Countless Simson Line Configurations

The Simson line property is normally associated with points on the circumcircle of a triangle. It is embodied by the following theorem.

Given any triangle *ABC* and a point *P* in the plane of the triangle, if perpendiculars from *P* on to the sides *BC*, *CA*, *AB* meet those sides at *L*, *M*, *N* respectively then *L*, *M*, *N* are collinear if and only if *P* lies on the circumcircle of triangle *ABC*. The line *LMN* is then known as the Simson line of *P*.

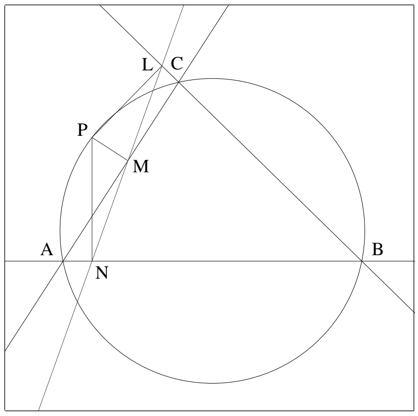

Fig 1: *The Simson line of P*

It is the word "perpendicular" that gives the impression that this type of configuration is somehow particular and that the Simson line property is not therefore capable of generalization. In this article we show that this is not the case and we demonstrate that the above theorem is simply one case of a more general theorem. Indeed it turns out that every transversal of a triangle is a Simson line in a more general sense, and we show how to associate these transversals with different configurations.

The generalization comes about by rephrasing the usual Simson line theorem as follows.

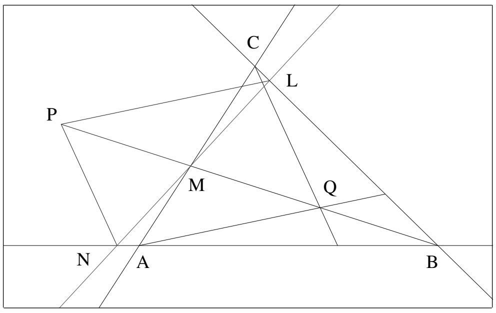

Fig 2: *The Simson line of P with respect to Q*

Given any triangle *ABC* and points *P*, *Q* in the plane of the triangle, if lines through *P* parallel to *AQ*, *BQ*, *CQ* meet the sides *BC*, *CA*, *AB* at *L*, *M*, *N* respectively then *L*, *M*, *N* are collinear if and only if *P* lies on a certain circumscribing conic of triangle *ABC*. This conic we term the Simson conic of *Q* and the line *LMN* as the Simson line of *P* with respect to *Q*.

The Simson conic of *Q* depends only on *Q* − otherwise of course, there would be no significance in the generalization. In terms of the reformulation the original Simson line theorem is recovered by identifying *Q* with *H*, the orthocentre of triangle *ABC*, since any lines parallel to *AH*, *BH*, *CH* are perpendicular to *BC*, *CA*, *AB* respectively. The Simson conic of *H* is the circumcircle of triangle *ABC*.

Areal Coordinates

In order to make the working more accessible we give first a brief introduction to areal coordinates for points in a plane containing a triangle *ABC*. A point *P* has coordinates (x, y, z) if and only if

$$x = \frac{[PBC]}{[ABC]} \qquad y = \frac{[APC]}{[ABC]} \qquad z = \frac{[ABP]}{[ABC]}$$

where $[XYZ]$ denotes the area of triangle *XYZ*. Areas are signed so that x is positive if *P* is on the same side of *BC* as *A*, and so on. With this convention $x + y + z = 1$.
The equation of the line joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$(y_1z_2 - z_1y_2)x + (z_1x_2 - x_1z_2)y + (x_1y_2 - y_1x_2)z = 0$$

Thus $P_3(x_3, y_3, z_3)$ lies on P_1P_2 if and only if

$$\det\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = 0$$

A homogeneous equation of the second degree represents a conic, and the equation of any conic circumscribing triangle ABC has the form

$$fyz + gzx + hxy = 0$$

When $f = a^2$, $g = b^2$, $h = c^2$, where $a = BC$, $b = CA$, $c = AB$, the conic is the circumcircle.

A property of areal coordinates that comes as a surprise when one first studies them is that it is always possible, whether triangle ABC is acute, right-angled or obtuse, to choose an origin O and vectors

$$\overrightarrow{OA} = \mathbf{i},\ \ \overrightarrow{OB} = \mathbf{j},\ \ \overrightarrow{OC} = \mathbf{k}$$

so that any point P in the plane of triangle ABC with areal coordinates (x, y, z) where $x + y + z = 1$, has vector position

$$\overrightarrow{OP} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

Furthermore there is a metric which provides the distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in the plane of the triangle according to the formula

$$(P_1P_2)^2 = {}^1\!/_2\{(x_1 - x_2)^2\,(b^2 + c^2 - a^2) + (y_1 - y_2)^2\,(c^2 + a^2 - b^2) + (z_1 - z_2)^2\,(a^2 + b^2 - c^2)\}$$

where $a = BC$, $b = CA$, $c = AB$ as usual.

The plane of the triangle has equation $x + y + z = 1$ and the vector displacement $\overrightarrow{P_1P_2}$ then has components $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ as one would expect to occur once the above construction has been made.

It is important to realize that the areal coordinates used within the framework of this vector algebra should be kept normalized so that displacements are represented by vectors, the sum of whose components is zero. We have no cause to use the above formula for the metric in this article but we move from areals to vectors in working that follows when it is suitable to do so, and all the usual rules of vector algebra hold fast.

The Case $Q = G$

Before proving the general theorem we consider what happens when H is replaced by G, the centroid. We consider this case in detail as it is the simplest of all the cases and yet illustrates all the features we want. For this section we use the following set of areal coordinates.

A,	Triangle vertex	$(1, 0, 0)$
B,	Triangle vertex	$(0, 1, 0)$
C,	Triangle vertex	$(0, 0, 1)$
G,	Centroid	$({}^1\!/_3, {}^1\!/_3, {}^1\!/_3)$
H,	Orthocentre	$(\cot B \cot C, \cot C \cot A, \cot A \cot B)$
O,	Circumcentre	$(\sin 2A, \sin 2B, \sin 2C) / 4 \sin A \sin B \sin C$
I,	Incentre	$(a, b, c) / (a + b + c)$

We now prove that the Simson conic of G is the ellipse with centre G passing through A, B, C. Also the line with equation

$$px + qy + rz = 0$$

is the Simson line of a point on this ellipse if and only if

$$p(q - r)^2 + q(r - p)^2 + r(p - q)^2 = 0$$

With $A(1, 0, 0)$ and $G(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})$ we have $\overrightarrow{AG} = (-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3})$.
If P is a general point (x, y, z) then points on the line through P parallel to $\overrightarrow{AG}$ have coordinates

$$\left(x - \frac{2k}{3},\ y + \frac{k}{3},\ z + \frac{k}{3}\right)$$

The point L on this line where it intersects BC, $x = 0$, has coordinates $(0, y + x/2, z + x/2)$.
Similarly if the lines through P parallel to BG, CG meet CA, AB at M, N respectively then M has coordinates $(x + y/2, 0, z + y/2)$ and N has coordinates $(x + z/2, y + z/2, 0)$. The condition for L, M, N to be collinear is thus

$$\det \begin{pmatrix} 0 & 2y + x & 2z + x \\ 2x + y & 0 & 2z + y \\ 2x + z & 2y + z & 0 \end{pmatrix} = 0$$

Adding up the columns one sees that $2(x + y + z) = 2$ is a factor, so the equation of the locus of P is homogeneous quadratic and not of the third degree. In fact the quadratic concerned is easily seen to be

$$xy + yz + zx = 0$$

This is a conic circumscribing triangle ABC. Also, since $x + y + z = 1$ it follows that $x^2 + y^2 + z^2 = 1$. So the conic is bounded and therefore an ellipse.

Finally the points $A'(-\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3})$, $B'(\tfrac{2}{3}, -\tfrac{1}{3}, \tfrac{2}{3})$, $C'(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3})$ lie on the conic and G is the midpoint of AA', BB', CC'. We can therefore identify the locus of P, when LMN is a transversal, to be the ellipse, centre G, circumscribing triangle ABC.

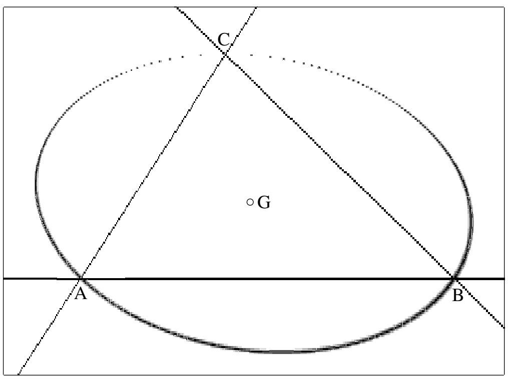

Fig 3: *The locus of P is a circumscribing conic*

It is worth noting that if one adds the rows of the determinant one gets $(3x + 1, 3y + 1, 3z + 1)$ and the significance of this is that the point with coordinates $\tfrac{1}{2}(x + \tfrac{1}{3}, y + \tfrac{1}{3}, z + \tfrac{1}{3})$ lies on the line LMN. That is, the Simson line of P with respect to G passes through the midpoint of PG. Those familiar with the usual Simson line case will remember that the Simson line of P with respect to H passes through the midpoint of PH. This turns out to be a general property and is illustrated in Figure 2 by the line PQ which appears to be bisected by its Simson line. Figure 3 demonstrates the circumscribing ellipse, centre G which is computer generated by the locus of P.

The General Case

Before determining the Simson lines with respect to points on the Simson conic of G we move to the general case and prove the following theorems.

Theorem 1

Let Q be a general point in the plane of triangle ABC. Then the Simson conic of Q is the (unique) conic passing through A, B, C with centre Q_0, where Q_0 lies on GQ and is such that $\overrightarrow{GQ} = 2\overrightarrow{Q_0G}$.

Theorem 2

If P lies on the Simson conic of Q, then Q lies on the Simson conic of P.

Theorem 3

If P lies on the Simson conic of Q, then the Simson line of P with respects to Q bisects PQ.

Theorem 4

The **fixed** line l with equation

$$px + qy + rz = 0$$

is a Simson line with respect to Q for all points Q on the conic circumscribing ABC with equation

$$p(q - r)^2yz + q(r - p)^2zx + r(p - q)^2xy = 0$$

Equivalently, if Q is the **fixed** point (u, v, w), only those transversals are Simson lines with respect to Q whose p, q, r values satisfy this equation with $x = u, y = v, z = w$.

For example l is a Simson line of H if and only if p, q, r satisfy

$$p(q - r)^2 \cot A + q(r - p)^2 \cot B + r(p - q)^2 \cot C = 0$$

Proofs

Let Q have coordinates (u, v, w) and let P be the general point (x, y, z) then

$$\overrightarrow{AQ} = (u - 1, v, w) = (-(v + w), v, w)$$

Points on the line through P parallel to $\overrightarrow{AQ}$ have coordinates of the form $(x - k(v + w), y + kv, z + kw)$. The point L on this line and on BC has coordinates

$$L\left(0\ ,\ \ y + \frac{xv}{v + w},\ \ z + \frac{xw}{v + w}\right)$$

Similarly M, N have coordinates

$$M\left(x + \frac{yu}{w + u},\ 0\ ,\ \ z + \frac{yw}{w + u}\right) \qquad N\left(x + \frac{zu}{u + v},\ \ y + \frac{zv}{u + v},\ \ 0\right)$$

The condition for LMN to be a straight line is therefore

$$\det\begin{pmatrix} 0 & y(v+w)+xv & z(v+w)+xw \\ x(w+u)+yu & 0 & z(w+u)+yw \\ x(u+v)+zu & y(u+v)+zv & 0 \end{pmatrix} = 0$$

Adding up the columns shows once again that $(x + y + z)$ is a factor of this determinant, so the locus of P for L, M, N to be collinear is a conic. A short computation shows its equation to be

$$u(v + w)yz + v(w + u)zx + w(u + v)xy = 0$$

This is a conic passing through A, B, C centre the point Q_0 with coordinates $^1/_2(1 - u, 1 - v, 1 - w)$. The reason for this is that A', B', C' with coordinates $(-u, 1 - v, 1 - w)$, $(1 - u, -v, 1 - w)$, $(1 - u, 1 - v, -w)$ lie on the conic and Q_0 is the midpoint of AA', BB', CC'. This establishes Theorem 1.

Theorem 2 follows immediately from the fact that the last equation may be rewritten in the form

$$x(y + z)vw + y(z + x)wu + z(x + y)uv = 0$$

Adding up the rows of the determinant one gets $(u + x, v + y, w + z)$, since $x + y + z = u + v + w = 1$. The significance of this is that the point with coordinates $^1/_2(u + x, v + y, w + z)$ lies on the Simson line of P with respect to Q. This establishes Theorem 3.

Finally, let us classify the Simson lines of points lying on the Simson conic of Q. If $px + qy + rz = 0$ is one of these lines then the conditions for L, M, N to lie on this line are

$$\frac{q}{r} = -\frac{vz + w(z + x)}{wy + v(x + y)} = -\frac{vz + w(1 - y)}{wy + v(1 - z)}$$

$$\frac{r}{p} = -\frac{wx + u(1 - z)}{uz + w(1 - x)}$$

and

$$\frac{p}{q} = -\frac{uy + v(1 - x)}{vx + u(1 - y)}$$

That is

$$qv + rw + (q - r)(wy - vz) = 0$$
$$rw + pu + (r - p)(uz - wx) = 0$$
$$pu + qv + (p - q)(vx - uy) = 0$$

Since $x + y + z = 1$ we have four equations for three unknowns and there is hence a compatibility condition for solution, which is

$$p(q - r)^2vw + q(r - p)^2wu + r(p - q)^2uv = 0$$

We have now proved Theorem 4.

Figures 4—9 below illustrate Simson conics for various points on the Euler line of triangle ABC. The loci of the conics were generated at a fixed tolerance which best illustrates the whole path. It should be understood that the instances at which the plot of the locus intermittently disappears are due to the tolerance at which the computer was working and are not necessarily points at which no locus exits.
The case $u = v = w = {}^1/_3$ covers the case of G, mentioned earlier.

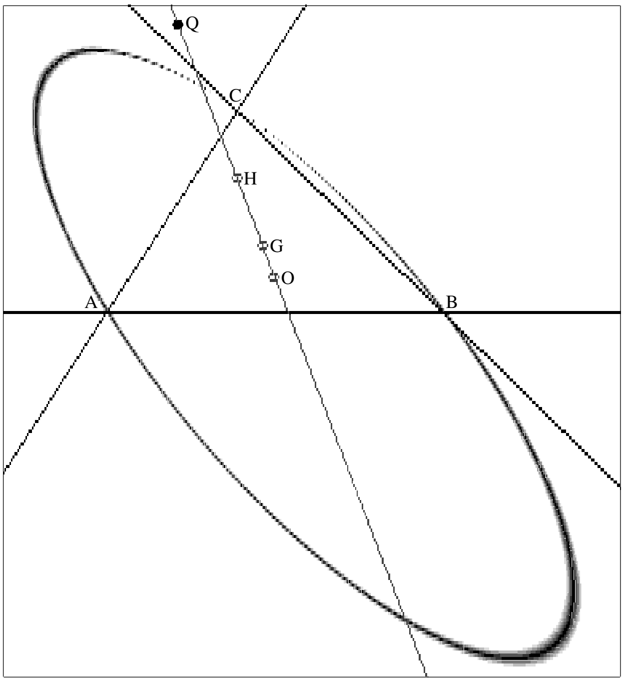

Fig 4: *Q is external to all produced sides of the triangle*

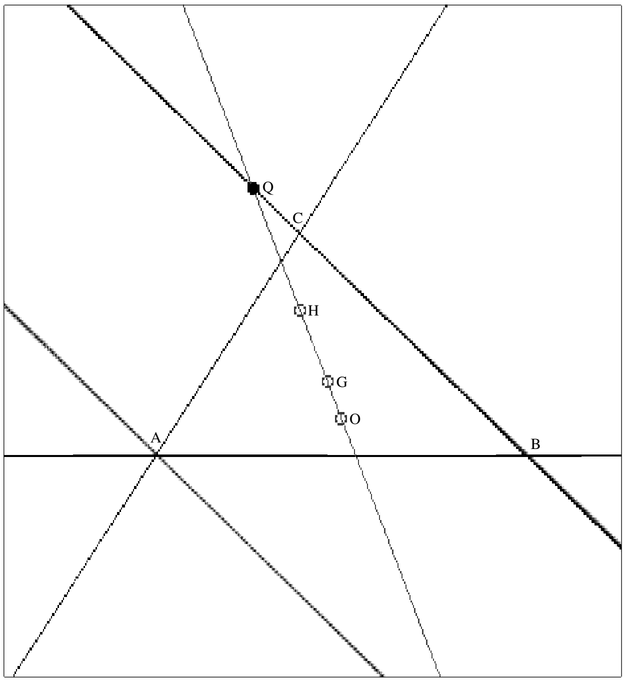

Fig 5: *Q on BC produces degenerate parallel lines*

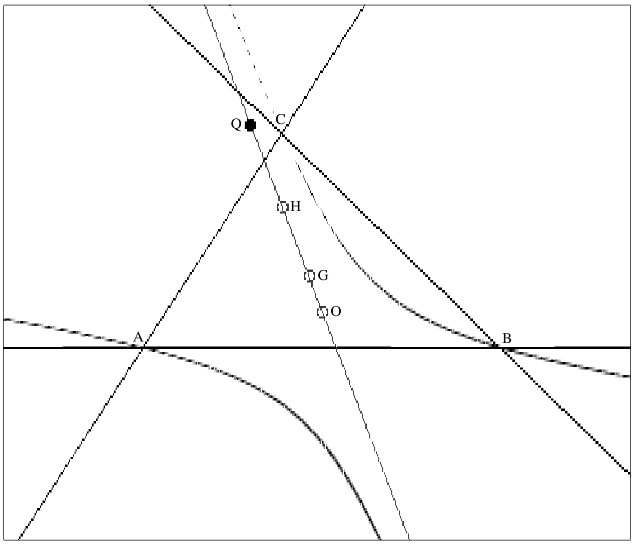

Fig 6: *Q between BC and CA generates a hyperbola*

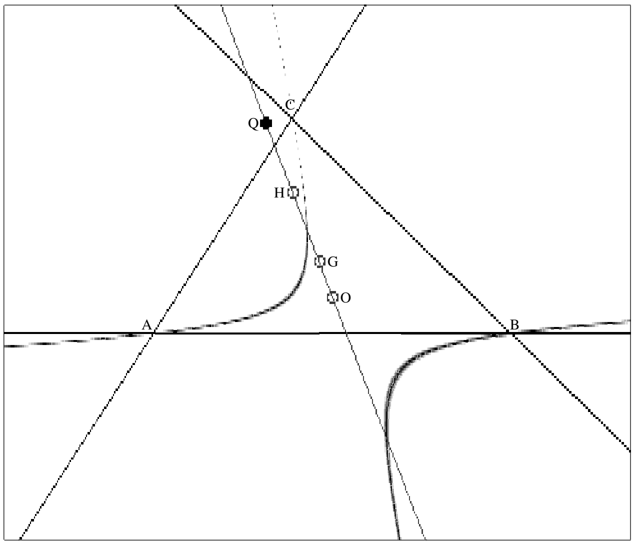

Fig 7: *A small change in Q produces a different hyperbola*

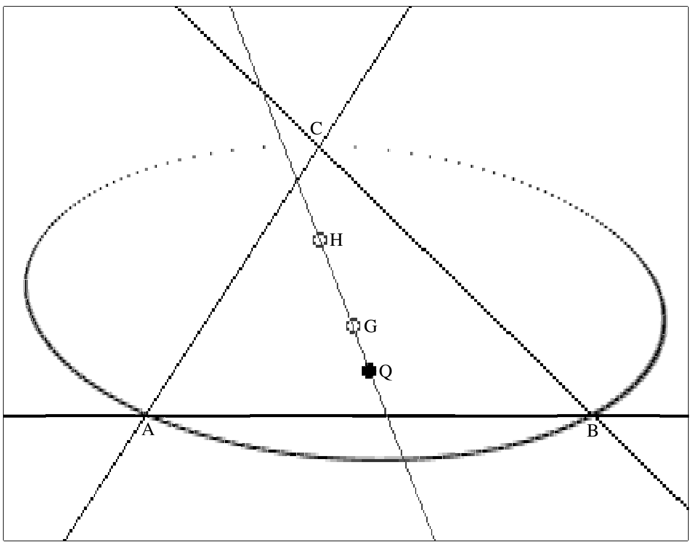

Fig 8: *Q reaches the circumcentre O*

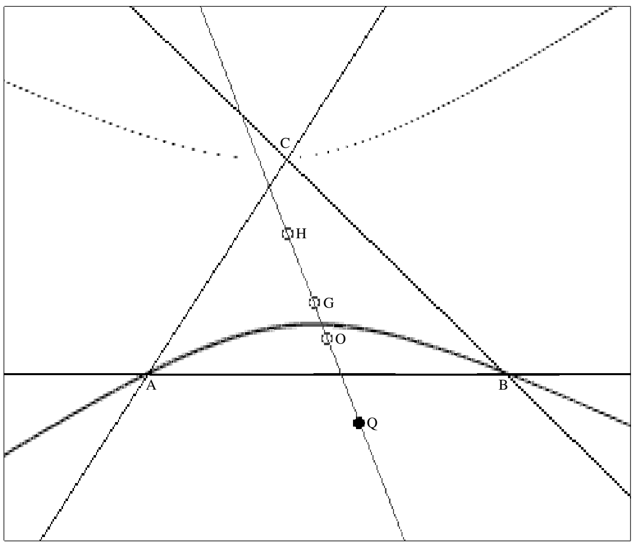

Fig 9: *Q outside AB produces the third type of hyperbola*

The Simson Line Deltoid for $Q = G$

It is known that the Simson lines with respect to H envelope a deltoid with three cusps. The triangle with vertices at the cusps is equilateral and thus independent of the shape of triangle ABC. Its orientation is the same as the Morley triangle formed by intersections of the angle trisectors. The deltoid also touches the sides of ABC. It seemed to us to be interesting in conclusion to see what happens when H is replaced by G. The general case is rather beyond us, but would doubtless show similar features. We have carried out the calculation using Cartesian coordinates and give only main results leaving it as an exercise for the reader to fill in the details.

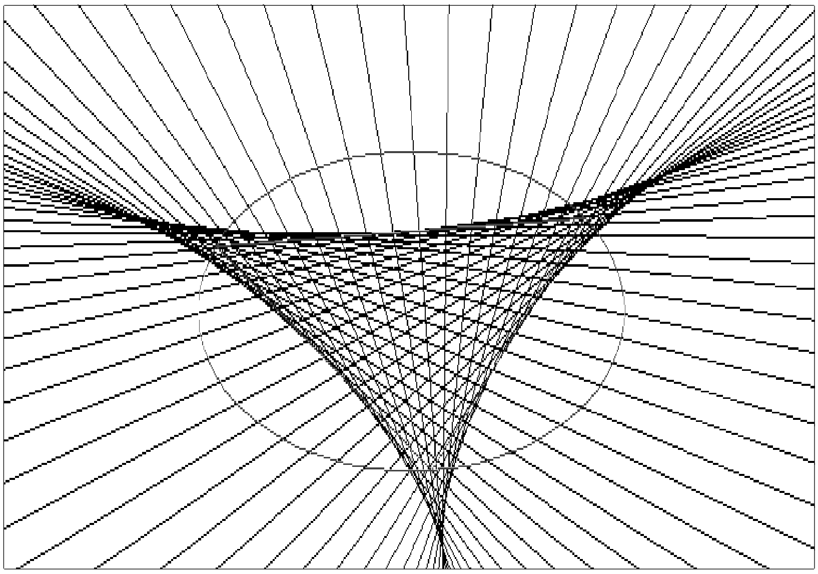

Fig 10: *The envelope of Simson Lines for $Q = G$*

We take the ellipse to have equation

$$\frac{x^2}{l^2} + \frac{y^2}{m^2} = 1$$

and A, B, C with coordinates

$$A(l\cos\theta,\ m\sin\theta),$$
$$B(l\cos(\theta + {}^{2\pi}/_{3}),\ m\sin(\theta + {}^{2\pi}/_{3}))$$
$$C(l\cos(\theta + {}^{4\pi}/_{3}),\ m\sin(\theta + {}^{4\pi}/_{3}))$$

Since

$$\mathrm{e}^{i\theta} + \mathrm{e}^{i\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{i\left(\theta + \frac{4\pi}{3}\right)} = 0$$

this choice puts the centroid of triangle ABC at the origin, and is a perfectly general choice. We take P to have coordinates $(l\cos\alpha, m\sin\alpha)$.

The equation BC is

$$mx\cos\theta + ly\sin\theta = -{}^1/_2 lm$$

The equation of the line through P parallel to $\overrightarrow{AG}$ is

$$mx\sin\theta - ly\cos\theta = lm\sin(\theta - \alpha)$$

Whereupon L has coordinates

$$x = l({}^1/_2\cos\alpha - \cos{}^1/_2(\theta - \alpha)\cos{}^1/_2(3\theta - \alpha))$$
$$y = m({}^1/_2\sin\alpha - \cos{}^1/_2(\theta - \alpha)\sin{}^1/_2(3\theta - \alpha))$$

satisfying,

$$l(y - {}^1/_2 m\sin\alpha)\cos{}^1/_2(3\theta - \alpha) = m(x - {}^1/_2 l\sin\alpha)\sin{}^1/_2(3\theta - \alpha)$$

Since $\tan\frac{1}{2}(3\theta - \alpha)$ is unaltered in value when θ is increased by $2\pi/3$ or $4\pi/3$ this line also contains M and N and so is the equation of the Simson line of P. Note also that the line passes through $(\frac{1}{2}l\cos\alpha, \frac{1}{2}m\sin\alpha)$, the midpoint of PG. The equation of the Simson line may be rewritten as

$$mx\sin\tfrac{1}{2}(3\theta - \alpha) - ly\cos\tfrac{1}{2}(3\theta - \alpha) = \tfrac{1}{2}lm\sin\tfrac{3}{2}(\theta - \alpha)$$

Differentiating this with respect to α and solving the resulting simultaneous equations for x, y we obtain the parametric form of the envelope of the Simson lines, as P moves around the ellipse, to be

$$x = l\cos\alpha + \tfrac{1}{2}l\cos(3\theta - 2\alpha)$$
$$y = m\sin\alpha + \tfrac{1}{2}m\sin(3\theta - 2\alpha)$$

This is the typical form of a three-cusped deltoid, with cusps at $\alpha = \theta$, $\theta + 2\pi/3$, $\theta + 4\pi/3$, but since $l \neq m$ we do not have an equilateral deltoid. However at $\alpha = \theta$ we have $x = \frac{3}{2}l\cos\theta$, $y = \frac{3}{2}m\sin\theta$ and similarly at $\alpha = \theta + 2\pi/3$, $\alpha = \theta + 4\pi/3$ we have $x = \frac{3}{2}l\cos\alpha$, $y = \frac{3}{2}m\sin\alpha$, in each case.

It follows that the triangle with vertices at the cusps is similar and similarly situated to triangle ABC, the scale factor being $3/2$. It is also easy to show that the deltoid touches the sides BC, CA, AB at their midpoints.

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