A tandem network of fluid queues with on-off arrivals

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Abstract

Fluid models have for some time been used to approximate stochastic networks with discrete state. These range from traditional 'heavy traffic' approximations to the recent advances in bio-chemical system models. Here we obtain an exact solution for a pair of two queues linked in tandem with on-off arrivals at the first and departures from the second. The solution method of the resulting vector differential equation is via Laplace transforms, which yields joint moments (e.g. covariance) directly and can be inverted to give the steady state joint probability distribution of the fluid levels at the two queues. The exactness of this result can be used to validate simulations, which in turn can be used to validate more complex models, and also suggests a route to analytical solution of larger networks.

1 Introduction

Stochastic fluid flow models have for long been used to describe networks of nodes that provide service to traffic of some sort that flows amongst them. Such models can exactly describe systems with *continuous state*, for example volumes in literal fluid flows, but more commonly are used to approximate *discrete state systems* of *traffic flows*. Traffic is measured in integer units of work such as packets in an IP network, jobs in a computer system or vehicles on roads. The motivation for these continuous approximations is that the numbers of states in discrete systems rapidly become prohibitively large as the complexity of the systems increase. Moreover, it can often be proved that as the rates of the discrete traffic inputs and of processing at the nodes jointly tend to infinity, the system's behaviour approaches that of a corresponding continuous model – a so-called 'fluid limit' [1]. Of course in practice, it is not a pre-requisite for such a limit to exist when entertaining a fluid-based model since all models are just abstractions of a real system. It may be that no analogous discrete model has been considered at all or, if it has, for that model to be inherently superior.

The adoption of fluid models has recently been taken up in the field of stochastic process algebra where large numbers of cooperating components in a concurrent system are approximated by a 'volume' of fluid [2]. Process algebras are notorious for their general profligacy in state space, but recent applications in biochemistry increase traditional discrete state space sizes by several orders of magnitude. Hence the representation of a very large number of cooperating

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identical components by a single non-negative real number is attractive and has met with considerable success [3].

In general, the differential equations arising from a fluid model have to be solved numerically, for example by stochastic simulation [4, 5, 6]. However, for a single fluid queue with Markovian on-off arrivals, the equilibrium solution (when it exists) for the probability distribution of the volume of fluid in the queue is exponential, closely resembling that of a traditional M/M/1 queue. This motivates the study of networks of $m \geq 2$ such fluid queues. However, the resulting differential equations are coupled, or equivalently are for a vector field in m variables representing the fluid levels in m queues. Moreover, the number of components in the vector is equal to the product of the numbers of states in the modulating arrival processes – e.g. 2^a if there are a external on-off arrival processes.

In this short paper, we consider the simplest case: a tandem network of 2 fluid queues with on-off arrival to the first queue which feeds into the second. We obtain the exact solution to this network via Laplace transforms and the entirely constructive derivation suggests how more complex networks might be analysed. In the full paper, we also consider an approximate method in which the nodes are considered independent, with appropriately chosen on-off arrival processes that match the average arrival rates. This approximation is validated against simulation.

2 The differential equation

2.1 Review of the single fluid queue

First consider a single fluid queue, comprising a server that outputs fluid, a reservoir where input fluid is stored and a Markov modulated input (or arrival) stream. Suppose that:

- there are n phases, or states, in the continuous time Markov chain, which has generator matrix $Q = (q_{ij} \mid 1 \leq i, j \leq n)$ and equilibrium probabilities $\vec{\pi}$ (so that $\pi Q = \vec{0}$ and $\pi \vec{e}^{\mathrm{T}} = 1$, where $e = (1, 1, \ldots, 1)$);
- the arrival rate in phase i is the constant λ_i volume-units of fluid per unit time;
- the rate at which the server outputs fluid when its reservoir is non-empty is μ volume-units of fluid per unit time.

We define the diagonal (net input) *rate matrix* $R = \mathrm{diag}(r_1, \ldots, r_n)$, where $r_i = \lambda_i - \mu$ for $1 \leq i \leq n$. The following, rather unrigorous, argument leads to a differential equation for the equilibrium fluid level probability distribution. Let the phase and the fluid level in the reservoir at time t be denoted by N_t and X_t respectively and define $\vec{F}(x, t) = (F_1(x, t), \ldots, F_n(x, t))$, where

$$F_i(x, t) = \mathbb{P}(N_t = i, X_t \leq x)$$

Now consider the infinitesimal interval $(t, t+h]$ for some small h. Then we have, to first order in h,

$$F_i(x, t+h) = (1 + q_{ii}) F_i(x - r_i h, t) + \sum_{j \neq i} F_j(x, t) q_{ji} h + o(h)$$

Thus we have:

$$\begin{aligned}
\frac{F_i(x,t+h)-F_i(x,t)}{h} &= F_i(x,t)q_{ii} - r_i\frac{\partial F_i(x,t)}{\partial x} + \sum_{j\neq i} F_j(x,t)q_{ji} + O(h) \\
&= -r_i\frac{\partial F_i(x,t)}{\partial x} + \sum_{j=1}^{n} F_j(x,t)q_{ji} + O(h)
\end{aligned}$$

so that in the limit $h \to 0$

$$\frac{\partial \vec{F}}{\partial t} = -\frac{\partial \vec{F}}{\partial x}R + \vec{F}Q$$

and, at equilibrium when this exists,

$$\vec{F}_x R = \vec{F}Q$$

where we use the subscript x to denote a partial derivative with respect to x. In addition, we have the boundary conditions that $F_i(0) = 0$ if $r_i > 0$, which reflects the fact that the reservoir cannot be empty when there is a positive net input. We solve this equation, under the further boundary condition at infinity that $\vec{F}(\infty) = \vec{\pi}$, by taking Laplace transforms in the next main section.

2.2 Tandem fluid queues

The vector differential equation describing two queues in tandem – and indeed more complex networks – follows immediately from the arguments used for a single fluid queue in the previous section. The model is therefore specified by:

- a Markov modulated on-off arrival process at node 1 with generator matrix $Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$ and equilibrium probabilities $\vec{\pi}$;
- constant on-rate λ in phase 1 and zero in phase 2 for arrivals at node 1, no external arrivals at node 2;
- server 1 outputs fluid at constant rate μ_1 when its reservoir is non-empty, sending it to node 2, with no fluid output from an empty reservoir;
- similarly, server 2 receives fluid input at constant rate μ_1 when node 1 has a non-empty reservoir, has no input when node 1 has an empty reservoir and outputs fluid at constant rate μ_2 when its own reservoir is non-empty;
- diagonal rate matrices $R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ and $S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$, where $r_1 = \lambda - \mu_1, r_2 = -\mu_1$ and $s_1 = s_2 = \mu_1 - \mu_2$.

Let the phase of the arrival process to node 1 and the fluid levels in reservoirs 1 and 2 at time t be denoted by N_t, X_t and Y_t respectively and define $\vec{F}(x,y,t) = (F_1(x,y,t), F_2(x,y,t))$, where

$$F_i(x,y,t) = \mathbb{P}(N_t = i, X_t \le x, Y_t \le y)$$

As for the single fluid queue, consider the infinitesimal interval $(t, t+h]$ for some small h. Then we have, for $x, y > 0$, to first order in h,

$$F_i(x,y,t+h) \quad = \quad (1+q_{ii})F_i(x-r_ih, y-s_ih, t) + \sum_{j\neq i} F_j(x,y,t)q_{ji}h + o(h)$$

Thus we have:

$$\frac{F_i(x,y,t+h) - F_i(x,y,t)}{h} \quad = \quad -r_i\frac{\partial F_i(x,y,t)}{\partial x} - s_i\frac{\partial F_i(x,y,t)}{\partial y} + \sum_{j=1}^{n} F_j(x,y,t)q_{ji} + O(h)$$

so that in the limit $h \to 0$

$$\frac{\partial \vec{F}}{\partial t} = -\frac{\partial \vec{F}}{\partial x}R - \frac{\partial \vec{F}}{\partial y}S + \vec{F}Q$$

and at equilibrium, when this exists,

$$\vec{F}_x R + \vec{F}_y S = \vec{F}Q \qquad (x, y > 0)$$

The interesting difference from the single fluid queue model is that when reservoir 1 is empty, there is no input to queue 2, we have the boundary equation at $x = 0$:

$$\frac{\partial F(\vec{0},y)}{\partial x}R + \frac{\partial F(\vec{0},y)}{\partial y}S' = F(\vec{0},y)Q$$

where $F_1(0,y) = 0$ for all $y \geq 0$ and $F_1(x,0) = F_2(x,0) = 0$ for $x > 0$.

Finally, we have the further boundary condition at infinity that $\vec{F}(\infty,\infty) = \vec{\pi}$.

3 Conclusion

We have presented a very preliminary analysis of networks of fluid queues. The main contribution is the exact result for the joint fluid level probability distribution, but the method appears to be extensible to larger networks. Such extension could enable fluid networks to become analogous to queueing networks in terms of their steady states, although the existence of product-forms is by no means certain. The intention is to find approximate scalable solutions, but our current, simple independence approach does not look too promising according to our simulations.

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