Extracting QoS Measures from Laplace Transforms

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Contents

Motivation

- From an uninverted Laplace transform:
  - Distribution moments
  - Average statistics
- Inverting a Laplace transform can give:
  - Performance statistics
  - Reliability statistics
  - Passage-time distributions
  - Transient distributions

Inversion Formulae/Method

- Bromwich/Euler
- Post-Widder
- Laguerre

Bromwich Contour Integral

$$f(t) = \frac{1}{2\pi i} \oint_{b-i\infty}^{b+i\infty} \hat{f}(z) e^{zt} \, \mathrm{d}z$$

...can be turned into a real integral

$$f(t) = \frac{2e^{bt}}{\pi} \int_0^\infty \mathrm{Re}(\hat{f}(b+iu)) \cos(ut) \, \mathrm{d}u$$

Post-Widder

$$f(t) = \lim_{n \longrightarrow \infty} g(n,t) \widehat{f}^{(n)} \left(\frac{n+1}{t} \right)$$

$$g(n,t) = \frac{(-1)^n}{n!} \left(\frac{n+1}{t} \right)^{n+1}$$

Laguerre Polynomials

- Formula

$$f(t) = \sum_{n=0}^{\infty} q_n l_n(t)$$

- Laguerre functions

$$l_n(t) = e^{-t/2} L_n(t)$$

- Laguerre polynomials

$$L_n(t) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \frac{(-t)^k}{k!}$$

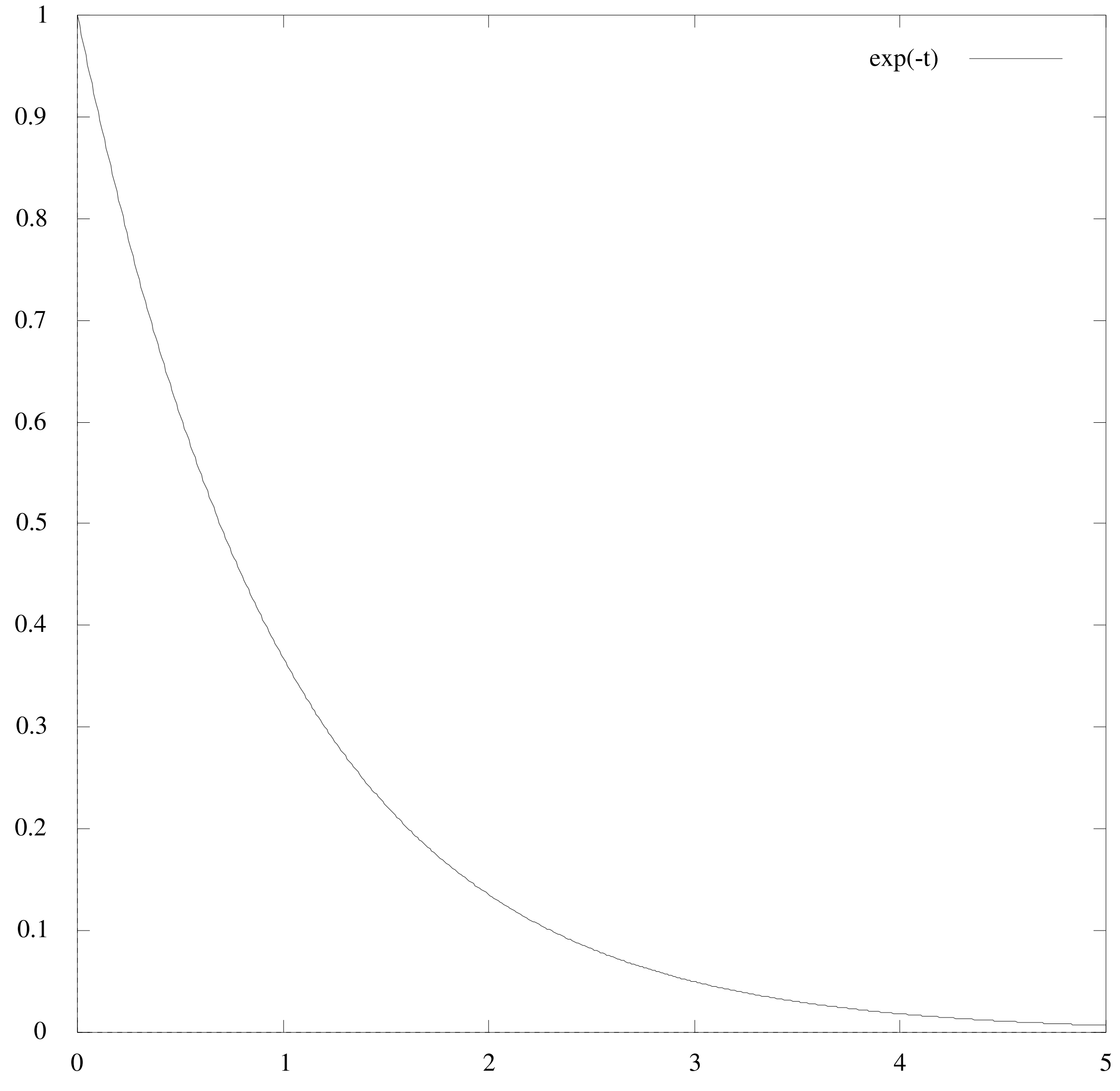
- Laguerre coefficients

$$\widehat{q}(z) = \sum_{n=0}^{\infty} q_n z^n = \frac{1}{1-z} \widehat{f}\left(\frac{1+z}{2(1-z)}\right)$$

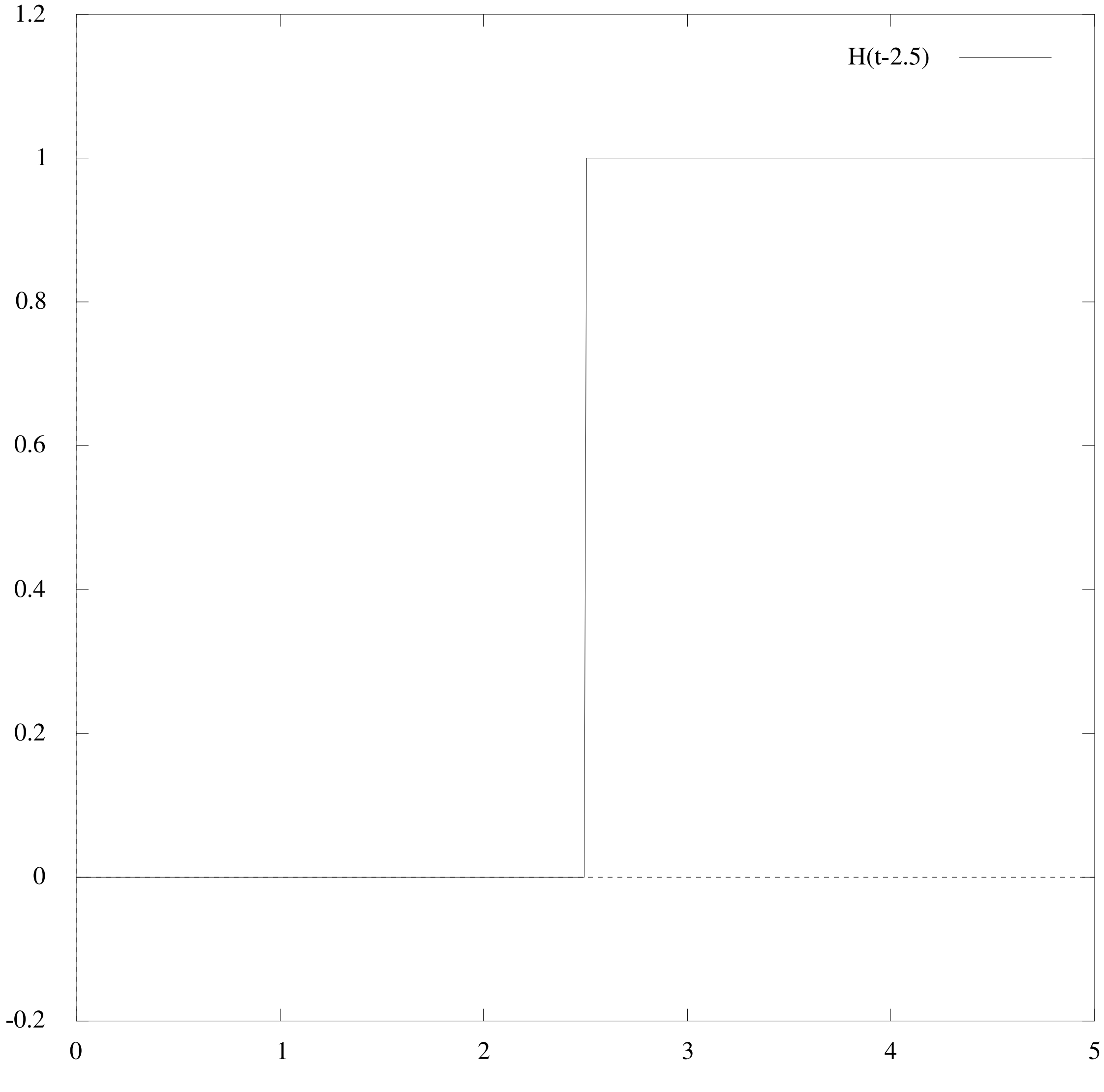
Test Functions

- Exponential
  - Continuous and continuously differentiable
- Heaviside
  - Not continuous and not continuously differentiable
- Transient
  - Continuous but not continuously differentiable

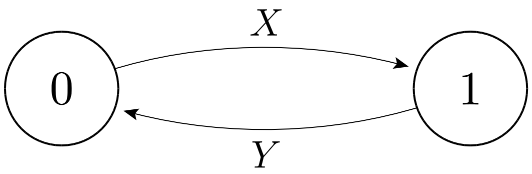
Exponential:



Heavside:

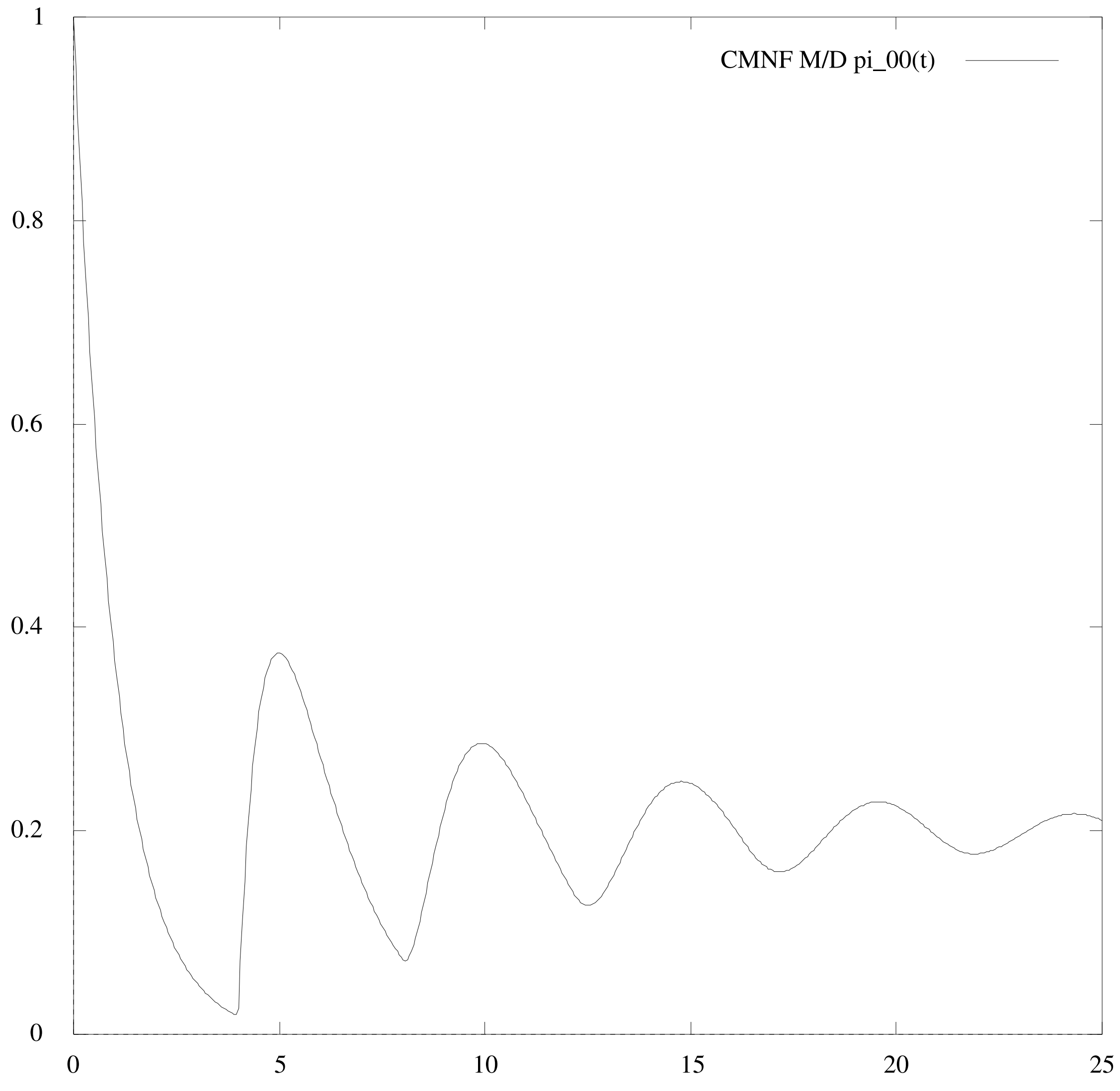


A Simple System

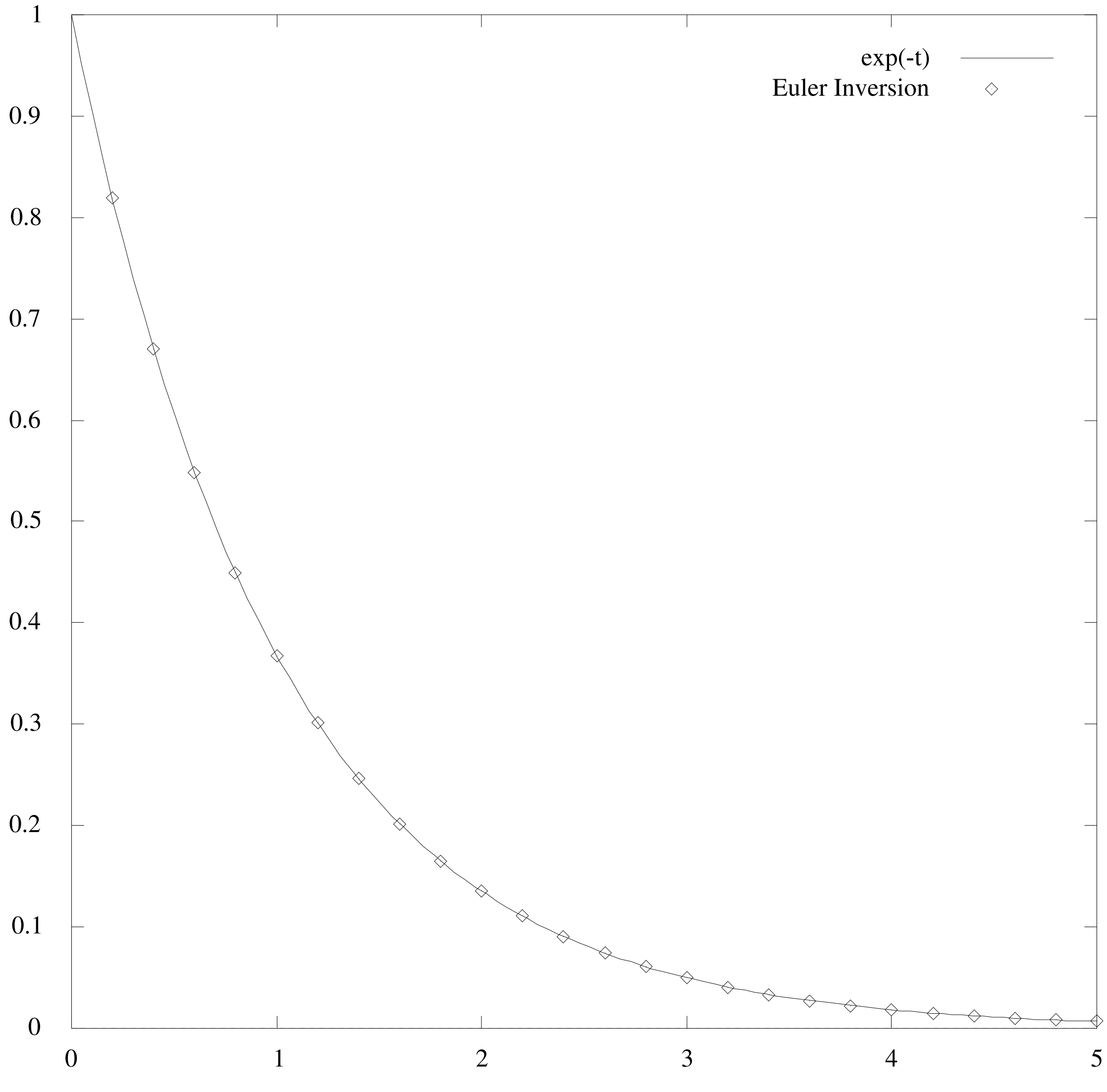


$$X \sim \text{exp}(1), Y \sim \delta(4)$$

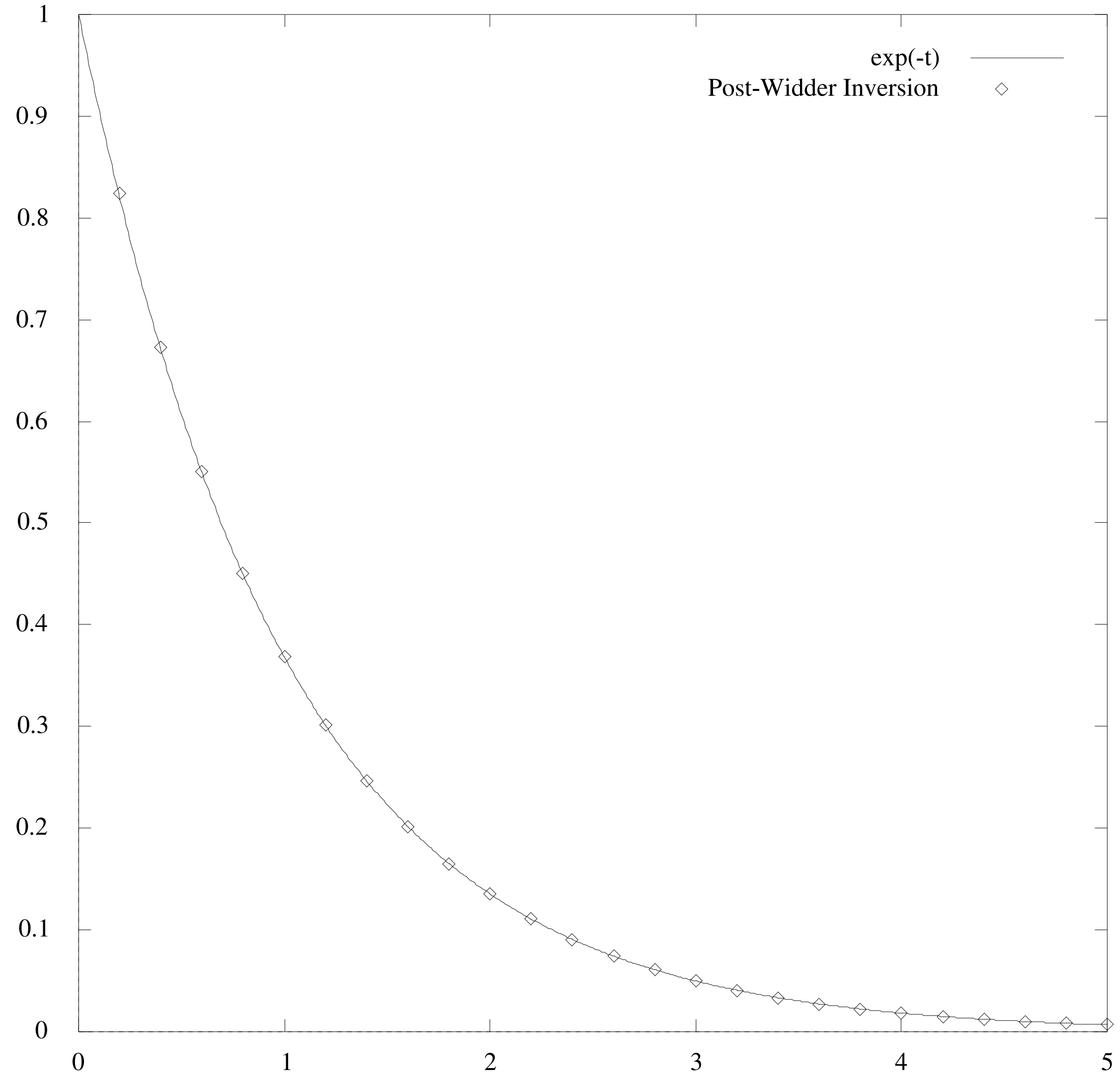
Transient $\pi_{00}(t)$:



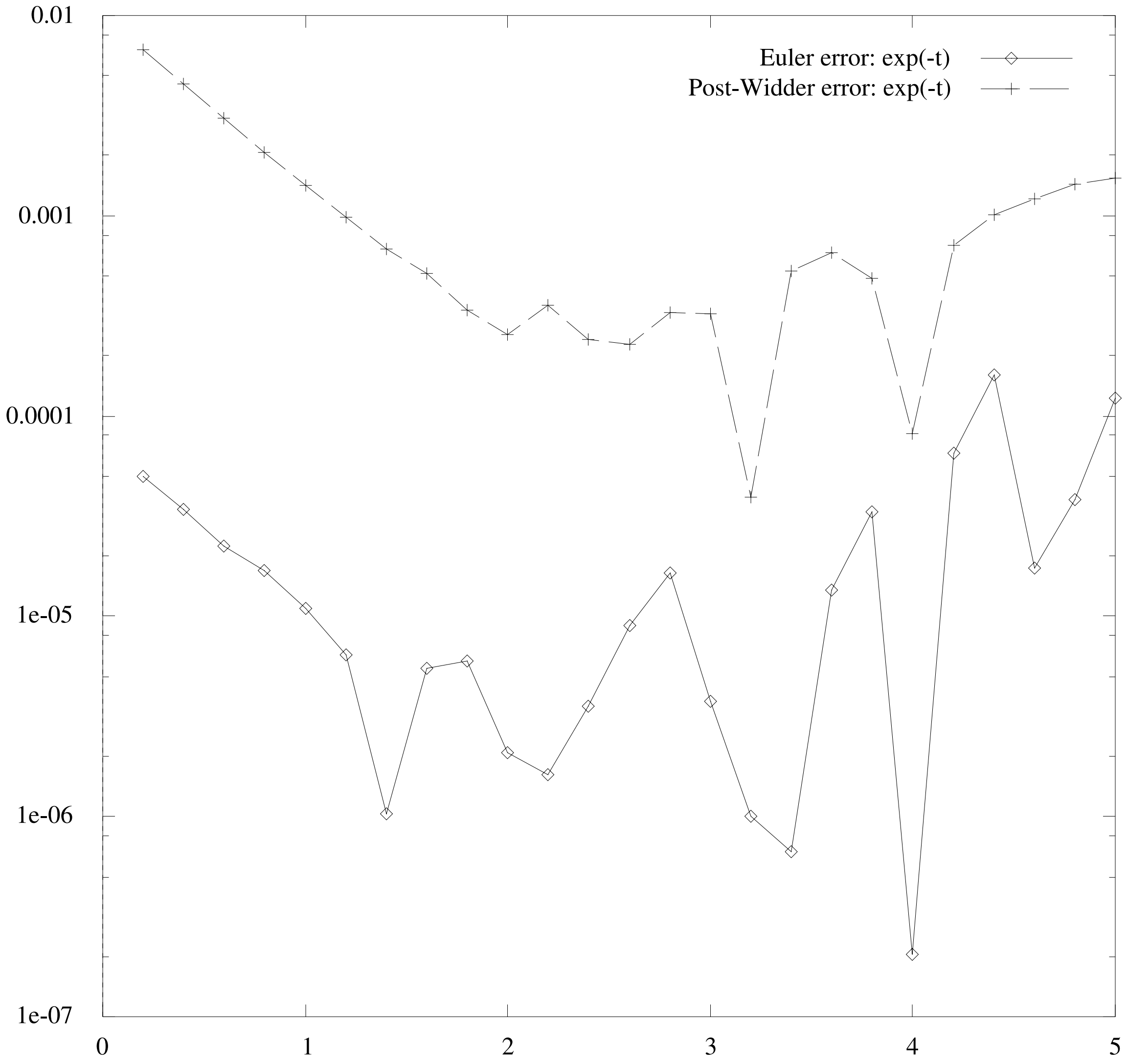
Exponential: Euler



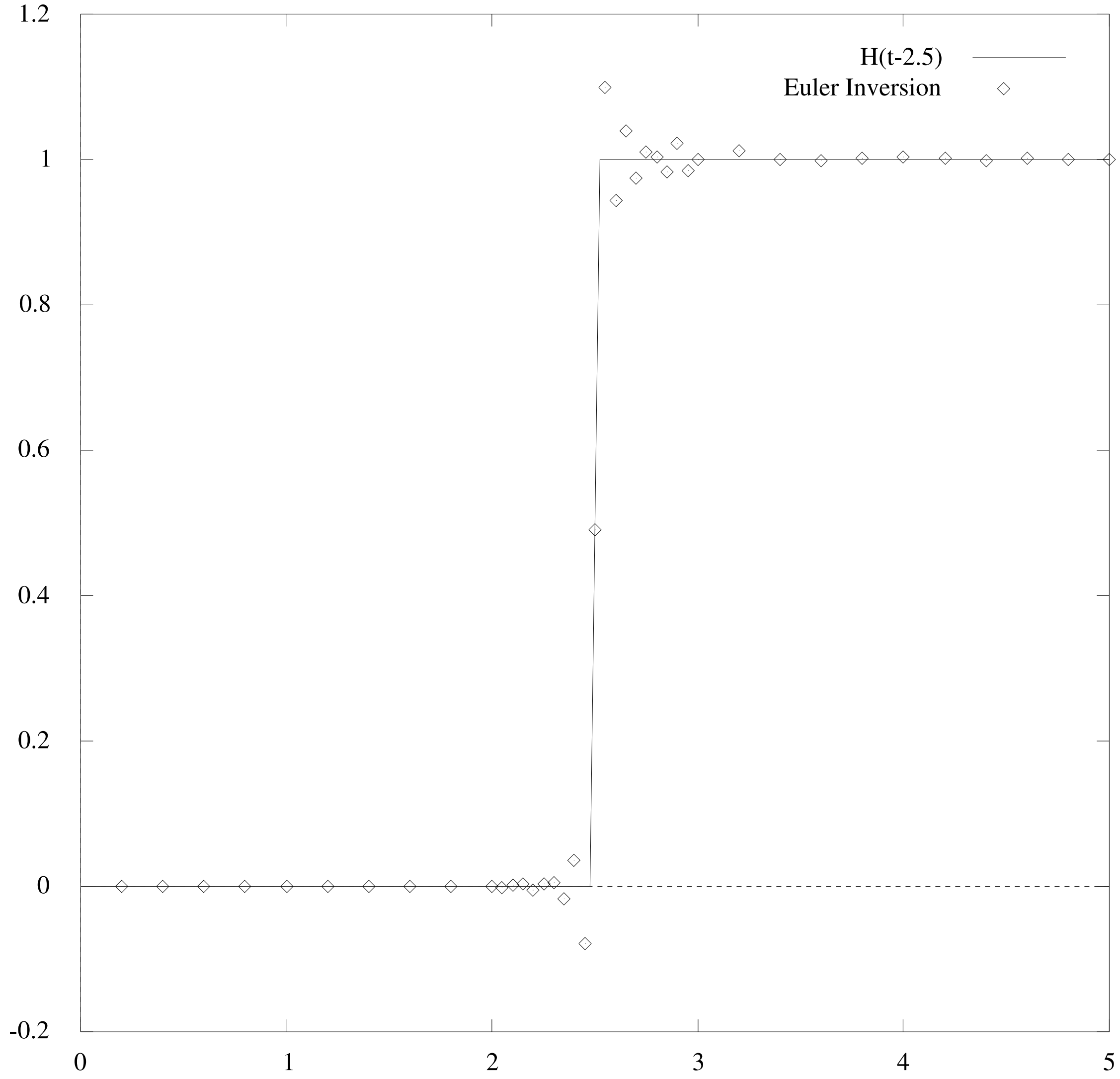
Exponential: Post-Widder



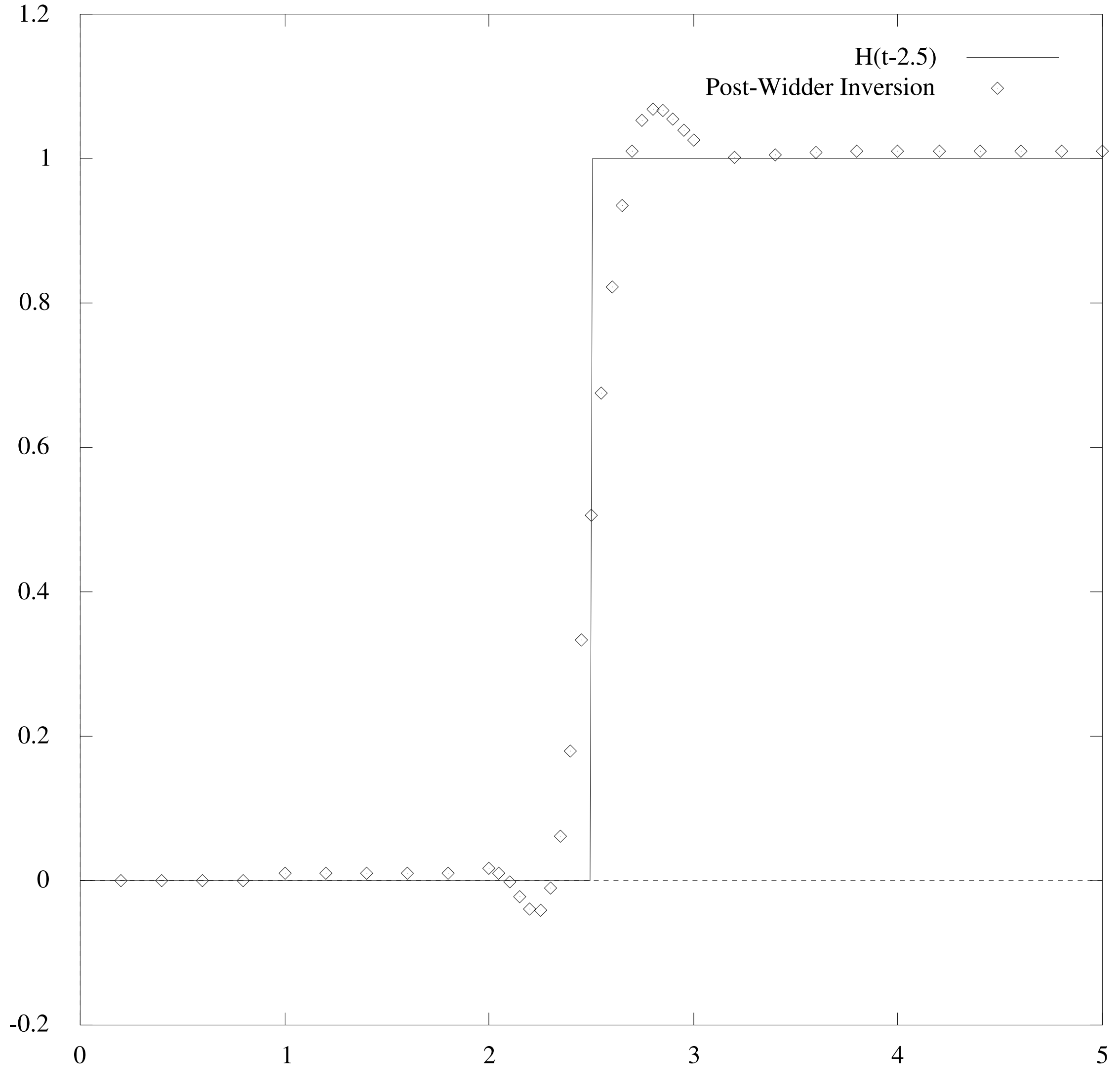
Exponential: Euler v Post-Widder



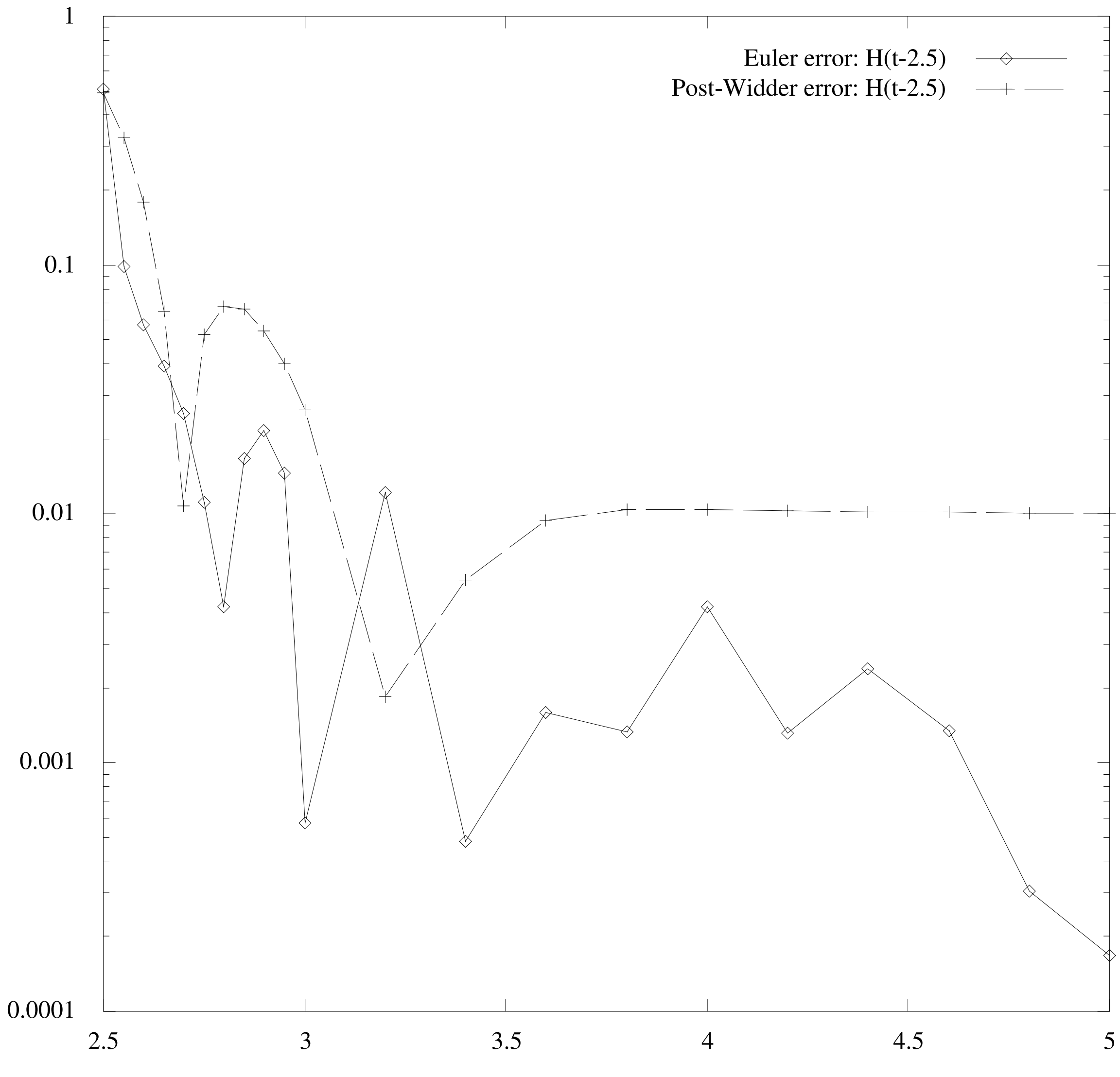
Heaviside: Euler



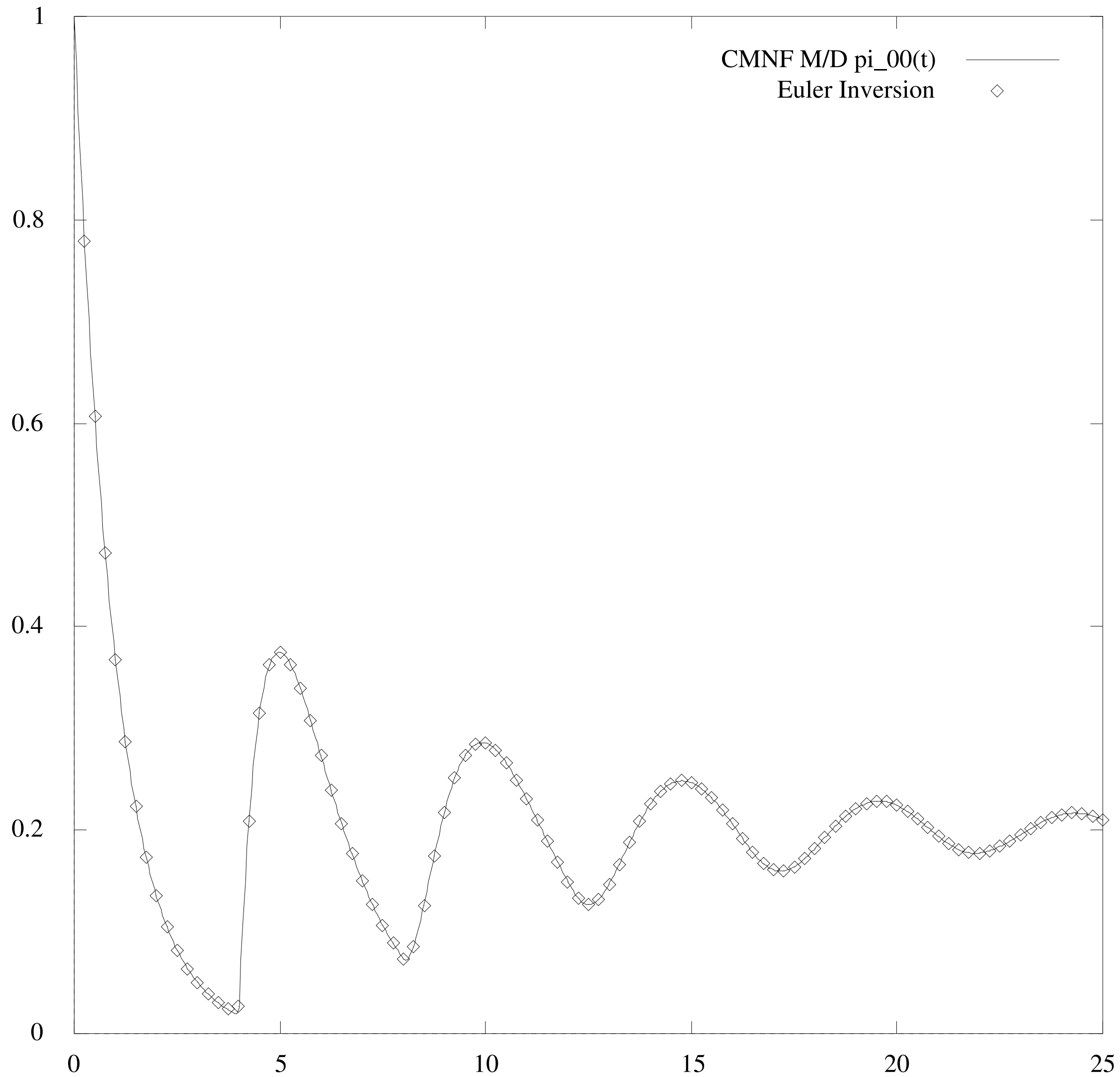
Heaviside: Post-Widder



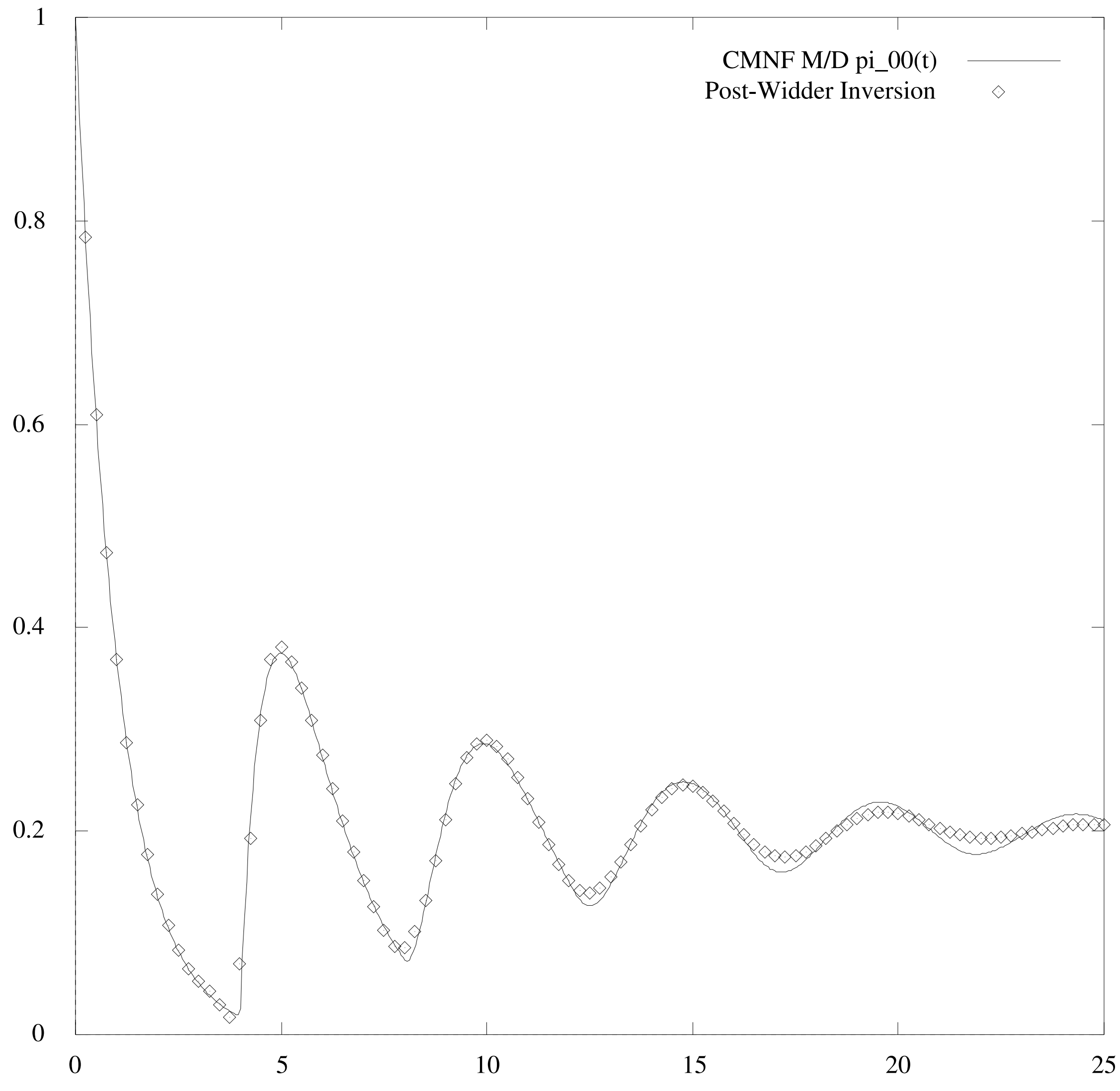
Heaviside: Euler v Post-Widder



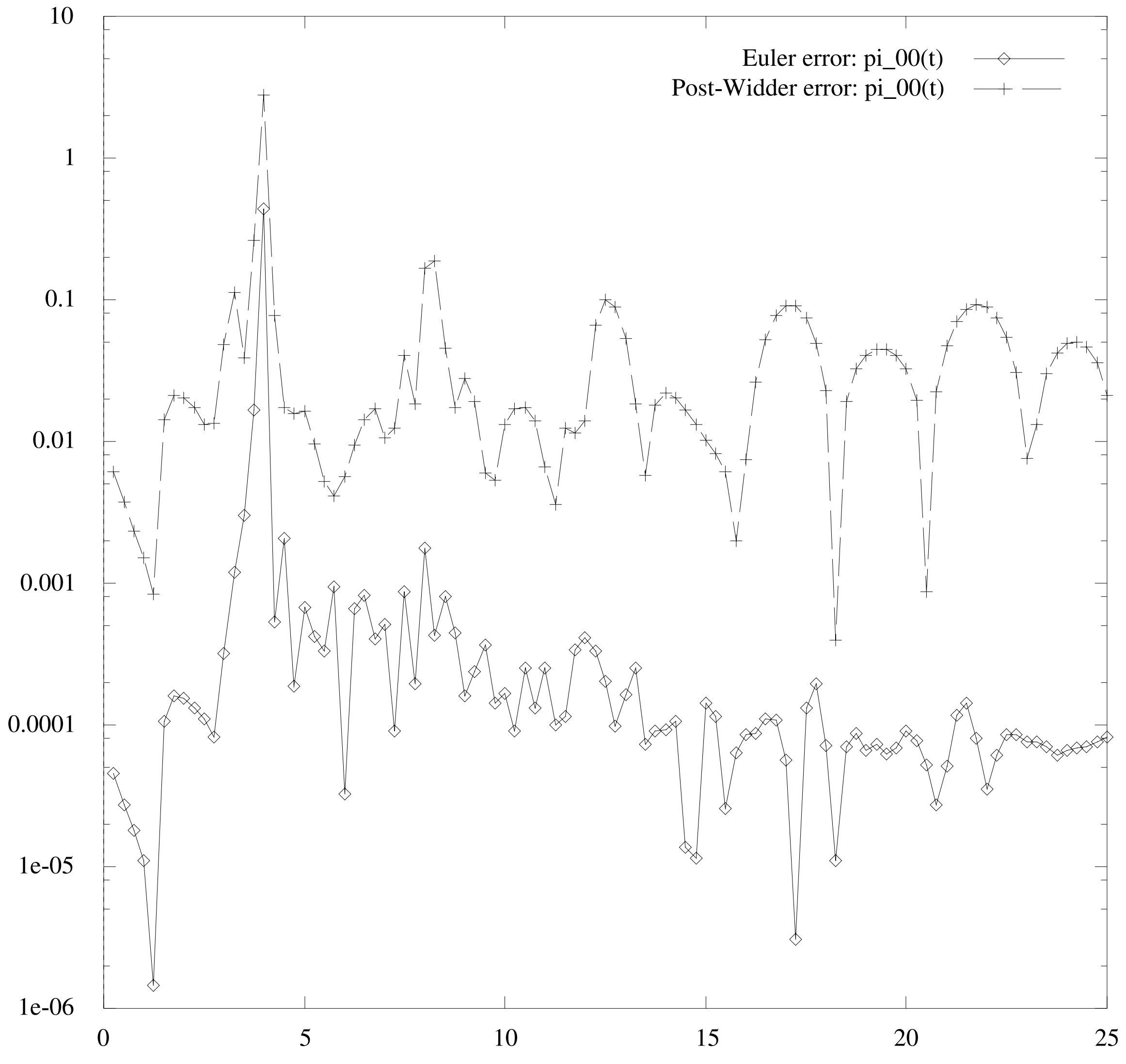
Transient $\pi_{00}(t)$: Euler



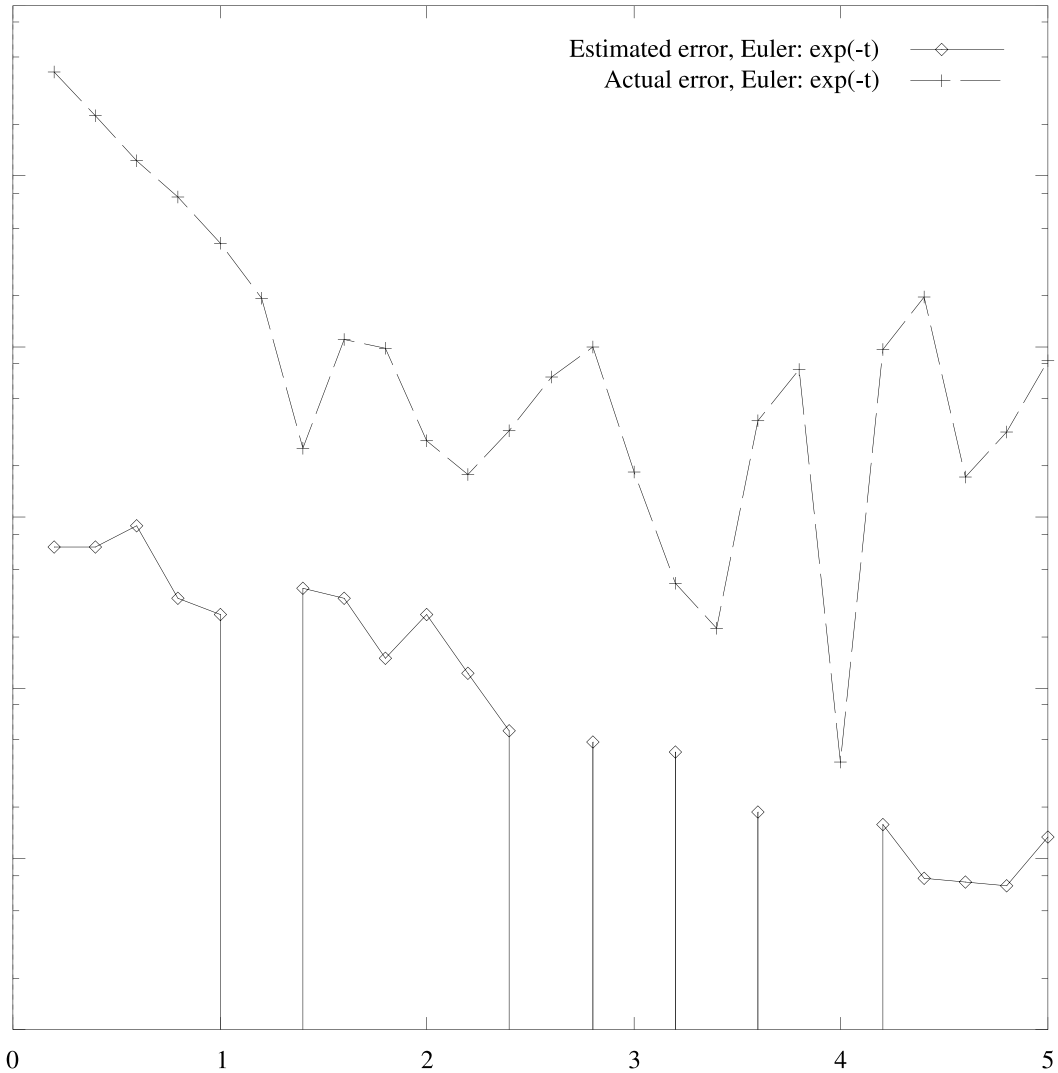
Transient $\pi_{00}(t)$: Post-Widder



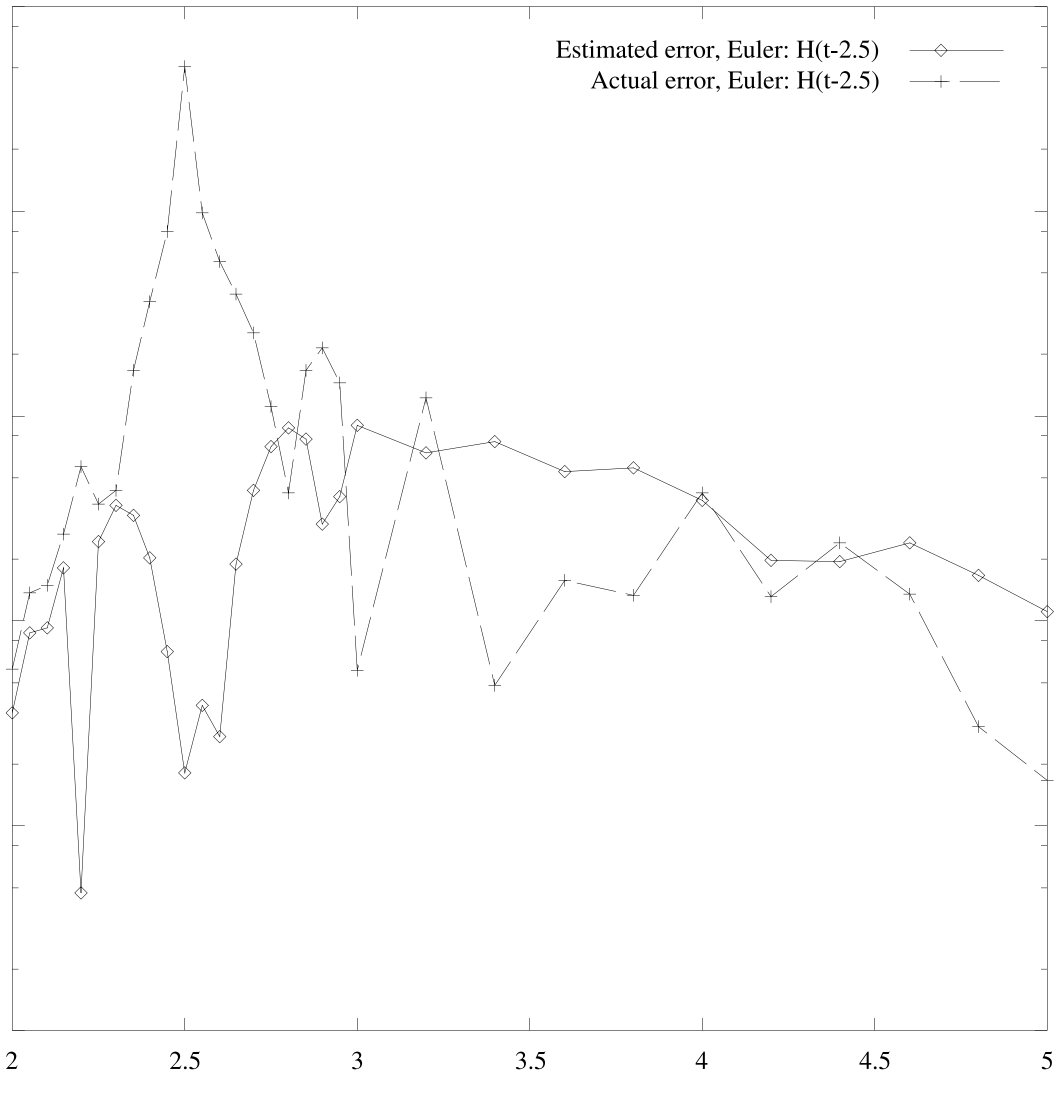
Transient $\pi_{00}(t)$: Euler v Post-Widder



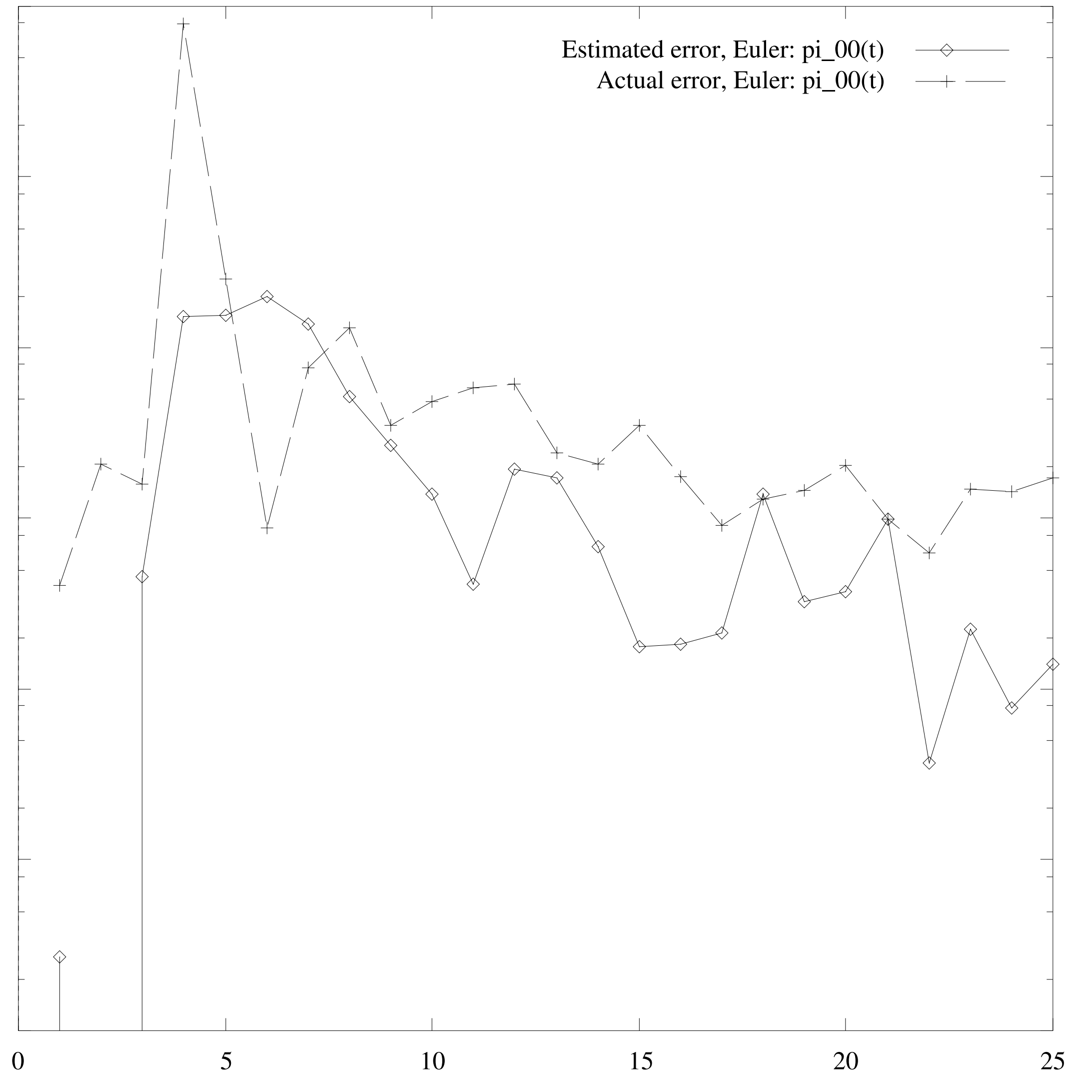
Exponential: Errors in Euler Method



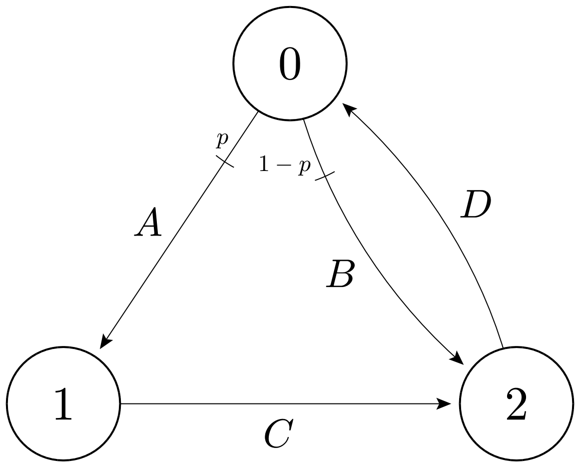
Heaviside: Errors in Euler Method



Transient $\pi_{00}(t)$: Errors in Euler Method



An Actual Example



...with $p = \frac{3}{10}$ and uses distributions:

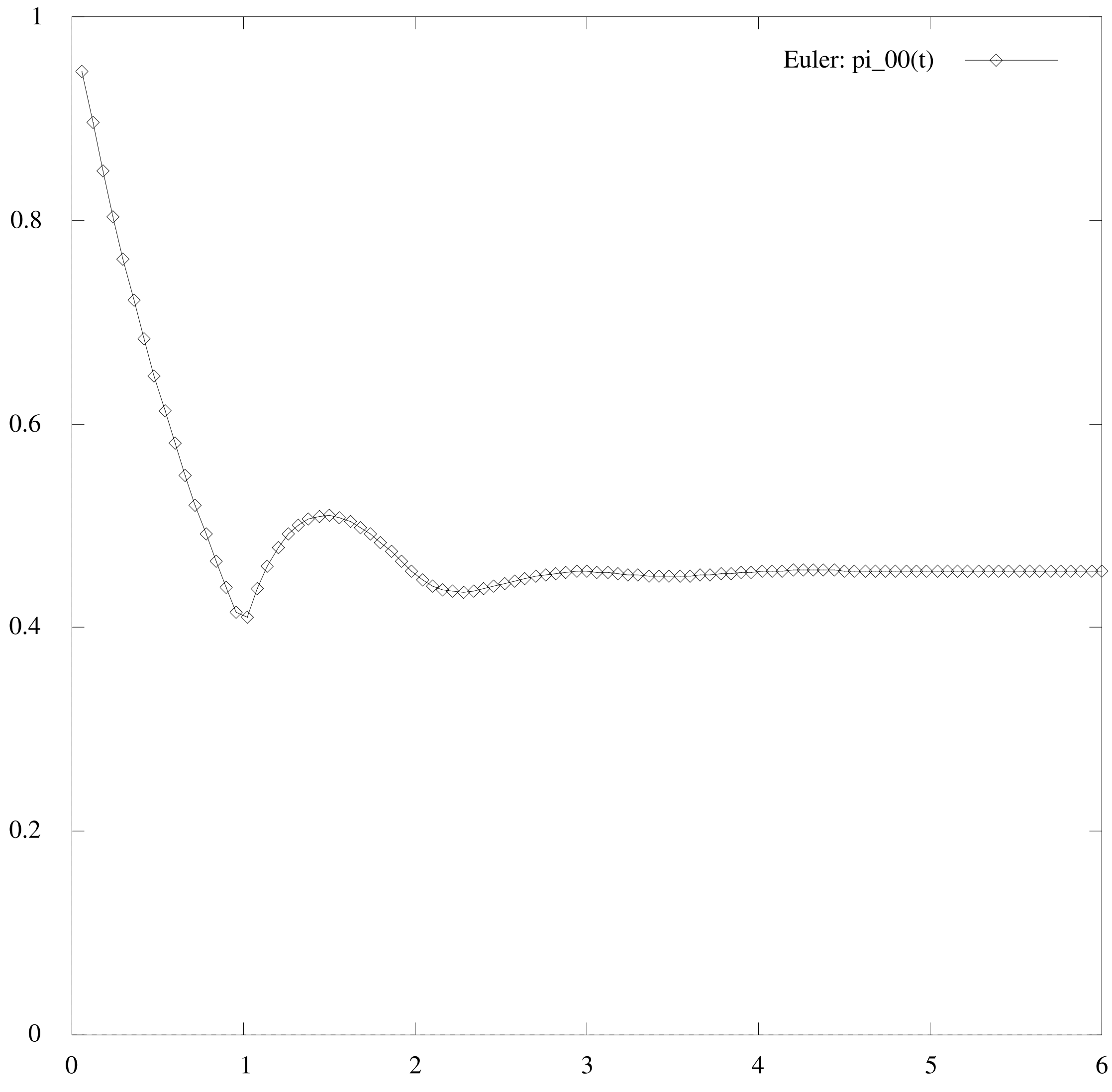
$A \sim \Gamma(\frac{5}{2}, \frac{3}{2})$

$B \sim \mathrm{Hyper}(\frac{2}{3}, \frac{1}{3}; \frac{3}{2}, 1)$

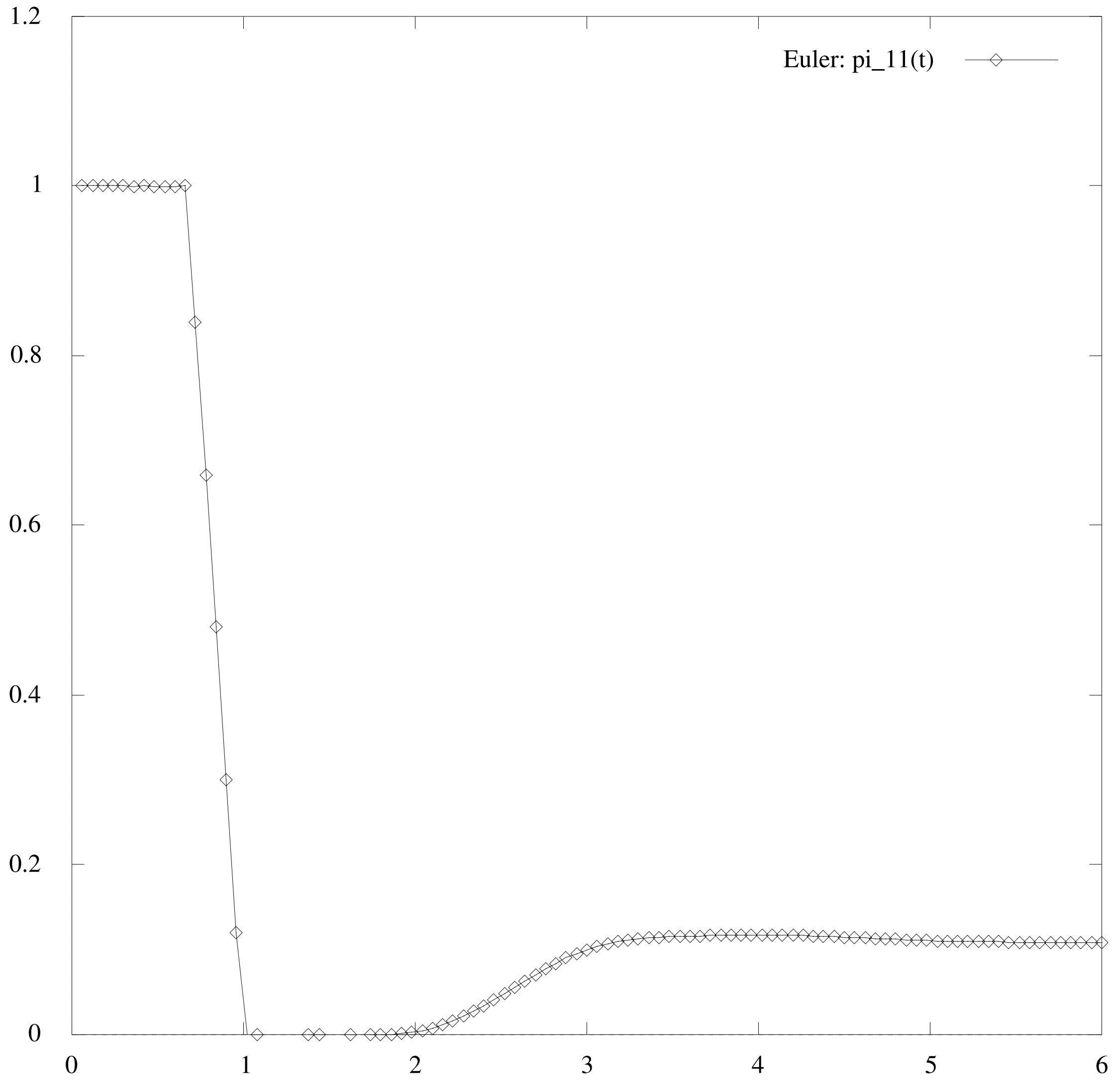
$C \sim \mathrm{U}\left[\frac{2}{3}, 1\right]$

$D \sim \delta(1)$

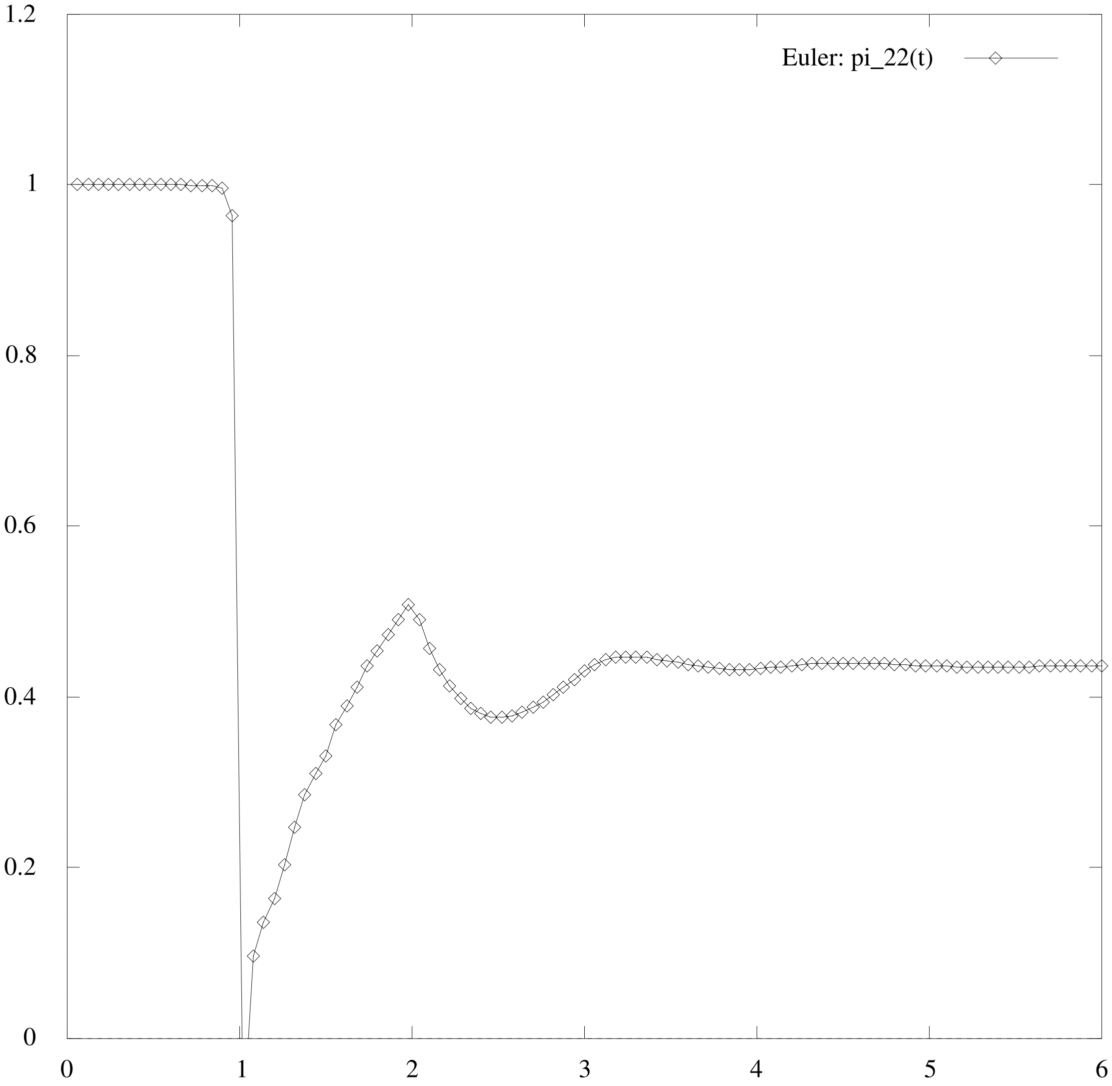
$\pi_{00}(t)$: Euler



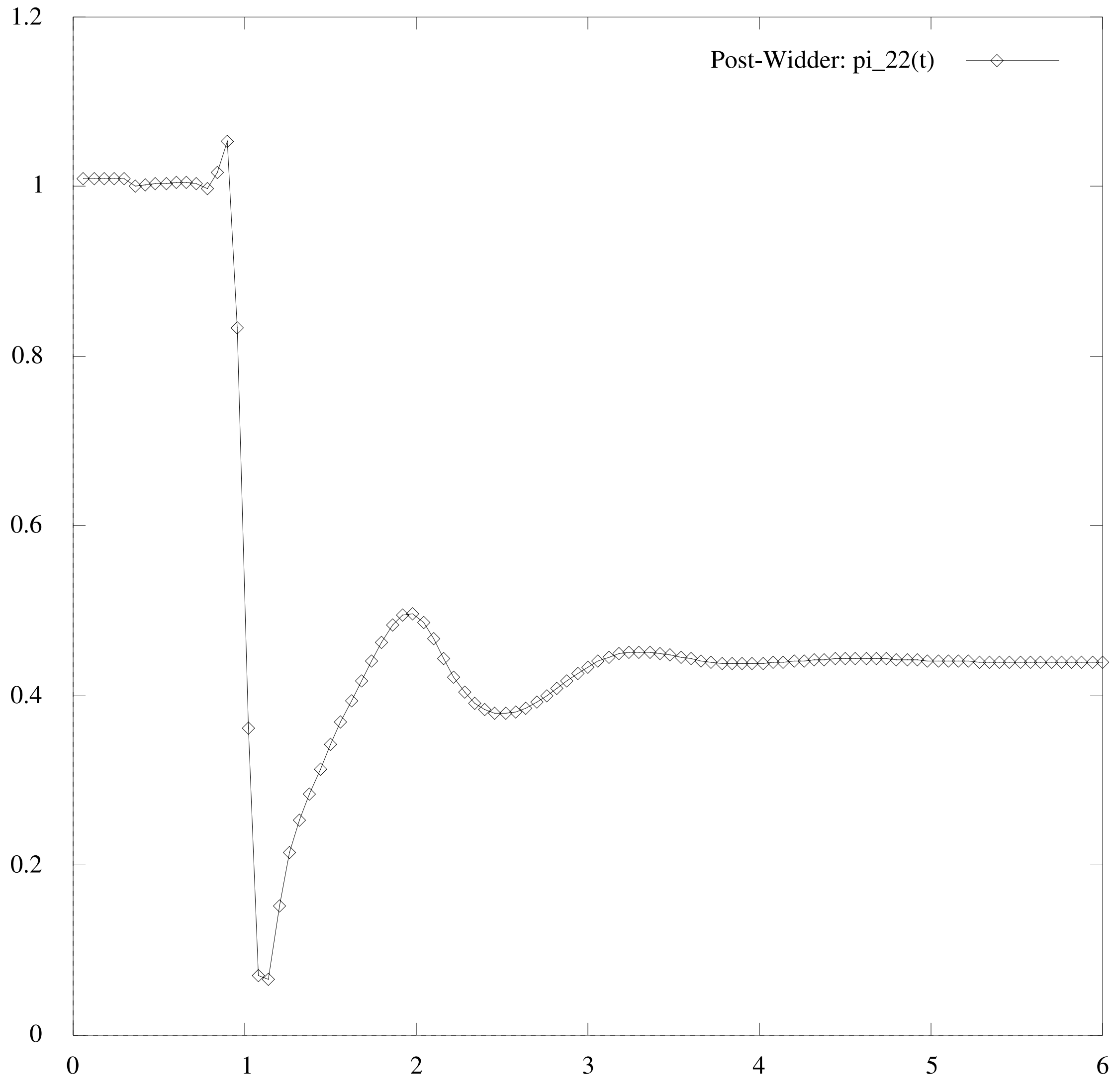
$\pi_{11}(t)$: Euler



$\pi_{22}(t)$: Euler



$\pi_{22}(t)$: Euler and Post-Widder



Haskell Library: LaplaceInvert.hs

- Exports

```
data InvAlgorithm = Euler | PostWidder | Laguerre
type LaplaceT = Complex Double -> Complex Double
type TimeF = Double -> Double
type DomainSpec = [((Double, Double), Int)]
```

- Interface

```
invertLaplace :: InvAlgorithm -> LaplaceT ->
  Maybe TimeF -> DomainSpec -> Matrix Double
```

- Output

2.5	0.082094	1.61469e-08
5.0	0.374363	0.00015557
7.5	0.106337	0.00010878
10.0	0.285637	1.38433e-05