



A tandem network of fluid queues with on-off arrivals

Peter Harrison and Paul Lonsdale

Email: pgh@doc.ic.ac.uk

Department of Computing, Imperial College London

Why fluid queues?

- ➔ exactly describe flow-systems with *continuous state*
 - e.g. volumes in literal fluid flows
 - applications in dynamics of *real* fluids
- ➔ Approximation in *discrete state systems* for *traffic flows*
 - state spaces in discrete systems become prohibitively large as complexity increases
 - more concise to represent state by a real number and transitions by derivatives
 - often a much simpler solution, either analytical or by stochastic simulation

Fluid limits

- ➔ As discrete rates tend to infinity, system's behaviour often approaches a fluid model
 - depends on the limiting regime, e.g. relative rates of increase between arrival rates and service rates
 - see the book by Whitt
- ➔ Used a lot ever since the 70s, e.g. papers by Gaver, Kobayashi, Gelenbe, Mitra, etc
- ➔ But in practice, not *necessary* for a fluid model to be the limit of any known discrete system

Single fluid queue

- ➔ Standard result, but solution usually ‘guessed’
- ➔ We derive the differential equation informally and obtain a solution constructively
 - simplified since the fluid model is *first order*
 - inclusion of random ‘noise’ would lead to a harder second-order equation using Ito calculus etc.
- ➔ Idea is that the approach then extends to simple *networks* and non-queueing systems

Problem definition

- ➔ n phases in CTMC, with generator matrix $Q = (q_{ij} \mid 1 \leq i, j \leq n)$ and equilibrium probabilities $\vec{\pi}$
- ➔ Arrival rate in phase i is λ_i
- ➔ Server outputs fluid when its reservoir is non-empty at rate μ
- ➔ Rate matrix $R = \text{diag}(r_1, \dots, r_n)$, where $r_i = \lambda_i - \mu$ for $1 \leq i \leq n$
- ➔ $\vec{F}(x, t) = (F_1(x, t), \dots, F_n(x, t))$, where

$$F_i(x, t) = \mathbb{P}(N_t = i, X_t \leq x)$$

Incremental analysis ($x > 0$)

- ➔ Consider the infinitesimal interval $(t, t + h]$ for some small h .

$$F_i(x, t + h) = (1 + q_{ii}h)F_i(x - r_ih, t) + \sum_{j \neq i} F_j(x, t)q_{ji}h + o(h)$$

- ➔ Terms on the r.h.s. correspond to
 - no phase change w.p. $1 + q_{ii}h$
 - flow of r_ih units of fluid into reservoir
 - phase change $j \rightarrow i$ w.p. $q_{ji}h$

Differential equation

➔ Rearranging:

$$\begin{aligned} & \frac{F_i(x, t + h) - F_i(x, t)}{h} \\ &= F_i(x, t)q_{ii} - r_i \frac{\partial F_i(x, t)}{\partial x} + \sum_{j \neq i} F_j(x, t)q_{ji} + O(h) \\ &= -r_i \frac{\partial F_i(x, t)}{\partial x} + \sum_{j=1}^n F_j(x, t)q_{ji} + O(h) \end{aligned}$$

Transient and steady states

- ➔ In the limit $h \rightarrow 0$

$$\frac{\partial \vec{F}}{\partial t} = -\frac{\partial \vec{F}}{\partial x} R + \vec{F} Q$$

- ➔ At equilibrium, when this exists,

$$\vec{F}_x R = \vec{F} Q$$

- ➔ Boundary conditions are

- $F_i(0) = 0$ if $r_i > 0$ (reservoir cannot be empty when there is a positive net input)
- $\vec{F}(\infty) = \vec{\pi}$ (phase probabilities at *any* fluid level)

Tandem fluid queues

➔ Two queues with on-off arrivals

- ➔ Generator matrix $Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$
- ➔ on-rate λ in phase 1 and 0 in phase 2, no external arrivals at node 2
- ➔ server 1 outputs fluid at rate μ_1 to node 2
- ➔ server 2 outputs fluid at rate μ_2
- ➔ rate matrices $R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ and $S = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$,
where $r_1 = \lambda - \mu_1$, $r_2 = -\mu_1$ and $s = \mu_1 - \mu_2$.

Incremental analysis ($x, y > 0$)

$$F_i(x, y, t + h) = (1 + q_{ii})F_i(x - r_i h, y - s_i h, t) + \sum_{j \neq i} F_j(x, y, t) q_{ji} h + o(h)$$

- ➔ Terms on the r.h.s. correspond to
 - no phase change w.p. $1 + q_{ii}h$
 - flow of λ or 0 units of fluid into node 1
 - flow of $\mu_1 h$ units of fluid from node 1 to node 2
 - flow of $\mu_2 h$ units of fluid from node 1 to node 2
 - phase change $j \rightarrow i$ w.p. $q_{ji}h$

Differential equation

$$\begin{aligned} & \frac{F_i(x, y, t + h) - F_i(x, y, t)}{h} = \\ & - r_i \frac{\partial F_i(x, y, t)}{\partial x} - s_i \frac{\partial F_i(x, y, t)}{\partial y} \\ & + \sum_{j=1}^n F_j(x, y, t) q_{ji} + O(h) \end{aligned}$$

so that as $h \rightarrow 0$

$$\frac{\partial \vec{F}}{\partial t} = - \frac{\partial \vec{F}}{\partial x} R - \frac{\partial \vec{F}}{\partial y} S + \vec{F} Q$$

Steady state

$$\vec{F}_x R + \vec{F}_y S - \vec{F} Q = 0 \quad (x, y > 0)$$

- But now, when reservoir 1 is empty, there is no input to queue 2
- Boundary equation at $x = 0$:

$$\frac{\partial \vec{F}(0, y)}{\partial x} R + \frac{\partial \vec{F}(0, y)}{\partial y} S' - \vec{F}(0, y) Q = 0$$

where $S' = -\mu_2 I$, $F_1(0, y) = 0$ for all $y \geq 0$ and $F_1(x, 0) = F_2(x, 0) = 0$ for $x > 0$.

- Boundary condition at infinity $\vec{F}(\infty, \infty) = \vec{\pi}$.

Solution for the single fluid queue

- Laplace transform $\vec{F}^*(\theta) = \int_0^\infty e^{-\theta x} \vec{F}(x) dx$
- Multiply o.d.e. by $e^{-\theta x}$ and integrate 0^+ to ∞

$$\vec{F}^*(\theta)(\theta R - Q) = \vec{F}(0)R$$

- Multiply by θ and set $\theta = 0$ gives $\vec{f}^*(0)Q = 0$
so $\vec{F}(\infty)Q = 0$, cf. $\vec{F}(\infty) = \vec{\pi}$

$$\vec{F}^*(\theta) = \frac{\vec{F}(0)R}{\theta(r_1 r_2 \theta + r_1 b + r_2 a)} \begin{pmatrix} r_2 \theta + b & a \\ b & r_1 \theta + a \end{pmatrix}$$

Solution (2)

- ➔ Multiplying by θ , we get

$$\vec{f}^*(\theta) = \frac{r_2 F_2(0)}{r_1 r_2 \theta + r_1 b + r_2 a} (b, r_1 \theta + a)$$

- ➔ At $\theta = 0$,

$$\vec{\pi} = \frac{r_2 F_2(0)}{r_1 b + r_2 a} (b, a)$$

- ➔ But $\vec{\pi} = \frac{(b, a)}{a+b}$ so (after simplification)

$$F_2(0) = 1 - \pi_1 \lambda / \mu_1$$

- ➔ So $\pi_1 \lambda / \mu_1$ is the utilisation, as expected.

Solution (3)

➔ Full solution is:

$$\vec{f}^*(\theta) = (0, F_2(0)) + \frac{\pi_1 \alpha}{\theta + \alpha} (1, (\lambda - \mu_1)/\mu_1)$$

where $\alpha = -\frac{r_1 b + r_2 a}{r_1 r_2}$ and $F_2(0) = \frac{-r_1 \alpha}{a+b}$.

➔ Easily inverted to give

$$\vec{f}(x) = (0, F_2(0))\delta(x) + (1, (\lambda - \mu_1)/\mu_1)\pi_1 \alpha e^{-\alpha x}$$

or

$$\vec{F}(x) = (0, F_2(0)) + (1, (\lambda - \mu_1)/\mu_1)\pi_1 [1 - e^{-\alpha x}]$$

Solution for tandem nodes

- ➔ So far, so good, now for the hard part!
- ➔ Same method yields

$$\vec{f}^*(\theta, \phi)(\theta R + \phi S - Q) = \phi \vec{f}^{*\cdot}(\theta, 0)S + \vec{f}^{\cdot*}(0, \phi)(\theta R + \phi S'')$$

where

$$S'' = S - S' = \mu_1 I, \quad \vec{f}^{*\cdot}(\theta, y) = \int_0^\infty e^{-\theta x} \frac{\partial \vec{F}}{\partial x} dx$$

$$\text{and } \vec{f}^{\cdot*}(x, \phi) = \int_0^\infty e^{-\theta y} \frac{\partial \vec{F}}{\partial y} dy$$

- ➔ Note the marginal distributions

$$\vec{f}^{*\cdot}(0, y) = F_Y(y) \text{ and } \vec{f}^{\cdot*}(x, 0) = F_X(x)$$

Marginal distribution at node 1

- ➔ When $\phi = 0$,

$$\vec{f}^*(\theta, 0)(\theta R - Q) = \vec{f}^*(0, 0)\theta R$$

i.e.

$$\vec{f}_X^*(\theta)\theta^{-1}(\theta R - Q) = \vec{F}_X^*(\theta)(\theta R - Q) = \vec{F}_X(0)R$$

- ➔ Exactly the single fluid queue result, so

$$\vec{F}_X(0) = (0, 1 - \pi_1 \lambda / \mu_1)$$

Marginal distribution at node 2

➔ When $\theta = 0$,

$$\begin{aligned} & \vec{f}^*(0, \phi)(\phi S - Q) \\ &= \phi(\vec{f}^*(0, 0)S + \vec{f}^*(0, \phi)S'' - \vec{F}(0, 0)S'') \\ &= \phi(\vec{f}_Y(0)(S - S'') + \vec{f}^*(0, \phi)S'') \\ &= \phi(\vec{f}_Y(0)S' + \vec{f}^*(0, \phi)S'') \end{aligned}$$

since X must be 0 whenever $Y = 0$

Marginal distribution at node 2 (2)

➔ Proceeding as before,

$$\vec{f}^*(0, \phi) = \frac{\vec{f}_Y(0)S' + \vec{f}^*(0, \phi)S''}{s(\phi s + a + b)} \begin{pmatrix} \phi s + b & a \\ b & \phi s + a \end{pmatrix}$$

where $s = \mu_1 - \mu_2$

Marginal distribution at node 2 (3)

➔ Thus, at $\phi = 0$,

$$\vec{f}^*(0, 0) = \vec{\pi} = \frac{\beta(b, a)}{s(a + b)} = \frac{\beta}{s} \vec{\pi}$$

where $\beta = \mu_1 - \pi_1 \lambda - \mu_2 f_{Y_2}(0)$

➔ So we must have $\beta = s$, i.e.

$$\mu_1 - \pi_1 \lambda - \mu_2 f_{Y_2}(0) = \mu_1 - \mu_2$$

giving

$$\vec{f}_Y(0) = 1 - \frac{\pi_1 \lambda}{\mu_2}$$

Utilisation of node 2

- ➔ At equilibrium, arrival rate to node 2 is the same as to node 1, viz. $\pi_1 \lambda$
- ➔ So the utilisation of the second queue is

$$\pi_1 \lambda / \mu_2$$

- ➔ Again as expected

Full solution?

- ➔ Still does not give the whole solution
- ➔ Any ideas gratefully received
- ➔ Assuming this can be done

What's the point?

- ➔ For large networks, make approximating assumptions, e.g. that nodes behave independently
 - Validate the approximations against simulation
 - Validate the simulation against exact results in small networks, e.g. tandem
- ➔ Search for separable solutions á la RCAT
 - product-form looks highly unlikely
 - maybe some special structure in the Laplace transform??

Independence approximation

- ➔ Need to choose an appropriate arrival process at each node
 - MM fluid arrival process with rate in each phase equal to the total rate from all upstream non-empty nodes
 - transition rate due to an upstream node becoming active is the transition rate at that node from empty to non-empty state
 - in our on-off model, the rate a
 - rates in opposite directions similarly, using upstream nodes, or chosen to match utilisation

Validation

- ➔ Paul implemented a simulator in his final year project
- ➔ Unfortunately found pretty poor agreement for 2 or 3 nodes in tandem
 - could be due to parameter matching against utilisation rather than upstream node
 - simulator could not be validated against exact results

Plans

- ➔ Investigate independence assumption
 - can calculate covariance directly from the Laplace transform ... when we get this to work!
 - certainly can check it via simulation, for arbitrary network topologies
- ➔ Find better arrival models for individual nodes
 - more phases in Markov modulated arrival processes
 - non-Markovian arrival processes