')$ iff $t <_T t'$ and $u <_U u'$. Note that $\mathfrak{T} \otimes \mathfrak{U}$ is an ω-tree, and if $\mathfrak{T}$ and $\mathfrak{U}$ are countable then so is $\mathfrak{T} \otimes \mathfrak{U}$.

A *bundle* on an ω-tree $\mathfrak{T}$ is a set $\mathcal{H}$ of full branches of $\mathfrak{T}$ such that $\bigcup \mathcal{H} = T$.

Models. In this paper, we deal with the 'non-local' variant of $\mathcal{PCTL}^*$ in which truth values of atoms can depend on the branch of evaluation. Thus, a *bundled model* has the form $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$, where $\mathfrak{T}$ is an ω-tree, $\mathcal{H}$ is a bundle on $\mathfrak{T}$, and $h : \mathcal{P} \to \wp(\{(\beta, t) : t \in \beta \in \mathcal{H}\})$ is a valuation in $\mathfrak{T}$; here, $\mathcal{P}$ is the ambient set of propositional atoms and $\wp$ denotes 'power set'. $\mathfrak{M}$ is said to be a *full tree model* if $\mathcal{H}$ is the set of all full branches of $\mathfrak{T}$; in this case we write simply $\mathfrak{M} = \langle \mathfrak{T}, h \rangle$. The truth-relation $\models$ in $\mathfrak{M}$ is now defined as follows, where $t \in \beta \in \mathcal{H}$:

[1] This contrasts with the behaviour of process logic, the local version of which is decidable, while the non-local one is undecidable [4].

- for an atom p, $(\mathfrak{M},\beta,t) \models p$ iff $(\beta,t) \in h(p)$;
- the booleans are defined as usual;
- $(\mathfrak{M},\beta,t) \models \varphi \mathsf{U} \psi$ iff there is $u > t$ with $u \in \beta$, $(\mathfrak{M},\beta,u) \models \psi$, and $(\mathfrak{M},\beta,v) \models \varphi$ for all $v \in (t,u)$, where $(t,u) = \{v \in T : t < v < u\}$;
- $(\mathfrak{M},\beta,t) \models \varphi \mathsf{S} \psi$ iff there is $u < t$ with $(\mathfrak{M},\beta,u) \models \psi$ and $(\mathfrak{M},\beta,v) \models \varphi$ for all $v \in (u,t)$;
- $(\mathfrak{M},\beta,t) \models \mathsf{E}\varphi$ iff $(\mathfrak{M},\gamma,t) \models \varphi$ for some $\gamma \in \mathcal{H}$ with $t \in \gamma$.

As usual, we say that a $\mathcal{PCTL}^*$-formulas φ is *satisfiable* in a bundled or full model $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$ if $(\mathfrak{M},\beta,t) \models \varphi$ for some $t \in \beta \in \mathcal{H}$.

LEMMA 1. *If a $\mathcal{PCTL}^*$-formula φ is satisfiable in a full (bundled) tree model, then φ is satisfiable in a full (respectively, bundled) tree model based on a countable ω-tree.*

Proof. Let $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$ be a tree model in which φ is satisfiable. We may view $\mathfrak{M}$ as a two-sorted first-order structure, as follows. The two sorts are T and $\mathcal{H}$. We include (i) the tree order $<$, with sort (T,T), (ii) a binary relation $\in$ of sort $(T,\mathcal{H})$ representing '$t \in \beta$' for $t \in T$, $\beta \in \mathcal{H}$, and (iii) a binary relation P of sort $(T,\mathcal{H})$ for each atom $p \in \mathcal{P}$: '$P(t,\beta)$' represents $(\beta,t) \in h(p)$. Taking a countable elementary substructure [9, §2.5] of this yields a *bundled* tree model $\mathfrak{N} = \langle \mathfrak{T}_0, \mathcal{H}_0, h_0 \rangle$ whose ω-tree $\mathfrak{T}_0$ and bundle $\mathcal{H}_0$ are countable. Here, $h_0(p) = h(p) \cap \{(\beta,t) : t \in \beta \in \mathcal{H}_0\}$ for any atom p. It is easy to translate $\mathcal{PCTL}^*$-formulas to two-sorted first-order formulas with the same meaning. It follows that for any $\beta \in \mathcal{H}_0$, $t \in \beta$, and any $\mathcal{PCTL}^*$-formula ψ, we have

$$(\mathfrak{M},\beta,t) \models \psi \quad \text{iff} \quad (\mathfrak{N},\beta,t) \models \psi. \tag{1}$$

It follows that φ is satisfiable in $\mathfrak{N}$. This completes the proof for the bundled case.

Suppose now that $\mathcal{H}$ contains all full branches of $\mathfrak{T}$ and let $\overline{\mathfrak{N}} = \left\langle \mathfrak{T}_0, \overline{\mathcal{H}}_0, \overline{h}_0 \right\rangle$ be the full tree model based on $\mathfrak{N}$, where $\overline{\mathcal{H}}_0 \supseteq \mathcal{H}_0$ is the set of all full branches of $\mathfrak{T}_0$, and $\overline{h}_0(p) = h(p) \cap \{(t,\beta) : t \in \beta \in \overline{\mathcal{H}}_0\}$, for an atom p. We claim that for all $\mathcal{PCTL}^*$-formulas ψ, all full branches γ of $\mathfrak{T}_0$ and all $t \in \gamma$, we have

$$(\mathfrak{M},\gamma,t) \models \psi \quad \text{iff} \quad (\overline{\mathfrak{N}},\gamma,t) \models \psi.$$

The proof is by induction on ψ. The atomic, boolean, and temporal cases are trivial. Consider the case $\mathsf{E}\psi$ and inductively assume the result for ψ. If $(\mathfrak{M},\gamma,t) \models \mathsf{E}\psi$, pick $\beta \in \mathcal{H}_0$ containing t. Clearly, $(\mathfrak{M},\beta,t) \models \mathsf{E}\psi$, so by (1), $(\mathfrak{N},\beta,t) \models \mathsf{E}\psi$. It follows that there is $\beta' \in \mathcal{H}_0$ with $(\mathfrak{N},\beta',t) \models \psi$. By (1) again, $(\mathfrak{M},\beta',t) \models \psi$. Inductively, $(\overline{\mathfrak{N}},\beta',t) \models \psi$. So $(\overline{\mathfrak{N}},\gamma,t) \models \mathsf{E}\psi$, as required. The converse implication is easy. As before, it follows easily that φ is satisfiable in $\overline{\mathfrak{N}}$. $\square$

Fix a $\mathcal{PCTL}^*$-formula φ.

DEFINITION 2. Let $sub(\varphi)$ denote the set of subformulas of φ and their negations. A *type* for φ is a subset $\mathbf{t}$ of $sub(\varphi)$ such that $\psi \wedge \chi \in \mathbf{t}$ iff $\psi \in \mathbf{t}$ and $\chi \in \mathbf{t}$, for every $\psi \wedge \chi \in sub(\varphi)$, and $\neg\psi \in \mathbf{t}$ iff $\psi \notin \mathbf{t}$, for every $\neg\psi \in sub(\varphi)$. A set Σ of types is said to be *coherent* if it is non-empty and for all $\mathsf{E}\psi \in sub(\varphi)$, the three conditions $\mathsf{E}\psi \in \bigcap\Sigma$, $\mathsf{E}\psi \in \bigcup\Sigma$, and $\psi \in \bigcup\Sigma$ are equivalent.

Fix an ω-tree $\mathfrak{T} = \langle T, < \rangle$.

DEFINITION 3. Suppose that we are given a non-empty set Σ_t of types for each $t \in T$, and a full branch β of $\mathfrak{T}$. In this context, a *run in* β is a map $r : \beta \to \bigcup_{t\in\beta} \Sigma_t$ such that

- $r(t) \in \Sigma_t$ for each $t \in \beta$,
- for all $\psi \mathsf{U} \chi \in sub(\varphi)$ and all $t \in \beta$, we have $\psi \mathsf{U} \chi \in r(t)$ iff there is $u > t$ with $u \in \beta$, $\chi \in r(u)$, and $\psi \in r(v)$ for all $v \in (t,u)$,
- for all $\psi \mathsf{S} \chi \in sub(\varphi)$ and all $t \in \beta$, we have $\psi \mathsf{S} \chi \in r(t)$ iff there is $u < t$ with $\chi \in r(u)$ and $\psi \in r(v)$ for all $v \in (u,t)$.

DEFINITION 4. A family $(\Sigma_t : t \in T)$ of coherent sets of types is said to be an *full quasimodel* (or simply a *quasimodel*) *for* φ *over* $\mathfrak{T}$ if

1. $\varphi \in \mathbf{t}$ for some $t \in T$ and $\mathbf{t} \in \Sigma_t$,
2. for all $t \in T$ and $\mathbf{t} \in \Sigma_t$, there is a full branch β of $\mathfrak{T}$ containing t and a run r in β such that $r(t) = \mathbf{t}$,
3. for each full branch β of $\mathfrak{T}$, there exists a run in β.

$(\Sigma_t : t \in T)$ is a *bundled quasimodel* for φ over $\mathfrak{T}$ if it satisfies conditions 1 and 2.

LEMMA 5. *A* $\mathcal{PCTL}^*$*-formula* φ *is satisfiable in a (bundled) model iff there is a (bundled) quasimodel for* φ *over a countable* ω*-tree.*

Proof. Let $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$ be such that $(\mathfrak{M}, \beta_0, t_0) \models \varphi$ for some $\beta_0 \in \mathcal{H}$ and $t_0 \in \beta_0$. By Lemma 1, we can assume that $\mathfrak{T}$ is countable. For $\beta \in \mathcal{H}$ and $t \in \beta$, let

$$\mathrm{tp}(t,\beta) = \{\psi \in sub(\varphi) : (\mathfrak{M}, \beta, t) \models \psi\}.$$

Clearly, $\mathrm{tp}(t,\beta)$ is a type for φ. For $t \in T$, let

$$\Sigma_t = \{\mathrm{tp}(t,\beta) : t \in \beta \in \mathcal{H}\}.$$

Clearly, Σ_t is coherent. For any $\beta \in \mathcal{H}$, the map $r_\beta : t \mapsto \mathrm{tp}(t,\beta)$ is then a run in β. We claim that $(\Sigma_t : t \in T)$ is a quasimodel for φ over $\mathfrak{T}$ (a bundled one if $\mathfrak{M}$ is bundled, and a full one otherwise). As $(\mathfrak{M}, \beta_0, t_0) \models \varphi$, we have $\varphi \in \mathrm{tp}(t_0, \beta_0) \in \Sigma_{t_0}$. For each $t \in T$ and $\mathbf{t} \in \Sigma_t$, we have $\mathbf{t} = \mathrm{tp}(t,\beta)$ for some $\beta \in \mathcal{H}$ containing t, so $r_\beta(t) = \mathbf{t}$ and condition 2 of Definition 4 holds. Finally, for all $\beta \in \mathcal{H}$, r_β is a run in β, so if $\mathfrak{M}$ is a full tree model, it is clear that condition 3 holds.

Conversely, let $\mathcal{Q} = (\Sigma_t : t \in T)$ be a quasimodel for φ over a countable ω-tree $\mathfrak{T}$. By replacing $\mathfrak{T}$ by $\mathfrak{T} \otimes \langle {}^{<\omega}2, \lhd \rangle$ (see 'Trees' above), we can assume without loss of generality that $(*)$ for each $t \in T$ and $\mathbf{t} \in \Sigma_t$, there are infinitely many full branches β of $\mathfrak{T}$ containing t such that there is a run r in β with $r(t) = \mathbf{t}$.

Each Σ_t is finite, so there are countably many pairs $(t, \mathbf{t})$ with $t \in T$, $\mathbf{t} \in \Sigma_t$. Enumerate them as $(t_n, \mathbf{t}_n)$ $(n < \omega)$. Inductively, using $(*)$, choose a full branch $\beta_n \ni t_n$ for each $n < \omega$, such that (i) there is a run r_{β_n} in β_n with $r_{\beta_n}(t_n) = \mathbf{t}_n$, and (ii) $\beta_n \neq \beta_m$ for all $m < n$. If $\mathcal{Q}$ is a bundled quasimodel, let $\mathcal{H} = \{\beta_n : n < \omega\}$. This is clearly a bundle on $\mathfrak{T}$. If $\mathcal{Q}$ is a full quasimodel, let $\mathcal{H}$ be the set of all full branches of $\mathfrak{T}$, and further choose for each $\beta \in \mathcal{H} \setminus \{\beta_n : n < \omega\}$ a run r_β in β; this can be done by condition 3 of Definition 4. So we have defined a run r_β in β, for each $\beta \in \mathcal{H}$.

Now define a model $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$ where $h(p) = \{(\beta, t) : t \in \beta \in \mathcal{H},\ p \in r_\beta(t)\}$ for each atom p. $\mathfrak{M}$ is bundled if $\mathcal{Q}$ is a bundled quasimodel; otherwise, it is full.

CLAIM. For all $\beta \in \mathcal{H}$, all $t \in \beta$, and all $\psi \in sub(\varphi)$, we have $(\mathfrak{M}, \beta, t) \models \psi$ iff $\psi \in r_\beta(t)$.

PROOF OF CLAIM. The proof is by induction on ψ. For atomic $\psi = p$, we have $(\mathfrak{M},\beta,t) \models p$ iff $(\beta,t) \in h(p)$, iff $p \in r_\beta(t)$ as required. The boolean cases are trivial. For $\psi \mathsf{U} \chi \in sub(\varphi)$, we have $(\mathfrak{M},\beta,t) \models \psi \mathsf{U} \chi$ iff there is $u \in \beta$ with $u > t$, $(\mathfrak{M},\beta,u) \models \chi$, and $(\mathfrak{M},\beta,v) \models \psi$ for all $v \in (t,u)$. Inductively, this holds iff there is $u \in \beta$ with $u > t$, $\chi \in r_\beta(u)$, and $\psi \in r_\beta(v)$ for all $v \in (t,u)$. Since r_β is a run in β, this is iff $\psi \mathsf{U} \chi \in r_\beta(t)$, as required. The case of S is similar.

Finally, for $\mathsf{E}\psi \in sub(\varphi)$, we have $(\mathfrak{M},\beta,t) \models \mathsf{E}\psi$ iff $(\mathfrak{M},\gamma,t) \models \psi$ for some $\gamma \in \mathcal{H}$ with $t \in \gamma$. Inductively, this is iff $\psi \in r_\gamma(t)$ for some $\gamma \in \mathcal{H}$ with $t \in \gamma$. But evidently, $\Sigma_t = \{r_\gamma(t) : \gamma \in \mathcal{H},\ t \in \gamma\}$, so this is iff $\psi \in \bigcup \Sigma_t$. Since Σ_t is coherent, this is iff $\mathsf{E}\psi \in r_\beta(t)$, as required. The claim is proved.

By condition 1 of Definition 4, there is $t \in T$ such that $\varphi \in \mathbf{t}$ for some $\mathbf{t} \in \Sigma_t$. We may choose $n < \omega$ with $(t,\mathbf{t}) = (t_n,\mathbf{t}_n)$. Then $t \in \beta_n \in \mathcal{H}$ and $r_{\beta_n}(t) = \mathbf{t}$, so by the claim, $(\mathfrak{M},\beta_n,t) \models \varphi$. Thus, φ is satisfiable in $\mathfrak{M}$, which is bundled or full according as Q is. ☐

LEMMA 6. *Given a $\mathcal{PCTL}^*$-formula φ, it is decidable whether φ has an unbundled quasimodel over a countable ω-tree. The same holds for full quasimodels.*

Proof. Given φ, we can effectively construct the set $\mathcal{C}$ of all coherent sets of types. A quasimodel over an ω-tree $\mathfrak{T}$ has the form $(\Sigma_t : t \in T)$ where $\Sigma_t \in \mathcal{C}$ for each t; we will express this by unary relation variables P_Σ for each $\Sigma \in \mathcal{C}$, the aim being that P_Σ is true at t iff $\Sigma_t = \Sigma$. We then express the stipulations of Definition 4 in terms of the P_Σ, as follows. Let R_ψ ($\psi \in sub(\varphi)$) be unary relation variables. For a type $\mathbf{t}$ for φ, let

$$\chi_{\mathbf{t}}(x) \quad = \quad \bigwedge_{\psi \in \mathbf{t}} R_\psi(x) \ \wedge \bigwedge_{\psi \in sub(\varphi)\setminus \mathbf{t}} \neg R_\psi(x).$$

The formula $\chi_{\mathbf{t}}(x)$ says that the $R_\psi(x)$ define the type $\mathbf{t}$ at x. For a unary relation variable B, let $\rho(B)$ be the conjunction of:

- $\bigwedge_{\Sigma \in \mathcal{C}} \forall x \big(B(x) \wedge P_\Sigma(x) \to \bigvee_{\mathbf{t} \in \Sigma} \chi_{\mathbf{t}}(x)\big)$
- $\forall x \big(R_{\psi_1 \mathsf{U} \psi_2}(x) \leftrightarrow \exists y (B(y) \wedge x < y \wedge R_{\psi_2}(y) \wedge \forall z (x < z < y \to R_{\psi_1}(z)))\big)$, for all $\psi_1 \mathsf{U} \psi_2 \in sub(\varphi)$;
- $\forall x \big(R_{\psi_1 \mathsf{S} \psi_2}(x) \leftrightarrow \exists y (y < x \wedge R_{\psi_2}(y) \wedge \forall z (y < z < x \to R_{\psi_1}(z)))\big)$, for all $\psi_1 \mathsf{S} \psi_2 \in sub(\varphi)$.

So assuming that B defines a full branch, $\rho(B)$ says that the R_ψ define a run in B. (Note that the R_ψ also occur free in $\rho(B)$.) Let $\beta(B)$ be a monadic second-order formula expressing that B is a full branch (a maximal linearly-ordered set). Thus, the following monadic second-order sentence μ is effectively constructible from φ:

$$\exists_{\Sigma \in \mathcal{C}} P_\Sigma \Bigg(\forall x \bigvee_{\Sigma \in \mathcal{C}} \Big[P_\Sigma(x) \wedge \bigwedge_{\substack{\Sigma' \in \mathcal{C} \\ \Sigma \neq \Sigma'}} \neg P_{\Sigma'}(x)\Big] \ \wedge\ \exists x \bigvee_{\substack{\Sigma \in \mathcal{C} \\ \varphi \in \bigcup \Sigma}} P_\Sigma(x) \quad \wedge\ \forall B \Big[\beta(B) \to \exists_{\psi \in sub(\varphi)} R_\psi \, \rho(B)\Big]$$
$$\wedge\ \forall x \bigwedge_{\substack{\Sigma \in \mathcal{C} \\ \mathbf{t} \in \Sigma}} \Big[P_\Sigma(x) \to \exists B \Big(\beta(B) \wedge B(x) \wedge \exists_{\psi \in sub(\varphi)} R_\psi (\rho(B) \wedge \chi_{\mathbf{t}}(x))\Big)\Big] \Bigg).$$

(If we are dealing with bundled quasimodels, we omit the conjunct $\forall B[\beta(B) \to \exists_{\psi \in sub(\varphi)} R_\psi \, \rho(B)]$.) It should be clear that for any ω-tree $\mathfrak{T}$, we have $\mathfrak{T} \models \mu$ iff there is a quasimodel for φ over $\mathfrak{T}$ (bundled or full, as appropriate). It follows from decidability of S2S [17] that it is decidable whether a given monadic second-order sentence is true in some countable ω-tree. The lemma now follows. ☐

As a consequence of Lemmas 5 and 6 we finally obtain

THEOREM 7. *It is decidable whether a $\mathcal{PCTL}^*$-formula has a full tree model in the non-local semantics. The same holds for bundled models.*

3 Decidable fragments of quantified $\mathcal{PCTL}^*$

$\mathcal{QPCTL}^*$, the *quantified* CTL^* *with past operators*, is obtained in the standard way by extending the language $\mathcal{QL}$ of classical first-order logic (without equality, constants, and function symbols) with the temporal operators $\mathcal{S}$, $\mathcal{U}$, $\bigcirc$, and the path existential quantifier E. Note that now we regard $\bigcirc$ as a primitive operator.

$\mathcal{QPCTL}^*$ is interpreted in structures of the form $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, D, I \rangle$, where $\mathfrak{T} = \langle T, < \rangle$ is an ω-tree, $\mathcal{H}$ is a set (*bundle*) of full branches of $\mathfrak{T}$ such that $\bigcup \mathcal{H} = T$, D is a non-empty set called the *domain* of $\mathfrak{M}$, and I is a function associating with every moment of time $t \in T$ a first-order $\mathcal{QL}$-structure

$$I(t) = \left\langle D, P_0^{I(t)}, P_1^{I(t)}, \ldots \right\rangle,$$

the *state* of $\mathfrak{M}$ at moment t. (Here, the $P_i^{I(t)}$ are predicates on D interpreting the predicate symbols P_i of $\mathcal{QL}$. We allow for 0-ary predicate symbols — that is, propositional variables.) As before, if $\mathcal{H}$ contains all full branches of $\mathfrak{T}$, we say that $\mathfrak{M}$ is a *full tree model*, or simply a *tree model*.

An *assignment in D* is a function $\mathfrak{a}$ from the set of individual variables of $\mathcal{QL}$ to D. The *truth relation* $(\mathfrak{M}, \beta, t) \models^{\mathfrak{a}} \varphi$, for $t \in \beta \in \mathcal{H}$, (or simply $(\beta, t) \models^{\mathfrak{a}} \varphi$ if $\mathfrak{M}$ is understood) is defined inductively as follows:

- $(\beta, t) \models^{\mathfrak{a}} P_i(x_1, \ldots, x_\ell)$ iff $P_i^{I(t)}(\mathfrak{a}(x_1), \ldots, \mathfrak{a}(x_\ell))$ holds in $I(t)$ (here the x_i are individual variables),
- $(\beta, t) \models^{\mathfrak{a}} \forall x \psi$ iff $(\beta, t) \models^{\mathfrak{b}} \psi$ for every assignment $\mathfrak{b}$ in D that may differ from $\mathfrak{a}$ only on x,
- $(\beta, t) \models^{\mathfrak{a}} \chi \mathsf{S} \psi$ iff there exists $v < t$ such that $(\beta, v) \models^{\mathfrak{a}} \psi$ and $(\beta, u) \models^{\mathfrak{a}} \chi$ for every $u \in (v, t)$,
- $(\beta, t) \models^{\mathfrak{a}} \chi \mathsf{U} \psi$ iff there is $v \in \beta$ such that $v > t$, $(\beta, v) \models^{\mathfrak{a}} \psi$ and $(\beta, u) \models^{\mathfrak{a}} \chi$ for every $u \in (t, v)$,
- $(\beta, t) \models^{\mathfrak{a}} \mathsf{E}\psi$ iff $(\beta', t) \models^{\mathfrak{a}} \psi$ for some $\beta' \in \mathcal{H}$ such that $t \in \beta'$,

plus the standard clauses for the booleans. Note that in this definition we use the traditional '*local*' semantics of CTL^* in which the truth values of atoms do not depend on the branch β of evaluation. For a formula $\varphi(\overline{x})$ and a tuple $\overline{a}$ of elements of D such that $\mathfrak{a}(\overline{x}) = \overline{a}$, we write $(\mathfrak{M}, \beta, t) \models \varphi(\overline{a})$ if $(\mathfrak{M}, \beta, t) \models^{\mathfrak{a}} \varphi$. We set $\Box_F^+ \varphi = \varphi \wedge \Box_F \varphi$ and $\Diamond_F^+ \varphi = \varphi \vee \Diamond_F \varphi$.

DEFINITION 8. Let $\mathcal{QPCTL}^{w}_{\boxed{1}}$ be the set of all $\mathcal{QPCTL}^*$-formulas φ satisfying the following conditions:

- every subformula of φ of the form $\bigcirc\psi$ contains at most one free variable,
- every subformula of φ of the form $\mathsf{E}\psi$, $\psi_1 \mathsf{U} \psi_2$, or $\psi_1 \mathsf{S} \psi_2$ contains no free variables (i.e., is a sentence).

We call the formulas in $\mathcal{QPCTL}^{w}_{\boxed{1}}$ *weak monodic* and $\mathcal{QPCTL}^{w}_{\boxed{1}}$ itself the *weak monodic fragment*[2] of $\mathcal{QPCTL}^*$.

It should be clear from the definition that $\mathcal{QPCTL}^{w}_{\boxed{1}}$ contains full propositional PCTL^* as well as the full first-order (non-temporal) language. The latter means, in particular, that the weak monodic fragment of $\mathcal{QPCTL}^*$ is undecidable. The main aim of this section is to prove a satisfiability criterion for weak monodic formulas (Corollary 17) and then apply it in order to obtain various decidable fragments of $\mathcal{QPCTL}^{w}_{\boxed{1}}$.

Bundled and full satisfiability. We begin by observing that satisfiability in bundled tree models can be reduced to satisfiability in full tree models. Indeed, given a $\mathcal{QPCTL}^{w}_{\boxed{1}}$-formula φ, we take a propositional variable q not occurring in φ and denote by $\varphi^{\uparrow}$ the result of replacing each subformula of φ of the form $\mathsf{E}\psi$ by $\mathsf{E}(\Diamond_F \Box_F q \wedge \psi)$. Note that if φ is a $\mathcal{QPCTL}^{w}_{\boxed{1}}$-formula, then so is $\varphi^{\uparrow}$.

LEMMA 9. *φ is satisfiable in a bundled tree model iff $(\mathsf{E}\varphi)^{\uparrow}$ is satisfiable in a full tree model.*

Proof. The implication ($\Leftarrow$) is easy. We prove ($\Rightarrow$). Using a Löwenheim–Skolem argument (cf. [3], and Lemma 1 above), we may assume φ to be satisfiable in a model $\mathfrak{M}$ with a countable bundle $\mathcal{H}$. We assume that $\mathcal{H}$ is infinite, leaving the (easy) other case to the reader. Let $\beta_0, \beta_1, \ldots$ be an enumeration of $\mathcal{H}$. We convert $\mathfrak{M}$ into a full tree model $\mathfrak{M}^{\dagger}$ with the same underlying ω-tree, and define a truth-relation for q in it inductively as follows. Put $(\mathfrak{M}^{\dagger}, \beta_0, t) \models q$ for all $t \in \beta_0$. Suppose that we have already defined truth of q at (β_i, t), for all $i \leq n$ and $t \in \beta_i$. Consider β_{n+1}. There must be a $t \in \beta_{n+1}$ such that the distance from t to each β_i, $i < n+1$, is ≥ 2 (the distance is the length of the shortest path from t to a point in β_i). Then we put $(\mathfrak{M}^{\dagger}, \beta_{n+1}, t') \models q$ if $t' \geq t$ and $t' \in \beta_{n+1}$.

Say that a full branch β is *marked* if there is $t \in \beta$ such that $(\mathfrak{M}^{\dagger}, \beta, t') \models q$ for all $t' \geq t$, $t' \in \beta$. One can easily see that β is marked iff $\beta \in \mathcal{H}$. In particular, if $\beta \notin \mathcal{H}$ and for each n, t_n is the least element of $\beta \setminus \bigcup_{m<n} \beta_m$, then $(\beta, t_n) \not\models q$ and $\{t_0, t_1, \ldots\}$ is infinite, so β is not marked. Now one can prove by induction that for every subformula ψ of φ and every (β, t), $(\mathfrak{M}, \beta, t) \models \psi$ iff $(\mathfrak{M}^{\dagger}, \beta, t) \models \psi^{\uparrow}$. It follows that $(\mathsf{E}\varphi)^{\uparrow}$ is satisfiable in $\mathfrak{M}^{\dagger}$. $\square$

All the fragments of $\mathcal{QPCTL}^{w}_{\boxed{1}}$ we consider in this paper are closed under the map

$$\varphi \mapsto (\mathsf{E}\varphi)^{\uparrow}.$$

So it will be sufficient to consider satisfiability in full tree models, which from now on will be denoted by $\mathfrak{M} = \langle \mathfrak{T}, D, I \rangle$.

Quasimodels. As in [11, 13], and Lemma 5 above, the idea of the decidability proof is to encode models in structures called quasimodels.

Fix a $\mathcal{QPCTL}^{w}_{\boxed{1}}$-sentence φ. Denote by $sub(\varphi)$ the closure under negation of the set of subformulas of φ. Without loss of generality, we may identify ψ and $\neg\neg\psi$, so $sub(\varphi)$ is finite. For $n < \omega$, denote by $sub_n(\varphi)$ the set of formulas in $sub(\varphi)$ with at most n free variables. Fix a variable x not occurring in φ, and let

$$sub_x(\varphi) = \{\psi\{x/y\} : \psi(y) \in sub_1(\varphi)\}.$$

[2]The 'monodic' fragment of $\mathcal{QPCTL}^*$ consists of all formulas whose subformulas of the form $\bigcirc\psi$, $\mathsf{E}\psi$, $\psi_1 \mathsf{U} \psi_2$, or $\psi_1 \mathsf{S} \psi_2$ have at most one free variable. However, [13] showed that even the one-variable fragment of $\mathcal{QPCTL}^*$ is undecidable. Since this is certainly monodic, we need further restrictions to obtain decidable fragments. For an alternative approach to that of Definition 8, see [13], where decidable fragments were obtained by restricting quantification to 'state formulas'.

Note that $sub(\varphi)$ and $sub_x(\varphi)$ contain the same sentences. Define a *type for* φ to be a subset $\mathbf{t}$ of $sub_x(\varphi)$ such that $\psi \wedge \chi \in \mathbf{t}$ iff $\psi \in \mathbf{t}$ and $\chi \in \mathbf{t}$, for every $\psi \wedge \chi \in sub_x(\varphi)$, and $\neg\psi \in \mathbf{t}$ iff $\psi \notin \mathbf{t}$, for every $\psi \in sub_x(\varphi)$.

For simplicity, we may assume that any *subsentence* $\bigcirc\psi$ of φ is replaced by $\bot \mathsf{U} \psi$. Thus, $\bigcirc$ is only applied to formulas with exactly one free variable. Now, for every formula $\theta(y)$ of the form $\bigcirc\psi(y) \in sub(\varphi)$ we reserve fresh unary predicates P^i_θ, and for every θ of the form $\mathsf{E}\psi$, $\psi_1 \mathsf{U} \psi_2$, or $\psi_1 \mathsf{S} \psi_2$ in $sub(\varphi)$ we reserve fresh propositional variables p^i_θ, where $i = 0, 1, \ldots$. The formulas $P^i_\theta(y)$ and p^i_θ are called the *i-surrogates* of $\theta(y)$ and θ, respectively. For $\psi \in sub(\varphi)$, denote by ψ^i the result of replacing in ψ all its subformulas of the form $\bigcirc\psi$, $\psi \mathsf{U} \chi$, $\psi \mathsf{S} \chi$, or $\mathsf{E}\psi$ that are not within the scope of another occurrence of a non-classical operator by their i-surrogates. Thus, ψ^i is a purely first-order (non-temporal) formula. Let $\Gamma^i = \{\chi^i : \chi \in \Gamma\}$ for any set $\Gamma \subseteq sub(\varphi)$. Similarly define ψ^i and Γ^i for $\psi \in sub_x(\varphi)$, $\Gamma \subseteq sub_x(\varphi)$.

The idea behind these definitions is as follows. The formulas χ^i abstract from the temporal component of χ and can be evaluated in a first-order structure without taking into account its temporal evolution. Of course, later we have to be able to reconstruct the truth value of χ in temporal models from the truth values of the χ^i. In contrast to the linear time case, we need a list of abstractions $\chi^0, \chi^1, \ldots$, since the temporal evolution depends on branches. So, intuitively, each $i < \omega$ represents a branch. (Actually, we will see that finitely many $i < \omega$ are enough, since we have to represent branches only up to a certain equivalence relation. So more accurately, i represents a 'kind' or 'species' of branch.)

Let $\mathbf{n}(\varphi) = 4^{4^{|sub_x(\varphi)|}}$.

DEFINITION 10. A *state candidate* for φ is a pair $\Theta = (\mathcal{S}, \mathcal{T})$ in which:

(i) $\mathcal{S} = \{S_1, \ldots, S_k\}$, where each S_i is a set of types for φ such that, for every sentence ψ, we have $\psi \in \mathbf{t}$ iff $\psi \in \mathbf{t}'$, for any $\mathbf{t}, \mathbf{t}' \in S_i$, and for every $\mathsf{E}\psi \in sub(\varphi)$,

$$\mathsf{E}\psi \in \bigcap_{i \le k} \bigcap S_i \quad \text{iff} \quad \mathsf{E}\psi \in \bigcup_{i \le k} \bigcap S_i \quad \text{iff} \quad \psi \in \bigcup_{i \le k} \bigcap S_i.$$

(ii) $\mathcal{T}$ is a set of maps $\tau : \{1, \ldots, n_\Theta\} \to \bigcup_{i \le k} S_i$, called *traces*, where $n_\Theta \le \mathbf{n}(\varphi)$ is a natural number depending on Θ and such that $\big\{\{\tau(i) : \tau \in \mathcal{T}\} : i \le n_\Theta\big\} = \mathcal{S}$.

The set of sentences in $\bigcap\{\tau(i) : \tau \in \mathcal{T}\}$ will be denoted by $\Theta(i)$. For a trace τ, we set

$$\overline{\tau} = \bigcup_{i \le n_\Theta} (\tau(i))^i, \qquad \overline{\mathcal{T}} = \{\overline{\tau} : \tau \in \mathcal{T}\}.$$

State candidates represent states w of temporal models. The intuition behind this definition will be clear from the proof of the theorem below. Here we only say that, roughly, the components S_i of a state candidate $\Theta = \langle \mathcal{S}, \mathcal{T} \rangle$ represent the states of a moment w in different branches, and each trace $\tau \in \mathcal{T}$ shows the types of one element of the domain D in these states (i.e., its possible states at moment w but in different histories). In short, the i correspond to kinds of branches (histories), and the τ to domain elements. n_Θ corresponds to the number of different kinds of branch through w.

DEFINITION 11. Let $\Theta = \langle \mathcal{S}, \mathcal{T} \rangle$ be a state candidate for φ, and

$$\mathfrak{D} = \Big\langle D, P_0^{\mathfrak{D}}, P_1^{\mathfrak{D}}, \ldots \Big\rangle$$

a first-order structure in the signature of $\mathcal{QPCTL}^{\mathsf{w}}_{\boxed{1}}$. For every $a \in D$ we define the *trace* of a (with respect to Θ) as

$$tr(a) = \{\psi \in \bigcup_{i \le n_\Theta} (sub_x(\varphi))^i : \mathfrak{D} \models \psi[a]\}.$$

We say that $\mathfrak{D}$ *realises* Θ if $\overline{\mathcal{T}} = \{tr(a) : a \in D\}$. We say that Θ is *realisable* if some $\mathfrak{D}$ realises it.

It follows immediately from the definition that we have:

LEMMA 12. *A state candidate $\Theta = \langle \mathcal{S}, \mathcal{T} \rangle$ for φ is realisable iff the first-order sentence*

$$\alpha_\Theta = \bigwedge_{\tau \in \mathcal{T}} \exists x \overline{\tau} \wedge \forall x \bigvee_{\tau \in \mathcal{T}} \overline{\tau}$$

is satisfiable.

DEFINITION 13. A *connection* is a quadruple (Δ, Θ, R, N) consisting of realisable state candidates $\Delta = (\mathcal{S}, \mathcal{T})$ and $\Theta = (\mathcal{U}, \mathcal{V})$, a relation $R \subseteq \mathcal{T} \times \mathcal{V}$ with domain $\mathcal{T}$ and range $\mathcal{V}$, and a relation $N \subseteq \{1, \ldots, n_\Delta\} \times \{1, \ldots, n_\Theta\}$ with range $\{1, \ldots, n_\Theta\}$, such that for all $(i, j) \in N$, all $(\tau, \tau') \in R$, and all $\bigcirc\psi \in sub(\varphi)$, we have $\bigcirc\psi \in \tau(i)$ iff $\psi \in \tau'(j)$.

A connection describes how (the abstract representation Θ of) a state w is related to (the abstract representation Δ of) its immediate predecessor v. To this end, the relation R between the traces in both representations is fixed. Intuitively, $R(\tau, \tau')$ indicates that τ (a trace at v) and τ' (a trace at w) represent the same domain point. The fact that the domain D is constant gives rise to the restriction on the domain and range of R. $N(i, j)$ indicates that there is a branch through w (and hence v) 'of kind i' at v and 'of kind j' at w. The fact that all branches through w go through v, but not perhaps conversely, suggests the restriction on the range of N.

For an ω-tree $\mathfrak{T} = \langle T, < \rangle$ and $w \in T$, denote by $B(w)$ the set of full branches of $\mathfrak{T}$ coming through w.

DEFINITION 14. A *quasimodel* for φ over $\mathfrak{T} = \langle T, < \rangle$ is a map f associating with the root w_0 of $\mathfrak{T}$ a pair $f(w_0) = (\Theta_{w_0}, g_{w_0})$, where Θ_{w_0} is a realisable state candidate, and with every non-root $w \in T$ a pair $f(w) = (C_w, g_w)$, where $C_w = (\Delta_w, \Theta_w, R_w, N_w)$ is a connection, and all g_w, for $w \in T$, are functions from $B(w)$ onto $\{1, \ldots, n_{\Theta_w}\}$ such that the following hold:

1. if $v \in T$ is the immediate predecessor of w, then $\Theta_v = \Delta_w$ and $N_w = \{(g_v(\beta), g_w(\beta)) : \beta \in B(w)\}$;
2. for all $\beta \in B(w)$, $\chi \mathsf{U} \psi \in \Theta_w(g_w(\beta))$ iff there exists $u > w$ with $u \in \beta$, $\psi \in \Theta_u(g_u(\beta))$ and $\chi \in \Theta_v(g_v(\beta))$ for all $v \in (w, u)$ ($\Theta(i)$ was defined after Definition 10);
3. for all $\beta \in B(w)$, $\chi \mathsf{S} \psi \in \Theta_w(g_w(\beta))$ iff there exists $u < w$ with $\psi \in \Theta_u(g_u(\beta))$ and $\chi \in \Theta_v(g_v(\beta))$ for all $v \in (u, w)$.

We say that f *satisfies* φ if there is $w \in T$ such that $\Theta_w = (\mathcal{S}_w, \mathcal{T}_w)$ and $\varphi \in \bigcup S$ for some $S \in \mathcal{S}_w$.

While the connections take care of the truth values of 'local' formulas of the form $\bigcirc\chi$, quasimodels take care of the remaining 'global' temporal operators.

THEOREM 15. *φ is satisfiable in a full model iff there exists a quasimodel satisfying φ.*

Proof. ($\Rightarrow$) Suppose that φ is satisfiable in some model. We may replace its tree $\mathfrak{T}$ by $\mathfrak{T}^+ = \mathfrak{T} \otimes \langle {}^{<\omega}\omega, \lhd \rangle$, as in the proof of Lemma 5; every branch of $\mathfrak{T}$ is 'duplicated' ω times in $\mathfrak{T}^+$ at each node, and φ is still satisfiable in the resulting model $\mathfrak{M} = \langle \mathfrak{T}^+, D, I \rangle$. So $(\mathfrak{M}, \sigma, v) \models^{\mathfrak{a}} \varphi$ for some $v \in T^+$, $\sigma \in B(v)$ (defined with respect to $\mathfrak{T}^+$) and some assignment $\mathfrak{a}$. Given $w \in T^+$ and $\beta \in B(w)$, let

$$S(\beta, w) = \{\mathrm{tp}(\beta, w, a) : a \in D\},$$

where

$$\mathrm{tp}(\beta, w, a) = \{\psi \in sub_x(\varphi) : (\mathfrak{M}, \beta, w) \models \psi[a]\}.$$

Let $\mathcal{S}_w = \{S(\beta, w) : \beta \in B(w)\}$. We extract from $\mathfrak{T}^+$ a subtree $\mathfrak{T}' = \langle T', <' \rangle$ in which every node has at most $\mathbf{m}(\varphi) = 2^{2^{|sub_x(\varphi)|}}$ immediate successors. To this end, we inductively define $T'_n \subseteq T^+$ with this property. Set $T'_0 = \{w_0\}$, where w_0 is the root of $\mathfrak{T}^+$. Given T'_n, for each $w \in T'_n$ with $ht(w) = n$, and each $S \in \mathcal{S}_w$, we pick $\beta_S \in B(w)$ such that $S(\beta_S, w) = S$, and (we use the form of $\mathfrak{T}^+$ here) $\beta_S \cap T'_n = \beta_S \cap \beta_{S'} = \{t \in T^+ : t \leq w\}$ for distinct $S, S' \in \mathcal{S}_w$. Let $B_w = \{\beta_S : S \in \mathcal{S}_w\}$, and $T_w = \bigcup B_w$. We can assume that $\sigma \in B_{w_0}$. Note that $|B_w| \leq \mathbf{m}(\varphi)$. Now set $T'_{n+1} = T'_n \cup \bigcup\{T_w : w \in T'_n,\ ht(w) = n\}$. Finally define $T' = \bigcup_{n<\omega} T'_n$. Note that $\sigma \subseteq T'$ and $v \in T'$.

Let $\mathfrak{M}' = \langle \mathfrak{T}', D, I' \rangle$ and $\mathfrak{T}'$ be the restrictions of $\mathfrak{M}$ and $\mathfrak{T}^+$ to T'. One can easily show by induction on the construction of $\psi \in sub(\varphi)$ that $(\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \psi$ iff $(\mathfrak{M}', \beta, w) \models^{\mathfrak{a}} \psi$, for all full branches β in $\mathfrak{T}'$ and all $w \in \beta$. (For example, suppose $(\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \mathsf{E}\psi$. Then there is $\beta' \in B(w)$ in $\mathfrak{T}^+$ such that $(\mathfrak{M}, \beta', w) \models^{\mathfrak{a}} \psi$. Pick a full branch γ in $\mathfrak{T}'$ for which $S(\beta', w) = S(\gamma, w)$. Since ψ is a sentence, we have $(\mathfrak{M}, \gamma, w) \models^{\mathfrak{a}} \psi$. It follows by the induction hypothesis that $(\mathfrak{M}', \gamma, w) \models^{\mathfrak{a}} \psi$ and so $(\mathfrak{M}', \beta, w) \models^{\mathfrak{a}} \mathsf{E}\psi$.)

Thus $\mathfrak{M}'$ satisfies φ and we can work with this model instead of $\mathfrak{M}$. Define an equivalence relation $\sim_w$ on $B(w)$ (defined in $\mathfrak{T}'$ now), for $w \in T'$, by taking $\beta \sim_w \beta'$ when $(\mathfrak{M}', \beta, w) \models^{\mathfrak{a}} \psi$ iff $(\mathfrak{M}', \beta', w) \models^{\mathfrak{a}} \psi$, for every $\psi \in sub(\varphi)$ and every assignment $\mathfrak{a}$. The $\sim_w$-equivalence class generated by β will be denoted by $[\beta]_w$.

Since only $\bigcirc$ is applied to open formulas, we can show that the number of $\sim_w$-equivalence classes is bounded by $\mathbf{n}(\varphi)$. To prove this, for $w \in T'$, full branches β and β' in $B(w)$, and $d < \omega$, we put $\beta \sim_w^d \beta'$ if for all $t \in T'$ with $t \geq w$ and $ht(t) \leq ht(w) + d$, we have

1. $t \in \beta$ iff $t \in \beta'$,

2. if $t \in \beta$, then for all assignments $\mathfrak{a}$ and all $\psi \in sub(\varphi)$ with at most $ht(w) + d - ht(t)$ occurrences of $\bigcirc$, we have $(\beta, t) \models^{\mathfrak{a}} \psi$ iff $(\beta', t) \models^{\mathfrak{a}} \psi$.

An induction on d shows that the number $\sharp(d)$ of $\sim_w^d$-classes is at most $\mathbf{m}(\varphi)^d \cdot 2^{(d+1)|sub_0(\varphi)|}$ (for any w). For $d = 0$, one may check that if $(\beta, w) \models \psi$ iff $(\beta', w) \models \psi$ for each sentence $\psi \in sub(\varphi)$, then $\beta \sim_w^0 \beta'$. So $\sharp(0) \leq 2^{|sub_0(\varphi)|}$. Assume the result for d. One may check that if $\beta, \beta' \in B(w)$ contain a common immediate successor v of w, $(\beta, w) \models \psi$ iff $(\beta', w) \models \psi$ for each sentence $\psi \in sub(\varphi)$, and $\beta \sim_v^d \beta'$, then $\beta \sim_w^{d+1} \beta'$. Both checks involve an induction on ψ in (2) above. It follows that $\sharp(d+1) \leq \mathbf{m}(\varphi) \cdot 2^{|sub_0(\varphi)|} \cdot \sharp(d)$, and hence that $\sharp(d) \leq \mathbf{m}(\varphi)^d \cdot 2^{(d+1)|sub_0(\varphi)|}$ for all d, as required. Finally observe that if $\beta \sim_w^{|sub_x(\varphi)|} \beta'$ then $\beta \sim_w \beta'$, so that $\sim_w$ has at most $\sharp(|sub_x(\varphi)|) \leq \mathbf{n}(\varphi)$ classes.

Let $\beta_1^w, \dots \beta_{n_w}^w$ be some minimal list of full branches such that $\{[\beta_1^w]_w, \dots, [\beta_{n_w}^w]_w\}$ is the set of all $\sim_w$-equivalence classes. With each $a \in D$ we associate a trace

$$\tau_a^w : \{1, \dots, n_w\} \to \bigcup \mathcal{S}_w$$

by taking $\tau_a^w(i) = \mathrm{tp}(\beta_i^w, w, a)$. Denote the resulting set of traces by $\mathcal{T}_w$. Let $\Theta_w = (\mathcal{S}_w, \mathcal{T}_w)$ for all $w \in T'$. We are now in a position to define a quasimodel f over $\mathfrak{T}'$ satisfying φ. If w is not the root, then set $f(w) = ((\Theta_v, \Theta_w, R_w, N_w), g_w)$, where v is the immediate predecessor of w, and for root w_0 let $f(w_0) = (\Theta_{w_0}, g_{w_0})$, where

- $g_w(\beta) = i$ iff $\beta \in [\beta_i^w]_w$,
- $R_w = \{(\tau_a^v, \tau_a^w) : a \in D\}$,
- $N_w = \{(g_v(\beta), g_w(\beta)) : \beta \in B(w)\}$.

Let us show that f is a quasimodel. It should be clear that the first component of each $f(w)$ ($w_0 \neq w$) is a connection and that of $f(w_0)$ is a realisable state candidate. We will check only item 2 of Definition 14. Suppose $g_w(\beta) = i$ and $\chi \mathsf{U} \psi \in \Theta_w(i)$. Since $\chi \mathsf{U} \psi$ is a sentence, we have $\chi \mathsf{U} \psi \in \tau_a^w(i) = \mathrm{tp}(\beta_i^w, w, a) = \mathrm{tp}(\beta, w, a)$ for every $a \in D$. So there is $u > w$ with $u \in \beta$ and $\psi \in \mathrm{tp}(\beta, u, a)$ (from which $\psi \in \Theta_u(g_u(\beta))$ follows) and for all $v \in (w, u)$ we have $\chi \in \mathrm{tp}(\beta, v, a)$ (from which $\chi \in \Theta_v(g_v(\beta))$ follows). The converse implication is proved similarly.

($\Leftarrow$) Now suppose that f is a quasimodel for φ over $\mathfrak{T} = \langle T, < \rangle$ with root w_0. Let $f(w_0) = (\Theta_{w_0}, g_{w_0})$ and $f(w) = (C_w, g_w) = ((\Delta_w, \Theta_w, R_w, N_w), g_w)$ for non-root $w \in T$. Let $\Theta_w = (\mathcal{S}_w, \mathcal{T}_w)$ and $n_w = n_{\Theta_w}$.

A *run* r in f is a function associating with any $w \in T$ a trace $r(w) \in \mathcal{T}_w$ such that $(r(v), r(w)) \in R_w$ for any non-root w with immediate predecessor v. Using the condition that the range and domain of R_w coincide with $\{1, \ldots, n_w\}$ and $\{1, \ldots, n_v\}$, respectively, it is not difficult to see that, for any w and any $\tau \in \mathcal{T}_w$, there exists a run r with $r(w) = \tau$. Let $\mathcal{R}$ be the set of all runs.

Take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of $\mathcal{R}$. Then, by classical model theory (since the language is countable and without equality; cf. [11, Lemma 9]), for every $w \in T$ we can find a first-order structure $I(w)$ with domain

$$D = \{\langle r, \xi \rangle \mid r \in \mathcal{R}, \xi < \kappa\}$$

realising $\Theta_w = (\mathcal{S}_w, \mathcal{T}_w)$ and such that for all $i \in \{1, \ldots, n_w\}$, $r \in \mathcal{R}$, $\xi < \kappa$, and $\psi \in sub_x(\varphi)$,

$$\psi \in \big(r(w)\big)(i) \quad \text{iff} \quad I(w) \models \psi^i[\langle r, \xi \rangle]. \tag{2}$$

Let $\mathfrak{M} = \langle \mathfrak{T}, D, I \rangle$ and let $\mathfrak{a}$ be any assignment in D. We show by induction that for all $\psi \in sub(\varphi)$, all $w \in T$, and all $\beta \in B(w)$ with $g_w(\beta) = i$, say, we have

$$I(w) \models^{\mathfrak{a}} \psi^i \quad \text{iff} \quad (\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \psi.$$

The case of atomic ψ is easy, since $\psi^i = \psi$ and by definition of $\mathfrak{M}$. The booleans are also easy. Suppose that $\psi = \chi_1 \mathsf{U} \chi_2$. Then ψ is a sentence, and for all $r \in \mathcal{R}$ we have

$$\begin{aligned}
I(w) \models^{\mathfrak{a}} p_\psi^i \quad &\Leftrightarrow \quad \chi_1 \mathsf{U} \chi_2 \in r(w)(g_w(\beta)) \ \text{ (by (2))} \\
&\Leftrightarrow \quad \exists u > w \big(u \in \beta \wedge \chi_2 \in r(u)(g_u(\beta)) \wedge \\
&\qquad \forall v \in (w, u)\, (\chi_1 \in r_v(g_v(\beta)))\big) \ \text{ (since } r \text{ is a run and } \psi \text{ is a sentence)} \\
&\Leftrightarrow \quad \exists u > w (u \in \beta \wedge I(u) \models^{\mathfrak{a}} \chi_2^{g_u(\beta)} \wedge \forall v \in (w, u) I(v) \models^{\mathfrak{a}} \chi_1^{g_v(\beta)}) \ \text{ (by (2))} \\
&\Leftrightarrow \quad \exists u > w (u \in \beta \wedge (\mathfrak{M}, \beta, u) \models^{\mathfrak{a}} \chi_2 \wedge \forall v \in (w, u) (\mathfrak{M}, \beta, v) \models^{\mathfrak{a}} \chi_1) \ \text{ (by IH)} \\
&\Leftrightarrow \quad (\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \psi.
\end{aligned}$$

The case of $\psi = \chi_1 \mathsf{S} \chi_2$ is a mirror image. Now suppose $\psi = \bigcirc \chi$. Then for any $r \in \mathcal{R}$, $\xi < \kappa$:

$$\begin{aligned}
I(w) \models P_\psi^i[\langle r, \xi \rangle] \quad &\Leftrightarrow \quad \bigcirc \chi \in r(w)(g_w(\beta)) \ \text{ (by (2))} \\
&\Leftrightarrow \quad \chi \in r(v)(g_v(\beta)), \text{ for the immediate successor } v \text{ of } w \text{ in } \beta, \\
&\qquad \text{since } C_v \text{ is a connection and } r \text{ a run} \\
&\Leftrightarrow \quad I(v) \models \chi^{g_v(\beta)}[\langle r, \xi \rangle] \ \text{ (by (2))} \\
&\Leftrightarrow \quad (\mathfrak{M}, \beta, v) \models \chi[\langle r, \xi \rangle] \ \text{ (by IH)} \\
&\Leftrightarrow \quad (\mathfrak{M}, \beta, w) \models \bigcirc \chi[\langle r, \xi \rangle].
\end{aligned}$$

For $\psi = \mathsf{E}\chi$:

$$\begin{aligned} I(w) \models^{\mathfrak{a}} p^i_\psi \quad &\Leftrightarrow \quad \mathsf{E}\chi \in r(w)(g_w(\beta)) \ \ \text{(by (2))} \\ &\Leftrightarrow \quad \chi \in r(w)(g_w(\beta')), \ \text{for some } \beta' \in B(w), \text{ since } \Theta_w \text{ is} \\ &\qquad \text{a state candidate, } g_w \text{ is surjective and } \psi \text{ is a sentence} \\ &\Leftrightarrow \quad \exists \beta' \, (\beta' \in B(w) \wedge I(w) \models^{\mathfrak{a}} \chi^{g_w(\beta')}) \ \ \text{(by (2))} \\ &\Leftrightarrow \quad \exists \beta' \, (\beta' \in B(w) \wedge (\mathfrak{M}, \beta', w) \models^{\mathfrak{a}} \chi) \ \ \text{(by IH)} \\ &\Leftrightarrow \quad (\mathfrak{M}, \beta, w) \models \mathsf{E}\chi. \end{aligned}$$

Since $\varphi \in r(w)(g_w(\sigma))$ for some $w \in T$, $\sigma \in B(w)$ and $r \in \mathcal{R}$, we finally obtain $(\mathfrak{M}, \sigma, w) \models \varphi$. □

Now we construct a reduction of $\mathcal{QPCTL}^w_{\boxed{1}}$ to non-local $\mathcal{PCTL}^*$, by encoding quasimodels in non-local propositional tree models. Suppose again that a $\mathcal{QPCTL}^w_{\boxed{1}}$-sentence φ is fixed.

With every realisable state candidate $\Theta = (\mathcal{S}, \mathcal{T})$ for φ, every connection C, and every $i \leq \mathbf{n}(\varphi)$, we associate propositional variables p_Θ, p_C, and p_i, respectively. Let $\mathcal{R}(\varphi)$ and $\mathcal{C}(\varphi)$ be the sets of realisable state candidates and connections for φ, respectively. For a sentence $\psi \in sub(\varphi)$, define

$$\psi^\sharp = \Bigg(\bigvee_{\substack{\Theta \in \mathcal{R}(\varphi), \\ i \leq n_\Theta, \ \psi \in \Theta(i)}} (p_i \wedge p_\Theta \wedge \neg \Diamond_P \top) \Bigg) \vee \Bigg(\bigvee_{\substack{C = (\Delta, \Theta, R, N) \in \mathcal{C}(\varphi), \\ i \leq n_\Theta, \ \psi \in \Theta(i)}} (p_i \wedge p_C \wedge \Diamond_P \top) \Bigg).$$

Let

$$\varphi^\star = \varphi^\sharp \wedge \big((\nu \wedge \neg \Diamond_P \top) \vee \Diamond_P (\nu \wedge \neg \Diamond_P \top) \big),$$

where ν is the conjunction of the formulas (3)–(9) defined below.

$$\bigvee_{\Theta \in \mathcal{R}(\varphi)} \mathsf{A} p_\Theta \wedge \bigwedge_{\Theta \neq \Theta'} \mathsf{A}(p_\Theta \to \neg p_{\Theta'}), \qquad \mathsf{A}\Box_F \Big(\bigvee_{C \in \mathcal{C}(\varphi)} \mathsf{A} p_C \wedge \bigwedge_{C \neq C'} \mathsf{A}(p_C \to \neg p_{C'}) \Big). \tag{3}$$

The formulas in (3) say that the p_Θ and p_C are 'local' (so we can write $w \models p_\Theta$ and $w \models p_C$) and that precisely one p_Θ holds at the root and precisely one p_C holds at each non-root point.

Intuitively, $w \models p_C$ means that $f(w) = (C, g)$, for some g. Say that a pair of connections (C_1, C_2) is *suitable* if the second state candidate of C_1 coincides with the first state candidate of C_2. The set of all suitable pairs of connections is denoted by $\mathcal{C}_s(\varphi)$. A pair (Θ, C) is *suitable* if the first state candidate of C coincides with Θ. The set of all suitable pairs of this form is denoted by $\mathcal{R}_s(\varphi)$. The following formulas say that the pair induced by a point and its immediate predecessor is suitable:

$$\mathsf{A} \bigvee_{(\Theta, C) \in \mathcal{R}_s(\varphi)} (p_\Theta \wedge \bigcirc p_C), \qquad \mathsf{A}\Box_F \bigvee_{(C_1, C_2) \in \mathcal{C}_s(\varphi)} (p_{C_1} \wedge \bigcirc p_{C_2}). \tag{4}$$

Intuitively, the p_i code g_w: for i such that $1 \leq i \leq \mathbf{n}(\varphi)$, $(\beta, w) \models p_i$ is intended to mean $g_w(\beta) = i$. (Here we need the non-local semantics.) This is ensured by the formulas

$$\mathsf{A} \bigwedge_{1 \leq i < j \leq \mathbf{n}(\varphi)} (p_i \to \neg p_j), \quad \mathsf{A}\Box_F \bigwedge_{1 \leq i < j \leq \mathbf{n}(\varphi)} (p_i \to \neg p_j), \tag{5}$$

$$\bigwedge_{\Theta \in \mathcal{R}(\varphi)} \Big(p_\Theta \to \bigwedge_{1 \leq i \leq n_\Theta} \mathsf{E} p_i \wedge \mathsf{A} \bigvee_{1 \leq i \leq n_\Theta} p_i \Big), \qquad \mathsf{A}\Box_F \bigwedge_{C \in \mathcal{C}(\varphi)} \Big(p_C \to \bigwedge_{1 \leq i \leq n_\Theta} \mathsf{E} p_i \wedge \mathsf{A} \bigvee_{1 \leq i \leq n_\Theta} p_i \Big). \tag{6}$$

Here and below we assume that $C = (\Delta, \Theta, R, N)$. It remains to capture the conditions of Definition 14. First, we write down a formula which says that N in C is determined by the functions g_w:

$$\mathsf{A}\Box_F \bigwedge_{C \in \mathcal{C}(\varphi)} \Big(p_C \to \big(\bigwedge_{(i,j)\in N} \mathsf{E}(p_j \wedge \bigcirc_P p_i) \wedge \mathsf{A} \bigvee_{(i,j)\in N} (p_j \wedge \bigcirc_P p_i)\big)\Big). \tag{7}$$

Finally, to capture conditions 2 and 3 of Definition 14, we include, for every sentence $\psi \mathsf{U} \chi \in sub(\varphi)$,

$$\mathsf{A}\Box_F^+ \big((\psi \mathsf{U} \chi)^\sharp \leftrightarrow (\psi^\sharp \mathsf{U} \chi^\sharp)\big), \tag{8}$$

and for every sentence $\psi \mathsf{S} \chi \in sub(\varphi)$,

$$\mathsf{A}\Box_F^+ \big((\psi \mathsf{S} \chi)^\sharp \leftrightarrow (\psi^\sharp \mathsf{S} \chi^\sharp)\big). \tag{9}$$

THEOREM 16. *A $\mathcal{QPCTL}^{w}_{\Box}$-sentence φ is satisfiable in a full model iff the $\mathcal{PCTL}^*$-formula $\varphi^\star$ is satisfiable in a full non-local model.*

Proof. $(\Rightarrow)$ If φ is satisfiable, then by Theorem 15 there is a quasimodel f for φ based on an ω-tree $\mathfrak{T} = \langle T, < \rangle$. Let $f(w) = (C_w, g_w) = ((\Delta_w, \Theta_w, N_w, R_w), g_w)$ if w is not the root and $f(w_0) = (\Theta_{w_0}, g_{w_0})$ for the root w_0 of $\mathfrak{T}$. Define a (propositional) valuation h in $\mathfrak{T}$ by taking, for all $w \in T$ and $\beta \in B(w)$:

- $(\beta, w) \in h(p_\Theta)$ iff $\Theta = \Theta_w$, for every realisable state candidate Θ;
- $(\beta, w) \in h(p_C)$ iff $C = C_w$, for every connection C (where $w \neq w_0$);
- $(\beta, w) \in h(p_i)$ iff $g_w(\beta) = i$, for all $i < \mathbf{n}(\varphi)$.

Let us prove that $\varphi^\star$ is satisfiable in the full model $\mathfrak{M} = \langle \mathfrak{T}, h \rangle$.

It should be clear from the definitions that for any sentence $\psi \in sub(\varphi)$,

$$(\beta, w) \models \psi^\sharp \iff \psi \in \Theta_w(g_w(\beta)). \tag{10}$$

Since $\varphi \in \Theta_w(g_w(\beta))$, for some $w \in T$ and $\beta \in B(w)$, we have that $(\mathfrak{M}, \beta, w) \models \varphi^\sharp$.

Now we show that $(\beta, w) \models (\nu \wedge \neg \Diamond_P \top) \vee \Diamond_P(\nu \wedge \neg \Diamond_P \top)$, i.e., $(\beta, w_0) \models \nu$, where w_0 is the root of $\mathfrak{T}$. We only check formulas (7), (8) and (9).

(7) Suppose $(\gamma, v) \models p_C$, for a full branch γ of $\mathfrak{T}$ and some $v \in \gamma$ such that $v \neq w_0$. Then by definition of $h(p_C)$, we have $C = C_v$. According to Definition 14, for each pair $(i, j) \in N_v$ there is a branch in $B(v)$, say γ', such that $g_v(\gamma') = j$ and $g_u(\gamma') = i$, for the immediate predecessor u of v. Hence, $(\gamma', v) \models p_j$ and $(\gamma', u) \models p_i$, from which $(\gamma, v) \models \mathsf{E}(p_j \wedge \bigcirc_P p_i)$ for all $(i, j) \in N_v$. Definition 14 also says that for each branch $\gamma' \in B(v)$, there is a pair $(i, j) \in N_v$ such that $g_v(\gamma') = j$ and $g_u(\gamma') = i$, where u is the immediate predecessor of v. This means that $(\gamma', v) \models p_j$ and $(\gamma', u) \models p_i$. So $(\gamma, v) \models \mathsf{A} \bigvee_{(i,j)\in N_v} (p_j \wedge \bigcirc_P p_i)$.

(8) and (9) follow immediately from (10) and conditions 2 and 3 of Definition 14.

$(\Leftarrow)$ Conversely, suppose $\mathfrak{M} = \langle \mathfrak{T}, h \rangle$ satisfies $\varphi^\star$. Then ν is true at the root w_0 of $\mathfrak{T}$. Define a quasimodel f by taking $f(w) = (C_w, g_w) = ((\Delta_w, \Theta_w, N_w, R_w), g_w)$ if $w \neq w_0$, and $f(w_0) = (\Theta_{w_0}, g_{w_0})$, where

- Θ_{w_0} is the unique Θ for which $w_0 \models p_\Theta$ (this is independent of the branch of evaluation);
- for $w \neq w_0$, C_w is the unique C such that $w \models p_C$;
- $g_w(\beta) = i$ for the unique i for which $(\beta, w) \models p_i$.

We show that f is a quasimodel by checking the conditions of Definition 14. The first part of condition 1 follows from the definition of suitable pair and formulas (3) and (4). Suppose now that $(i, j) \in N_w$. We have $w \models p_{C_w}$. Hence, by (7), $w \models \mathsf{E}(p_j \wedge \bigcirc_P p_i)$. So there is $\sigma \in B(w)$ with $(\sigma, w) \models p_j \wedge \bigcirc_P p_i$. This implies $g_w(\sigma) = j$ and $g_v(\sigma) = i$, for the immediate predecessor v of w. Conversely, suppose $g_w(\sigma) = j$ and $g_v(\sigma) = i$, where v is the immediate predecessor of w. Then $(\sigma, v) \models p_i$ and $(\sigma, w) \models p_j$. Hence $w \models \mathsf{E}(p_j \wedge \bigcirc_P p_i)$ and so, by (7) and (5), $(i, j) \in N_w$. Thus, the second part of condition 1 of Definition 14 holds. To prove conditions 2 and 3, we just observe that (10) holds again; the conditions then follow from (8) and (9).

Finally, we check that f satisfies φ. By assumption, there is $w \in T$ and $\beta \in B(w)$ such that $(\beta, w) \models \varphi^\star$. Hence, $(\beta, w) \models \varphi^\sharp$. By (10), $\varphi \in \Theta_w(g_w(\beta))$. It follows that f satisfies φ. □

If $\mathcal{L} \subseteq \mathcal{QPCTL}^{w}_{\boxed{1}}$ is a fragment such that it is decidable whether a given state candidate for a given sentence $\varphi \in \mathcal{L}$ is realisable, then it is clear that the map

$$\cdot^\star : \mathcal{L} \to \mathcal{PCTL}^*$$

is effective. Hence, we obtain:

COROLLARY 17. *Let $\mathcal{L} \subseteq \mathcal{QPCTL}^{w}_{\boxed{1}}$ and suppose that for any sentence $\varphi \in \mathcal{L}$ it is decidable whether a given state candidate for φ is realisable. Then the satisfiability problem for $\mathcal{L}$-formulas in full models is decidable.*

If, moreover, $\mathcal{L}$ is closed under the map $\varphi \mapsto (\mathsf{E}\varphi)^\uparrow$ (with fresh propositional variable q), then the satisfiability problem for $\mathcal{L}$-sentences in bundled models is decidable as well.

Proof. The first part follows from Theorem 7 and Theorem 16. The second part follows from the first part and Lemma 9. □

4 Applications

Denote by $\mathcal{L}_2$ the *two-variable fragment* of $\mathcal{QPCTL}^{w}_{\boxed{1}}$ which consists of all $\mathcal{QPCTL}^{w}_{\boxed{1}}$-formulas containing only two variables, say x and y. Obviously, for any state candidate Θ for a sentence $\varphi \in \mathcal{L}_2$, the sentence α_Θ belongs to the two-variable fragment of $\mathcal{QL}$. The two-variable fragment of $\mathcal{QL}$ is known to be decidable [16]. So it is decidable whether a state candidate for a $\mathcal{L}_2$-sentence is realisable. Moreover, $\mathcal{L}_2$ is closed under the map $\varphi \mapsto (\mathsf{E}\varphi)^\uparrow$. As a consequence, we obtain from Corollary 17:

THEOREM 18. *The satisfiability problem for the two-variable fragment of $\mathcal{QPCTL}^{w}_{\boxed{1}}$ is decidable both in bundled and in full models.*

The *monadic fragment* of $\mathcal{QPCTL}^{w}_{\boxed{1}}$ consists of all $\mathcal{QPCTL}^{w}_{\boxed{1}}$-formulas containing only 0-ary and unary predicate symbols. In the same manner as above, we obtain from Corollary 17 and the decidability of the monadic fragment of first-order logic:

THEOREM 19. *The satisfiability problem for the monadic fragment of $\mathcal{QPCTL}^{w}_{\boxed{1}}$ is decidable both in bundled and in full models.*

The *guarded fragment* of $\mathcal{QPCTL}^{w}_{\boxed{1}}$ consists of all $\mathcal{QPCTL}^{w}_{\boxed{1}}$-formulas in which all uses of $\forall$ follow the 'guarded' pattern $\forall \overline{y}(\alpha(\overline{x}, \overline{y}) \to \varphi(\overline{x}, \overline{y}))$, where α is atomic and involves all variables occurring free in φ. It is defined as in the linear time case [11, Definition 72]. We now obtain from Corollary 17 and the decidability of the guarded fragment of first-order logic [1, 8]:

THEOREM 20. *The satisfiability problem for the guarded fragment of $\mathcal{QPCTL}^{w}_{\square}$ is decidable both in bundled and in full models.*

Similar results can be proven for the loosely guarded and packed (or clique-guarded) fragments of $\mathcal{QPCTL}^{w}_{\square}$ (see [2, 15, 7]). Moreover, equality can be added in these cases: cf. [10].

A simple extension of the above proof covers the case when the underlying first-order signature contains constants, interpreted rigidly in temporal models: cf. [11, 10].

5 Conclusion

The decidability result for the weak one-variable fragment of first-order PCTL^* can be used to obtain decidability results for certain spatio-temporal logics connecting PCTL^* with region connection calculus RCC-8 (see the survey paper [12] or the monograph [6]). From this viewpoint it has sufficient expressive power to be useful. However, there is still a gap between the undecidability of the one-variable fragment of first-order CTL^* and the decidability of its weak one-variable fragment. In particular, the following problems are still open. What happens if the path-quantifier E is applied to open formulas as well? Or, what happens if all temporal operators are applied to open formulas (but E only to sentences)?

Another open problem is the computational complexity of the logics considered here. From the reduction proofs provided in the present paper we obtain only non-elementary decision procedures. We do not believe that this is optimal.

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