Report on Extracting Transient Distributions from Semi-Markov Processes

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July 31, 2001

1 Introduction

Transient functions are very useful quantities in stochastic modelling in that they describe how a system evolves with time from a given start point. Unfortunately, for most types of general stochastic system, extracting transient state distributions in any analytic form is hard or impossible.

Analytic methods have existed for some time for extracting transient distributions from the general class of semi-Markov processes [6] (which include as a subset all Markovian processes). These have been woefully overlooked for stochastic system modelling purposes, possibly because of the perceived lack of techniques for inverting Laplace transforms. We address both these issues.

In this report we explore some basic transient state distributions of semi-Markov processes (SMPs). By relating these simple problems to previous work on stochastic transition system[1] manipulation [3] we show how transient results for arbitrary n-state SMPs can be deduced.

We compare the technique with previous analytic methods [6] and finish with an example of transient distribution calculation for a simple system.

In doing so we demonstrate both the power of the analytic transient result (which comes in the form of a Laplacian function) and many recent developments in the inversion of Laplace transforms [1].

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[1] An algebraic representation of the semi-Markov process.

2 Some Simple Transient Distribution Calculations

2.1 Transient State Probabilities

In general there is an $n \times n$ matrix of $\pi_{ij}(t)$ of transient-state probabilities for an n-state system, where:

$$\pi_{ij}(t) = \mathbb{P}(\text{system in state } i \text{ at time } t \mid \text{started from state } j) \tag{1}$$

where: $\sum_{i=1}^{n} \pi_{ij}(t) = 1$ for a given j.

In systems with *pleasant* distributions, we can express the steady-state probability as the limit of the transient probability (using property (a) of [6, p.171]):

we can relate the transient-state distributions to the steady-state distributions with:

$$\lim_{t \longrightarrow \infty} \pi_{ij}(t) = \pi_i \tag{2}$$

However a broader convergence definition can be obtained by looking at the proportion of time a system spends in a given state (by property (c), [6, p.172]):

$$\lim_{t \longrightarrow \infty} \frac{1}{t} \int_0^t \pi_{ij}(\tau)\, \mathrm{d}\tau = \pi_i \tag{3}$$

which includes the convergence of equation (2). A proof of this can be found in appendix A.

2.2 G/D CMNF Transient Solution

In figure 1, we assume that X is generally distributed and Y is a deterministic evolution with parameter d.

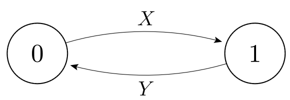

Fig. 1. The G/D Cox-Miller Normal Form where $Y \sim \delta(d)$

In this analysis we are assuming that the system start state coincides with one of the two states in the CMNF. This means that in the event that a system has been aggregated to a CMNF then the start state must also be the *preserved* state during aggregation. I do not think this is overly restrictive in general but it is worth bearing in mind that more complex normal forms will need to be considered if this is not the case.

2.2.1 Convolution Aside

It turns out that the following formula will provide a nice simplification in the general case of the G/D CMNF normal form:

$$\begin{aligned}
\overline{F}_{X+Y}(t) &= \mathbb{P}(X > t - Y) \\
&= \iint_{X>t-Y} f_X(x) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \overline{F}_X(t-y) f_Y(y)\,\mathrm{d}y \qquad (4)
\end{aligned}$$

If X and Y are positive random variables, we will assume that we can treat the probability density functions of such variables as if they have a doubly-infinite domain where $f_X(t) = 0$ for $t < 0$.

Thus:

$$\overline{F}_{X+Y}(t) = \int_0^t \overline{F}_X(t-y) f_Y(y)\,\mathrm{d}y + \int_t^{\infty} \underbrace{\overline{F}_X(t-y)}_{1} f_Y(y)\,\mathrm{d}y \qquad (5)$$

So:

$$\int_0^t \overline{F}_X(t-y) f_Y(y)\,\mathrm{d}y = \overline{F}_{X+Y}(t) - \overline{F}_Y(t) \qquad (6)$$

2.2.2 Calculating $\pi_{00}(t)$

As the system cycles round the loop it generates random variable X-samples $X_1, X_2, \ldots$ and Y-samples $Y_1, Y_2, \ldots$ respectively. We divide the time-line into slices of size d as before:

For $0 \le t < d$:

$$\begin{aligned}
\pi_{00}(t) &= \mathbb{P}(X_1 > t) \\
&= \overline{F}_{X_1}(t) \qquad (7)
\end{aligned}$$

For $d \le t < 2d$:

$$\begin{aligned}
\pi_{00}(t) &= \mathbb{P}(X_1 > t) + \underbrace{\mathbb{P}(X_1 + Y_1 < t, X_1 + Y_1 + X_2 > t)}_{\mathbb{P}(X_1 < t-d, X_1+X_2 > t-d)} \\
&= \overline{F}_{X_1}(t) + \int_0^{t-d} \overline{F}_{X_2}(t-d-x) f_{X_1}(x)\,\mathrm{d}x \\
&= \overline{F}_{X_1}(t) + \overline{F}_{X_1+X_2}(t-d) - \overline{F}_{X_1}(t-d) \qquad (8)
\end{aligned}$$

For $2d \le t < 3d$:

$$\begin{aligned}
\pi_{00}(t) &= \mathbb{P}(X_1 > t) + \mathbb{P}(X_1 + Y_1 < t, X_1 + Y_1 + X_2 > t) \\
&\quad + \underbrace{\mathbb{P}(X_1 + Y_1 + X_2 + Y_2 < t, X_1 + Y_1 + X_2 + Y_2 + X_3 > t)}_{\mathbb{P}(X_1+X_2 < t-2d, X_1+X_2+X_3 > t-2d)} \\
&= \overline{F}_{X_1}(t) + \overline{F}_{X_1+X_2}(t-d) - \overline{F}_{X_1}(t-d) \\
&\quad + \int_0^{t-2d} \overline{F}_{X_3}(t-2d-x) f_{X_1+X_2}(x)\,\mathrm{d}x \\
&= \overline{F}_{X_1}(t) + \overline{F}_{X_1+X_2}(t-d) - \overline{F}_{X_1}(t-d) \\
&\quad + \overline{F}_{X_1+X_2+X_3}(t-2d) - \overline{F}_{X_1+X_2}(t-2d) \qquad (9)
\end{aligned}$$

Now let $Z_n = \sum_{r=1}^{n} X_r$ and $G_n(t) = \sum_{r=1}^{n} \overline{F}_{Z_{r+1}}(t-rd) - \overline{F}_{Z_r}(t-rd)$ to look at the general case:

$$\pi_{00}(t) = \begin{cases} \overline{F}_{X_1}(t) & : 0 \le t < d \\ \overline{F}_{X_1}(t) + G_n(t) & : nd \le t < (n+1)d \end{cases} \qquad (10)$$

2.2.3 Calculating $\pi_{11}(t)$

Similarly, for $0 \le t < d$:

$$\begin{aligned}
\pi_{11}(t) &= \mathbb{P}(Y_1 > t) \\
&= 1 \qquad (11)
\end{aligned}$$

For $d \le t < 2d$:

$$\begin{aligned}
\pi_{11}(t) &= \underbrace{\mathbb{P}(Y_1 > t)}_{0} + \underbrace{\mathbb{P}(Y_1 + X_1 < t, Y_1 + X_1 + Y_2 > t)}_{\mathbb{P}(X_1 < t-d)} \\
&= F_{X_1}(t-d) \qquad (12)
\end{aligned}$$

For $2d \leq t < 3d$:

$$\begin{aligned}
\pi_{11}(t) &= \mathbb{P}(Y_1 > t) + \underbrace{\mathbb{P}(Y_1 + X_1 < t, Y_1 + X_1 + Y_2 > t)}_{\mathbb{P}(X_1 < t-d, X_1 > t-2d)} \\
&\quad + \underbrace{\mathbb{P}(Y_1 + X_1 + Y_2 + X_2 < t, Y_1 + X_1 + Y_2 + X_2 + Y_3 > t)}_{\mathbb{P}(X_1 + X_2 < t-2d)} \\
&= F_{X_1}(t-d) - F_{X_1}(t-2d) + F_{X_1+X_2}(t-2d) \qquad (13)
\end{aligned}$$

Using $H_n(t) = \sum_{r=1}^{n} F_{Z_r}(t - rd) - F_{Z_{r+1}}(t - rd)$ to give us the general case:

$$\pi_{11}(t) = \begin{cases} 1 & : 0 \leq t < d \\ F_{Z_1}(t-d) & : d \leq t < 2d \\ F_{Z_1}(t-d) - H_{n-1}(t-d) & : nd \leq t < (n+1)d \end{cases} \qquad (14)$$

We observe that $G_n(t) = H_n(t)$ and thus can also derive $\pi_{00}(t)$ from $\pi_{11}(t)$ directly.

$$\pi_{00}(t) = 1 - \pi_{11}(t + d) \qquad (15)$$

which represents a translation of $\pi_{01}(t)$ by $-d$, i.e. taking into account the extra Y transition.

2.3 M/D CMNF Special Case

For $X \sim \exp(\lambda)$, it implies that $Z_n \sim \text{erl}(n, \lambda)$, so:

$$f_{Z_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \qquad (16)$$

Now:

$$\overline{F}_{Z_n}(t) = e^{-\lambda t} \sum_{r=0}^{n-1} \frac{(\lambda t)^r}{r!} \qquad (17)$$

In order to construct $G_n(t)$, we examine:

$$\overline{F}_{Z_{n+1}}(\phi) - \overline{F}_{Z_n}(\phi) = \frac{(\lambda \phi)^n}{n!} e^{-\lambda \phi} \qquad (18)$$

Now we can say that for $nd \leq t < (n+1)d$:

$$\pi_{00}(t) = \sum_{r=0}^{n} \frac{(\lambda(t - rd))^n}{n!} e^{-\lambda(t-rd)} \qquad (19)$$

and:

$$\pi_{11}(t) = \begin{cases} 1 & : 0 \le t < d \\ 1 - \sum_{r=0}^{n-1} \frac{(\lambda(t-(r+1)d))^r}{r!} e^{-\lambda(t-(r+1)d)} & : nd \le t < (n+1)d \end{cases} \tag{20}$$

Plots of $\pi_{00}(t)$ and $\pi_{11}(t)$ are shown tending to their steady-state limits in figures 2 and 3. Its clear from the graphs that M/D CMNF transient probabilities satisfy the convergence of equation (2). However, a D/D system, for example, would not converge under this condition and we would need to employ equation (3) to obtain the steady-state.

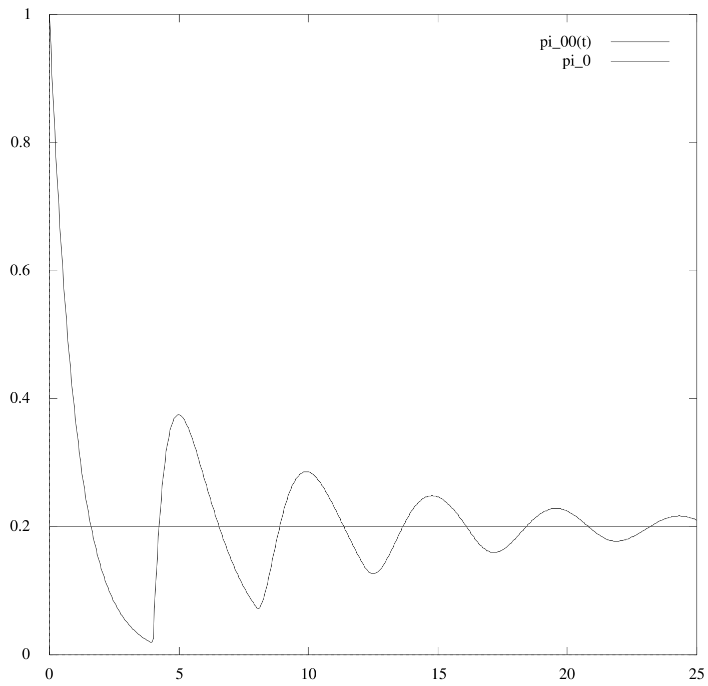

Fig. 2. Plot of $\pi_{00}(t)$ of M/D system against t for $\pi_0 = 0.2$

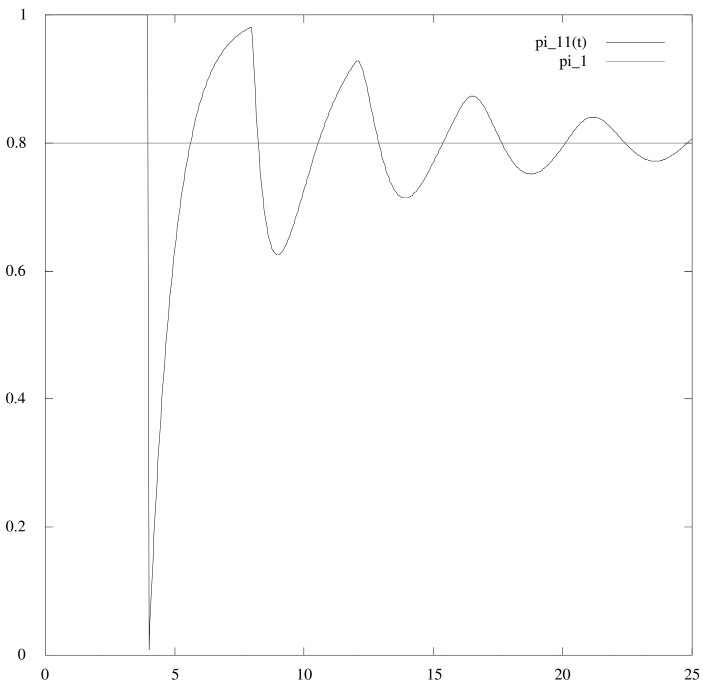

Fig. 3. Plot of $\pi_{11}(t)$ of M/D system against t for $\pi_1 = 0.8$

3 Generalising the Two State Transient Distribution Problem

3.1 CMNF Transient Distributions

Now generalising the method of section 2.2.2, we can construct a single formula for a CMNF with generally distributed X and Y.

$$\begin{aligned}\pi_{00}(t) &= \mathbb{P}(0 \leq t < X_1) + \mathbb{P}(X_1 + Y_1 \leq t, X_1 + Y_1 + X_2 > t) + \cdots \\ &\quad + \mathbb{P}(\underbrace{\sum_{i=0}^{r}(X_i + Y_i) \leq t}_{\psi_r \leq t}, \underbrace{\sum_{i=0}^{r}(X_i + Y_i) + X_{r+1} > t}_{X_{r+1} > t - \psi_r}) + \cdots \\ &= \overline{F}_{X_1}(t) + \sum_{r=1}^{\infty} \int_0^t \overline{F}_{X_{r+1}}(t-\tau) f_{\psi_r}(\tau)\,\mathrm{d}\tau \\ &= \overline{F}_{X_1}(t) + \sum_{r=1}^{\infty} \overline{F}_{\psi_r + X_{r+1}}(t) - \overline{F}_{\psi_r}(t) \qquad (21)\end{aligned}$$

where we take $\psi_r = \sum_{i=1}^{r}(X_i + Y_i)$.

Equation (21) is unwieldy given how difficult it is to find quantities like $\overline{F}_{\psi_r}$, itself within an infinite summation. By differentiating and taking the Laplace transform of the transient state function, $\pi_{00}(t)$, we can obtain an elegant expression which not only eliminates the infinite summation but removes the need to find expressions for $\psi_r + X_{r+1}$ and ψ_r.

Differentiating equation (21) then:

$$\pi'_{00}(t) = -f_{X_1}(t) + \sum_{r=1}^{\infty} f_{\psi_r}(t) - f_{\psi_r + X_{r+1}}(t) \tag{22}$$

Taking the Laplace transform gives:

$$\begin{aligned}
\mathcal{L}\left\{\pi'_{00}(t)\right\} &= -L_{X_1}(z) + \sum_{r=1}^{\infty} L_{\psi_r}(z) - L_{\psi_r}(z) \underbrace{L_{X_{r+1}}(z)}_{X_{r+1}=X} \\
&= -L_X(z) + \sum_{r=1}^{\infty} L_{\psi_r}(z)(1 - L_X(z)) \\
&= -1 + (1 - L_X(z)) \sum_{r=0}^{\infty} (L_X(z) L_Y(z))^r \\
&= \frac{1 - L_X(z)}{1 - L_X(z) L_Y(z)} - 1
\end{aligned} \tag{23}$$

Now from general properties of Laplace transforms [9, P1.1.4] we know that:

$$\mathcal{L}\left\{\pi'_{00}(t)\right\} = z\hat{\pi}_{00}(z) - \underbrace{\pi_{00}(0)}_{1} \tag{24}$$

Thus:

$$z\hat{\pi}_{00}(z) = 1 + \mathcal{L}\left\{\pi'_{00}(t)\right\} \tag{25}$$

Giving:

$$\hat{\pi}_{00}(z) = \frac{1}{z} \frac{1 - L_X(z)}{1 - L_X(z) L_Y(z)} \tag{26}$$

Equation (26) is particularly nice as it is solely a function of $L_X(z)$ and $L_Y(z)$, which we know we can systematically obtain through stochastic aggregation of any semi-Markov process.

We now look at how we can obtain the steady-state behaviour as a limit of the transient-state function. Expanding equation (25) we obtain:

$$z\hat{\pi}_{00}(z) = \pi_{00}(0) + \int_0^{\infty} \pi'_{00}(t) e^{-zt} \, \mathrm{d}t \tag{27}$$

Taking the limit:

$$\begin{aligned}\lim_{z\longrightarrow 0} z\hat{\pi}_{00}(z) &= \pi_{00}(0) + \lim_{z\longrightarrow 0}\int_0^\infty \pi'_{00}(t)e^{-zt}\,\mathrm{d}t \\ &= \pi_{00}(0) + [\pi_{00}(t)]_0^\infty \\ &= \lim_{t\longrightarrow\infty} \pi_{00}(t) \\ &= \pi_0 \end{aligned} \tag{28}$$

Applying this to equation (26) and using l'Hôpital's rule gives:

$$\begin{aligned}\pi_0 &= \lim_{z\longrightarrow 0}\frac{1-L_X(z)}{1-L_X(z)L_Y(z)} \\ &= \lim_{z\longrightarrow 0}\frac{-L'_X(z)}{-L'_X(z)L_Y(z)-L_X(z)L'_Y(z)} \\ &= \frac{\mathbb{E}(X)}{\mathbb{E}(X)+\mathbb{E}(Y)}\end{aligned} \tag{29}$$

This result agrees very nicely with the result for the steady-state distribution of the Cox-Miller normal form as presented in [3, p.39] which was derived by an entirely different method using the Strong Law of Large Numbers.

4 Further Two State Transient Functions

4.1 CMNF Transient Functions

We look at deriving the remaining transient functions, $\pi_{10}(t)$, $\pi_{11}(t)$, $\pi_{01}(t)$, based on the work of section 3 which found an expression for $\hat{\pi}_{00}(z)$.

We now have:

$$\mathcal{L}\left\{\pi_{00}(t)\right\} = \hat{\pi}_{00}(z) = \frac{1}{z}\,\frac{1-L_X(z)}{1-L_X(z)L_Y(z)} \tag{30}$$

under certain conditions (for instance, neither X or Y can be $\delta(0)$ distributions).

For all $t \geq 0$, we know:

$$\pi_{00}(t) + \pi_{10}(t) = 1 \tag{31}$$

Giving:

$$\hat{\pi}_{00}(z) + \hat{\pi}_{10}(z) = \frac{1}{z} \tag{32}$$

$$\hat{\pi}_{10}(z) = \frac{1}{z}\frac{L_X(z)(1 - L_Y(z))}{1 - L_X(z)L_Y(z)} \tag{33}$$

By symmetry, we also obtain:

$$\hat{\pi}_{11}(z) = \frac{1}{z}\frac{1 - L_Y(z)}{1 - L_X(z)L_Y(z)} \tag{34}$$

$$\hat{\pi}_{01}(z) = \frac{1}{z}\frac{L_Y(z)(1 - L_X(z))}{1 - L_X(z)L_Y(z)} \tag{35}$$

4.2 Alternative Derivation

This gives us an interesting alternative derivation for the $\hat{\pi}_{ij}(z)$. We note that:

$$\hat{\pi}_{01}(z) = L_Y(z)\hat{\pi}_{00}(z) \qquad \text{and} \qquad \hat{\pi}_{10}(z) = L_X(z)\hat{\pi}_{11}(z) \tag{36}$$

Together with equation (32) and its dual:

$$\hat{\pi}_{00}(z) + \hat{\pi}_{10}(z) = \frac{1}{z} \qquad \text{and} \qquad \hat{\pi}_{11}(z) + \hat{\pi}_{01}(z) = \frac{1}{z} \tag{37}$$

we can generate all of the relevant $\hat{\pi}_{ij}(z)$ (equations (30,33–35)). For example:

$$\begin{aligned}
\hat{\pi}_{01}(z) &= L_Y(z)\hat{\pi}_{00}(z) \\
&= L_Y(z)\left(\frac{1}{z} - \hat{\pi}_{10}(z)\right) \\
&= L_Y(z)\left(\frac{1}{z} - L_X(z)\hat{\pi}_{11}(z)\right) \\
&= L_Y(z)\left(\frac{1}{z} - L_X(z)\left(\frac{1}{z} - \hat{\pi}_{01}(z)\right)\right) \\
&= \frac{1}{z}\frac{L_Y(z)(1 - L_X(z))}{1 - L_X(z)L_Y(z)}
\end{aligned} \tag{38}$$

The physical interpretation of equation (36) is similar to that of equation (15) in section 2. This effectively related $\pi_{00}(t)$ to $\pi_{01}(t)$ in a M/D system and demonstrated that the two quantities differed by a translation of $-d$, i.e. the Y-transition. In general, we can obtain $\pi_{01}(t)$ from $\pi_{00}(t)$ by convolving it with a Y distribution (and similarly for $\pi_{10}(t)$ and $\pi_{11}(t)$).

4.3 Some Results

We present a couple of results based on having the four $\pi_{ij}(t)$ functions, for completeness.

4.3.1 Steady-state Values

Using the technique of equation 29 we also obtain the steady-state limits for the other transient functions:

$$\lim_{z \longrightarrow 0} z\hat{\pi}_{ij}(z) = \pi_i \qquad : i,j = 0,1 \tag{39}$$

4.3.2 Non-conditioned Transient Functions

Recall from section 2 that:

$$\pi_{ij}(t) = \mathbb{P}(\text{system in state } i \text{ at time } t \mid \text{started from state } j) \tag{40}$$

For purposes of compatibility with previously published transient distributions (for example, the M/M case in [7]), we will reformulate our transient functions in terms of unconditioned functions $p_i(t)$ for $i = 0, 1$, where:

$$p_i(t) = \mathbb{P}(\text{system in state } i \text{ at time } t) \tag{41}$$

By the Law of Total Probability:

$$p_0(t) = p_0(0)\pi_{00}(t) + p_1(0)\pi_{01}(t) \tag{42}$$

$$p_1(t) = p_1(0)\pi_{11}(t) + p_0(0)\pi_{10}(t) \tag{43}$$

where $p_0(0) + p_1(0) = 1$ and are the initial conditions of the system.

It is easy to show:

$$p_0(t) + p_1(t) = 1 \qquad \text{for all } t \geq 0 \tag{44}$$

Using the linear nature of the Laplacian:

$$\hat{p}_0(z) = p_0(0)\hat{\pi}_{00}(z) + p_1(0)\hat{\pi}_{01}(z) \tag{45}$$

and then equation (36):

$$\hat{p}_0(z) = \frac{1}{z}\,\frac{(1 - L_X(z))(p_0(0) + p_1(0)L_Y(z))}{1 - L_X(z)L_Y(z)} \tag{46}$$

Similarly:

$$\hat{p}_1(z) = \frac{1}{z}\,\frac{(1 - L_Y(z))(p_1(0) + p_0(0)L_X(z))}{1 - L_X(z)L_Y(z)} \tag{47}$$

5 Transient Method Comparisons

5.1 Complexity of Transient Analysis

Our motivation for concentrating so hard on the two state problem is that from it and the work in [3], we know that we can obtain any $\pi_{ii}(t)$ for any n-state semi-Markov system. This is achieved by reducing a general n-state system to two states using a process of stochastic aggregation, then it can be reasoned about in the same way as our two state example of section 3. From $\pi_{ii}(t)$ and the passage-time distribution $g_{ji}(t)$ (also obtained from stochastic aggregation), we can obtain the entire transient matrix, $\pi_{ij}(t)$[2]. The complexity of this process is dominated by the inversion of an $(n-1)\times(n-1)$ matrix, however in solving this system we can also obtain passage-time distributions and steady-state values.

5.2 Other Transient Techniques

In Cox and Miller [6, pp.352–355], they derive a matrix technique for solving a two state semi-Markov system for its transient distributions, which extends directly to n-states. The method yields all of the transient functions, $\pi_{ij}(t)$, not just the leading diagonal. Using the limit argument discussed earlier, it will also give us steady-state values. However it is not possible to obtain passage-time distributions from this method. The complexity of the technique is dominated by the inversion of an $n \times n$ matrix $(I - g^*)$ and so is of the same O-complexity as the method presented in this paper.

In Pyke [8], a different method is presented which provides a distinct formula for calculating the matrix of $\hat{\pi}_{ij}(z)$ of transient Laplace functions. As with the Cox and Miller method, it hinges around the inversion of a matrix quantity $(I - q)$ and is dominated y this calculation.

It is also of interest to note that [8] provides a formula for the calculation of passage-time distributions $\hat{g}_{ij}(z)$. It shows that if we have the passage-time distributions and the leading diagonal of the transient distribution matrix, $\hat{\pi}_{ii}(z)$, we can calculate the off-diagonal elements $\hat{\pi}_{ij}(z) = \hat{\pi}_{jj}(z)\hat{g}_{ij}(z)$.

[2] This is identical to the technique used in [8], mentioned later.

5.3 A Problem with Cox-Miller Technique?

We had problems with implementing the Cox-Miller technique for extracting transient distributions. It would appear that the method presented [6], does not seem to extend to n-states as claimed. We tested a simple 3-state Markov system and took the appropriate limit of the transient Laplace distribution to get the steady-state value for a given state (see equation 28). This appeared to differ from the steady-state result obtained through standard Markov chain theory.

The example system was defined by the CTMC generator matrix:

$$Q = \begin{pmatrix} -(a+d) & a & d \\ 0 & -b & b \\ c & 0 & -c \end{pmatrix} \tag{48}$$

We obtained the transient matrix and extracted $\hat{\pi}_{22}(z)$ and extracted the steady-state probability π_{22} as $z \longrightarrow 0$. From this calculation we obtained:

$$\pi_{22} = \frac{abc}{(a+d)(ca+cb+ab+db)} \tag{49}$$

From the standard equation $\pi Q = 0$, we obtained the equivalent result:

$$\pi_{22} = \frac{ca}{ca+cb+ab+db} \tag{50}$$

This gives us an anomaly in the Cox-Miller technique and care should be taken if using this method.

6 An Example of Transient Analysis

6.1 The System

We demonstrate the process of stochastic aggregation on the system in figure 4. For each state the system is reduced to a G/G CMNF and the transient state distribution $\hat{\pi}_{ii}(z)$ is calculated. We then apply the analysis of [1] to invert the Laplace transform and display the three distributions $\pi_{ii}(t)$.

The Laplace inversions were done using the Euler method in all cases, the Post-Widder technique was used to confirm the result for $\pi_{22}(t)$ as there was the possibility of some error due to the step function which occurs early in the evolution of the system (see [1, 4] for details).

In all cases the the final value of the damped oscillations of transient behaviour came within 3 or more significant figures of the theoretical steady state value. The steady state values are analytically calculated in [3, pp.45–49] for this system.

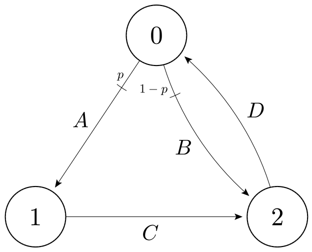

Fig. 4. A generally-distributed stochastic transition system.

The distributions used in the transition system of figure 4 are given by:

$$A \sim \Gamma(\alpha, \lambda) \tag{51}$$

$$B \sim \text{Hyper}\left(\frac{2}{3}, \frac{1}{3}; \lambda, \mu\right) \tag{52}$$

$$C \sim \text{U}\left[\frac{1}{\lambda}, \frac{1}{\mu}\right] \tag{53}$$

$$D \sim \delta\left[\frac{1}{\mu}\right] \tag{54}$$

where:

$$L_A(z) = \left(\frac{\lambda}{\lambda + z}\right)^{\alpha} \tag{55}$$

$$L_B(z) = \frac{2}{3}\frac{\lambda}{\lambda + z} + \frac{1}{3}\frac{\mu}{\mu + z} \tag{56}$$

$$L_C(z) = \frac{\lambda\mu}{(\lambda - \mu)z}\left[e^{-z/\lambda} - e^{-z/\mu}\right] \tag{57}$$

$$L_D(z) = e^{-z/\mu} \tag{58}$$

When we come to plot the graphs, the following values will be selected for the parameters: $\alpha = 5/2$, $\lambda = 3/2$, $\mu = 1$ and $p = 3/10$.

6.2 State 1

Figure 5 shows the aggregation of states 0 and 2 into a lumped state with a single out-distribution ψ_1. This can be achieved by hand using the techniques

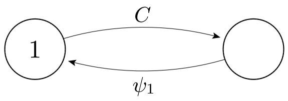

Fig. 5. Aggregating states 0 and 2 to give a Cox-Miller Normal Form for state 1.

of [3] or using the algorithm of [5].

$$L_{\psi_1}(z) = \frac{pL_A(z)L_D(z)}{1-(1-p)L_B(z)L_D(z)} \tag{59}$$

Having obtained the CMNF, we can now write down the appropriate Laplace of the transient function:

$$\hat{\pi}_{11}(z) = \frac{1}{z}\frac{1-L_C(z)}{1-L_C(z)L_{\psi_1}(z)} \tag{60}$$

Figure 6 shows the inversion of $\hat{\pi}_{11}(z)$. Note that the system stays in state 1 for $t \leq 2/3$ and then the probability descends linearly to 0 for $2/3 < t \leq 1$. This is consistent with the uniform C distribution which is the first one encountered when starting from state 1. The system is nicely predictable for the second transition (D from state 2) which is deterministic for 1 time unit; this is shown by a 0 probability for $1 < t \leq 5/3$ and a near zero probability for $5/3 < t \leq 2$ while the system resides in state 2 for the first time.

6.3 State 2

Figure 7 shows the aggregation around state 2 to give a G/G CMNF. The aggregate ψ_2 distribution is given by:

$$L_{\psi_2}(z) = pL_A(z)L_C(z) + (1-p)L_B(z) \tag{61}$$

which gives the following transient function for state 2:

$$\hat{\pi}_{22}(z) = \frac{1}{z}\frac{1-L_D(z)}{1-L_D(z)L_{\psi_2}(z)} \tag{62}$$

Figures 11 and 12 show the inversion of $\hat{\pi}_{22}(z)$. The first transition of the system from state 2 is shown very clearly by the step function as a result of the deterministic D distribution. The system stays in state 2 with probability 1 for $0 < t \leq 1$ before leaving. The Post-Widder generated graph of figure 12 confirms the qualitative behaviour of the first although it still contains more error. It should be noted that no pre-smoothing was carried out (as this would have altered the result) but nonetheless good results of a discontinuous function were obtained—this confirms our own experience of the inversion algorithms but is perhaps surprising given the remarks of [2].

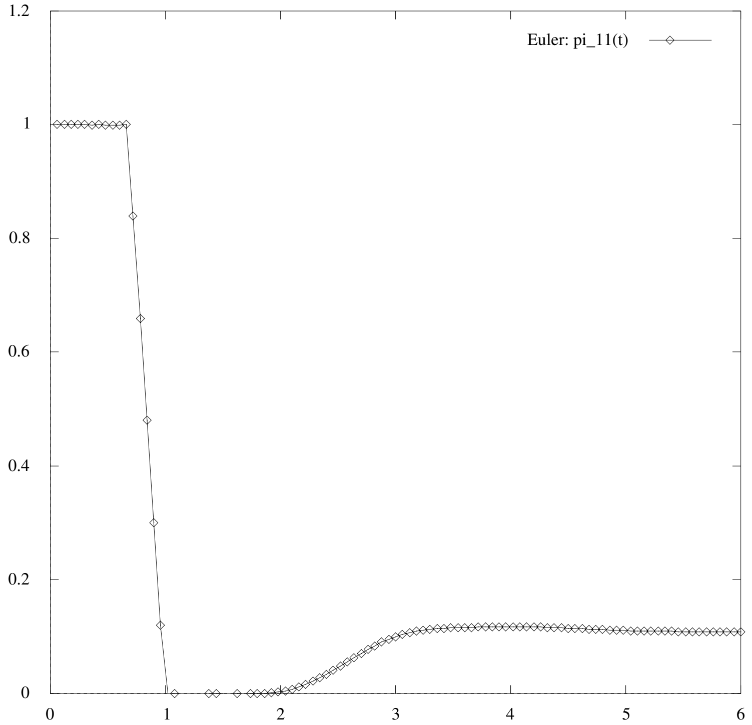

Fig. 6. The transient state distribution for $\pi_{11}(t)$

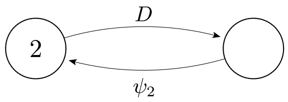

Fig. 7. Aggregating states 0 and 1 to give a Cox-Miller Normal Form for state 2.

6.4 State 0

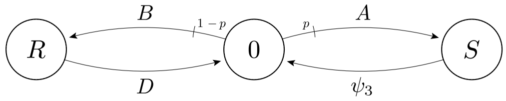

Fig. 8. Aggregating around state 0 to get a more complex normal form.

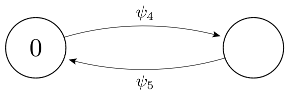

Fig. 9. Constructing a CMNF structure.

The system is first simplified to that of figure 8 where:

$$L_{\psi_3}(z) = L_C(z)L_D(z) \tag{63}$$

before being transformed to a CMNF as before in figure 9 with:

$$L_{\psi_4}(z) = pL_A(z) + (1-p)L_B(z) \tag{64}$$

$$L_{\psi_5}(z) = \frac{pL_A(z)L_C(z)L_D(z) + (1-p)L_B(z)L_D(z)}{pL_A(z) + (1-p)L_B(z)} \tag{65}$$

As before the transient function can now be written down:

$$\hat{\pi}_{00}(z) = \frac{1}{z}\,\frac{1 - L_{\psi_4}(z)}{1 - L_{\psi_4}(z)L_{\psi_5}(z)} \tag{66}$$

Figure 10 shows the inversion of $\hat{\pi}_{00}(z)$ to give $\pi_{00}(t)$.

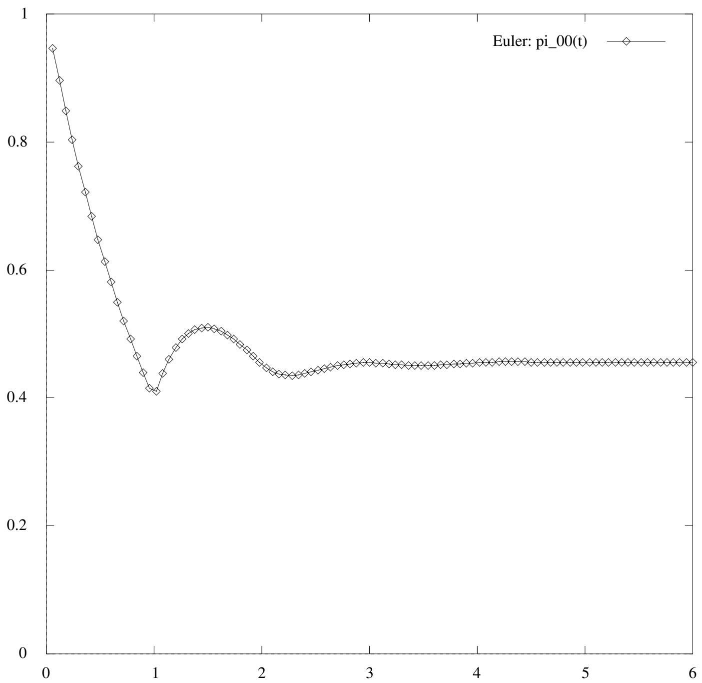

Fig. 10. The transient state distribution for $\pi_{00}(t)$

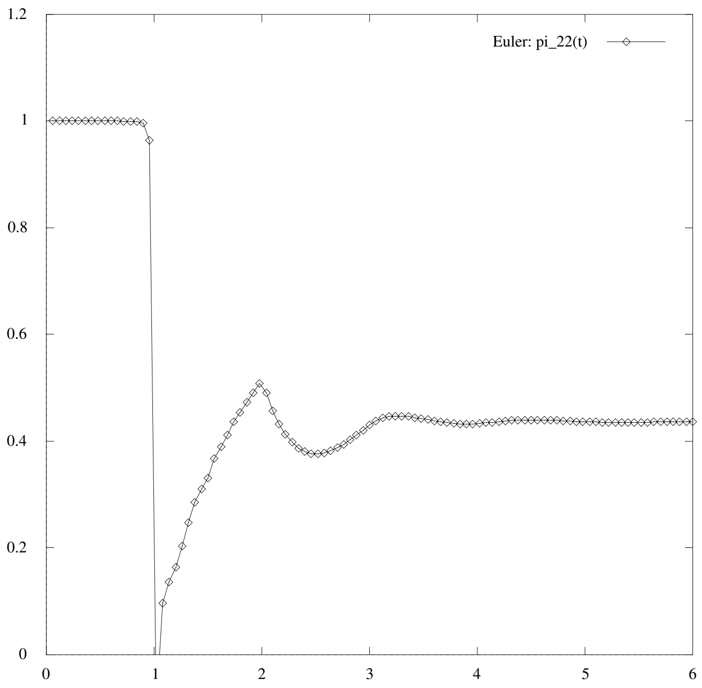

Fig. 11. The transient state distribution for $\pi_{22}(t)$

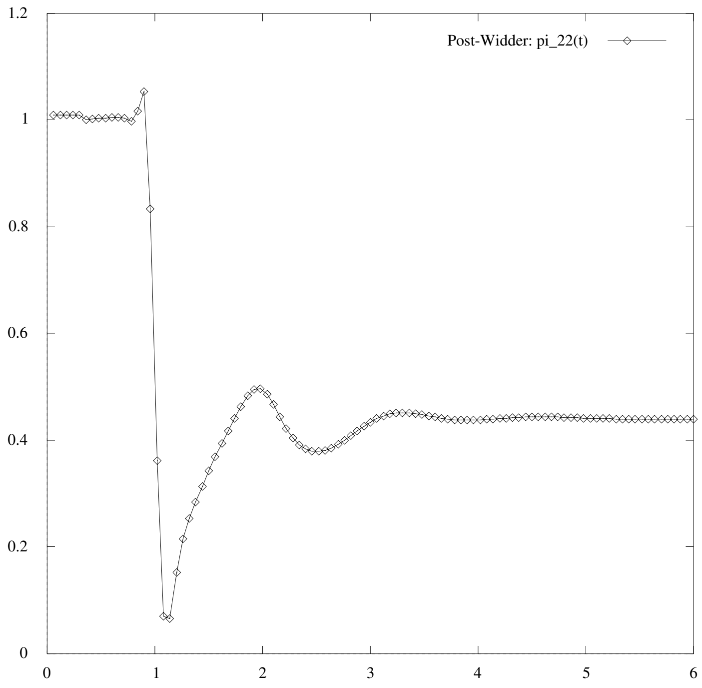

Fig. 12. The transient state distribution for $\pi_{22}(t)$, confirmed using the Post-Widder method.

7 Conclusion

In this report, we have shown how to derive entire transient distribution results for semi-Markov processes from an initial analysis of a two state M/D system. It introduces other techniques and presents a possible problem with the Cox-Miller method. An example is also analysed.

In deriving single transient results this is a good technique to use, as it can generate individual probabilities as required. For the entire transient matrix, the largely similar Pyke technique is preferable.

Acknowledgements

JTB is funded by the EPSRC QWiD project GR/N01491.

References

[1] Abate, J., Choudhury, G. L., and Whitt, W. An introduction to numerical transform inversion and its application to probability models. In *Computational Probability*, W. Grassman, Ed. Kluwer, Boston, 1999, pp. 257–323.

[2] Abate, J., and Whitt, W. The Fourier-series method for inverting transforms of probability distributions. *Queueing Systems 10*, 1 (1992), 5–88.

[3] Bradley, J. T. *Towards Reliable Modelling with Stochastic Process Algebras.* PhD thesis, Department of Computer Science, University of Bristol, Bristol BS8 1UB, UK, October 1999.

[4] Bradley, J. T. Report on the numerical inversion of Laplace tranforms of probabilistic functions: A Haskell implementation. Tech. Rep. 2001(6), Department of Computer Science, University of Durham, South Road, Durham, DH1 3LE, UK, July 2001.

[5] Bradley, J. T., and Davies, N. J. A matrix-based method for analysing stochastic process algebras. In *ICALP Workshops 2000, Process Algebra and Performance Modelling Workshop* (Geneva, July 2000), J. D. P. Rolim, Ed., University of Waterloo, Ontario, Canada, Carleton Scientific, pp. 579–590.

[6] Cox, D. R., and Miller, H. D. *The Theory of Stochastic Processes.* Methuen, 1965.

[7] Gross, D., and Harris, C. M. *Fundamentals of Queueing Theory*, 3rd ed. John Wiley & Sons, 1998.

[8] Pyke, R. Markov renewal processes with finitely many states. *Annals of Mathematical Statistics 32*, 4 (December 1961), 1243–1259.

[9] Watson, E. J. *Laplace Transforms and Applications.* Van Nostrand Reinhold, 1981.

A Establishing Compatibility of Convergence Definitions

Statement of problem

We have proposed two definitions for the convergence of the transient probabilities to the steady-state probabilities:

i. Well-behaved case

$$\lim_{t \longrightarrow \infty} \pi_{ij}(t) = \pi_i \tag{67}$$

ii. General case

$$\lim_{t \longrightarrow \infty} \frac{1}{t} \int_0^t \pi_{ij}(\tau)\, d\tau = \pi_i \tag{68}$$

Proof that (ii) includes (i)

Suppose the transient probability converges in the sense of (i):

$$\lim_{t \longrightarrow \infty} \pi_{ij}(t) = \pi_i \tag{69}$$

Then given any $\varepsilon > 0$, there exists a time T such that $|\pi_{ij}(t) - \pi_i| < \varepsilon$ for all $t \geq T$. We split the integral from (ii) into two ranges:

$$\frac{1}{t} \int_0^t \pi_{ij}(\tau)\, d\tau = \underbrace{\frac{1}{t} \int_0^T \pi_{ij}(\tau)\, d\tau}_{\longrightarrow 0 \text{ as } t \longrightarrow \infty} + \frac{1}{t} \int_T^t \pi_{ij}(\tau)\, d\tau \tag{70}$$

$$\lim_{t \longrightarrow \infty} \frac{1}{t} \int_0^t \pi_{ij}(\tau)\, d\tau = \lim_{t \longrightarrow \infty} \frac{1}{t} \int_T^t \pi_{ij}(\tau)\, d\tau \tag{71}$$

But when $t \geq T$, we know that $\pi_i - \varepsilon < \pi_{ij}(t) < \pi_i + \varepsilon$, so:

$$\frac{1}{t} \int_T^t (\pi_i - \varepsilon)\, d\tau < \frac{1}{t} \int_T^t \pi_{ij}(\tau)\, d\tau < \frac{1}{t} \int_T^t (\pi_i + \varepsilon)\, d\tau \tag{72}$$

$$(\pi_i - \varepsilon) \left(1 - \frac{T}{t}\right) < \frac{1}{t} \int_T^t \pi_{ij}(\tau)\, d\tau < (\pi_i + \varepsilon) \left(1 - \frac{T}{t}\right) \tag{73}$$

$$\pi_i - \varepsilon < \lim_{t \longrightarrow \infty} \frac{1}{t} \int_T^t \pi_{ij}(\tau)\, d\tau < \pi_i + \varepsilon \tag{74}$$

Finally, from equation (71) we have that for any $\varepsilon > 0$:

$$\pi_i - \varepsilon < \lim_{t \longrightarrow \infty} \frac{1}{t} \int_0^t \pi_{ij}(\tau)\, d\tau < \pi_i + \varepsilon \tag{75}$$

and it follows that $\lim_{t \longrightarrow \infty} \frac{1}{t} \int_0^t \pi_{ij}(\tau)\, d\tau = \pi_i$. ■