

Matrix-Vector Splitting for Eigenvector Solution

Douglas de Jager

<dvd03@doc.ic.ac.uk>

Department of Computing, Imperial College London

Esteemed Supervisors

Dr Jeremy Bradley & Dr William Knottenbelt

Presentation overview

- Preamble
 - An example problem
 - A “nicer” problem
 - How many solutions to a problem?
- Theorem
 - How to find “nicer” problems
 - Upper bound on number of solutions
- What counts as “nice”?
- A “nicer” PageRank problem

A problem $\rightarrow$ A “nice” problem


$$\vec{x} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 1 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}$$

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$$x_1 = \frac{1}{3}$$


$$x_3 = x_1 = \frac{1}{3}$$


$$x_2 = x_3 + \frac{2}{3} = 1$$

How many solutions?


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Theorem

(P) $\mathbf{M} \in \mathbb{R}^{n \times n}$ is irreducible;
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Then,

- (C1) There is at most one solution to $\vec{x} = \mathbf{M}\vec{x}$.
(C2) This cannot be other than the *unique* fixed-point $\vec{x}$
(which comprises no zero entries) in

$$\vec{x} = (\mathbf{M} - \mathbf{V}^{(K)})\vec{x} + \vec{v}, \text{ where}$$

$$\max\{|\lambda| : \lambda \text{ is an eigenvalue of } |\mathbf{M} - \mathbf{V}^{(K)}|\} < 1.$$

$$\vec{x} = (\mathbf{M} - \mathbf{V}^{(K)})\vec{x} + \vec{v}$$

• For $\vec{v} \in \mathbb{R}^n$, $K \subseteq \{1, 2, \dots, n\}$ we define $\mathbf{V}^{(K)} \in \mathbb{R}^{n \times n}$ by

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$$V_{ij}^{(K)} := \begin{cases} v_i & \text{if } j \in K \\ 0 & \text{if } j \notin K \end{cases}.$$

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$$V_{ij}^{(K)} := \begin{cases} v_i & \text{if } j \in K \\ 0 & \text{if } j \notin K \end{cases}.$$

• Both $\begin{cases} \forall i, j (0 \leq V_{ij}^{(K)} \leq M_{ij} \text{ or } M_{ij} \leq V_{ij}^{(K)} \leq 0); \\ \exists i, j (V_{ij}^{(K)} \neq 0). \end{cases}$

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• E.g. $\mathbf{M} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} \\ 1 & 0 & 0 \end{pmatrix}$ and $\mathbf{V}^{(K)} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$

“Nicer”?

$$\begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ versus } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Sparsity;
Spectral radius;
Interdependency;
Asynchronous solver.

Asynchronous solution for $\vec{x}$

- Class of asynchronous iterations for $\vec{y} = \mathbf{A}\vec{y} + \vec{b}$

$$y_i^{(s+1)} := \begin{cases} \sum_{j=1}^n a_{ij} y_j^{(s-d(s,i,j))} + b_i & \text{if } i \in u(s) \\ y_i^{(s)} & \text{if } i \notin u(s) \end{cases}$$

- u gives entries to be updated at step s .
- d gives relative age of entries being used in update.

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- Restrictions:
 - Each entry updated infinitely often.
 - $\forall s, i, j \in \mathbb{N}, \begin{cases} (s - d(s, i, j)) \rightarrow \infty \text{ as } s \rightarrow \infty; \\ d(s, i, j) \leq s. \end{cases}$

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- Convergence:

- $\max\{|\lambda| : \lambda \text{ is an eigenvalue of } |\mathbf{A}|\} < 1.$ ✓
- $\max\{|\lambda| : \lambda \text{ is an eigenvalue of } |\mathbf{A}|\} \geq 1.$ ✗

Asynchronous solution for $\vec{x}$

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- Convergence:

- $\vec{x} = (\mathbf{M} - \mathbf{V}^{(K)})\vec{x} + \vec{v}$. ✓

- $\vec{x} = \mathbf{M}\vec{x}$. ✗

“Nicer” PageRank equation

Definitions:

- $P_{ij} := \begin{cases} \frac{1}{\deg(j)} & \text{if there is a hyperlink from } j \text{ to } i \\ 0 & \text{otherwise} \end{cases}$
- $\vec{v}$ is a probability vector
- $D_{ij} := \begin{cases} v_i & \text{if } \deg(j)=0 \\ 0 & \text{otherwise} \end{cases}$
- $E_{ij} := v_i$
- $0 < \alpha < 1$

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PageRank Equation:

- $\vec{x} = (\alpha \mathbf{P} + \alpha \mathbf{D} + (1 - \alpha) \mathbf{E}) \vec{x}$

“Nicer” PageRank equation

● *Removing D, E:*

$$\begin{aligned}\vec{y} &= (\alpha \mathbf{P} + \alpha \mathbf{D} + (1 - \alpha) \mathbf{E} - \alpha \mathbf{D}) \vec{y} + \alpha \vec{v} \\ &= (\alpha \mathbf{P} + (1 - \alpha) \mathbf{E}) \vec{y} + \alpha \vec{v} \\ &= \alpha \mathbf{P} \vec{y} + (1 - \alpha) \mathbf{E} \vec{y} + \alpha \vec{v} \\ &= \alpha \mathbf{P} \vec{y} + \beta \vec{v} + \alpha \vec{v} \\ &= \alpha \mathbf{P} \vec{y} + \gamma \vec{v}\end{aligned}$$

where $\beta > 0$

where $\gamma > 0$

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where $\beta > 0$

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● $\vec{z} = \alpha \mathbf{P} \vec{z} + \vec{v}$

A happy ending

Thank you!!