Aggregation Strategies
in Semi-Markov Performance Models

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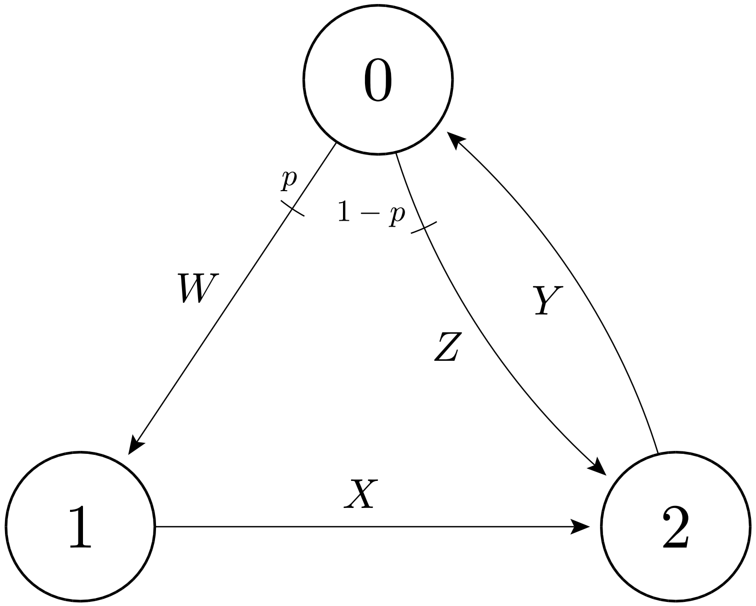
What is an SMP?



Distributions:
$W \sim \Gamma(\alpha, \lambda)$, $Z \sim \text{Hyper}\left(\frac{2}{3}, \frac{1}{3}; \lambda, \mu\right)$, $X \sim \text{U}\left[\frac{1}{\lambda}, \frac{1}{\mu}\right]$, $Y \sim \delta\left[\frac{1}{\mu}\right]$

What is an SMP?

- Represented by *discrete-time Markov chain*, P_{ij}, and state sojourn-time distribution matrix, $H_{ij}(t)$

- Kernel of process:

$$\begin{aligned} R_{ij}(t) &= \mathbb{P}(X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i) \\ &= P_{ij} H_{ij}(t) \end{aligned}$$

- X_n is state at nth state change, T_n is time of nth state change

Passage-time Analysis

- Traditional SMP passage-time and transient analysis requires matrix inversion e.g.

$$\begin{pmatrix} 1 & -\widehat{R}_{12}(s) & \cdots & -\widehat{R}_{1n}(s) \\ 0 & 1-\widehat{R}_{22}(s) & \cdots & -\widehat{R}_{2n}(s) \\ 0 & -\widehat{R}_{32}(s) & \cdots & -\widehat{R}_{3n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -\widehat{R}_{n2}(s) & \cdots & 1-\widehat{R}_{nn}(s) \end{pmatrix} \begin{pmatrix} \widehat{L}_{11}(s) \\ \widehat{L}_{21}(s) \\ \widehat{L}_{31}(s) \\ \vdots \\ \widehat{L}_{n1}(s) \end{pmatrix} = \begin{pmatrix} \widehat{R}_{11}(s) \\ \widehat{R}_{21}(s) \\ \widehat{R}_{31}(s) \\ \vdots \\ \widehat{R}_{n1}(s) \end{pmatrix}$$

- Representation of $\widehat{R}_{ij}(s)$ is crucial to solution tractability

SMP Aggregation

Aggregation of a semi-Markov process has 3 steps:

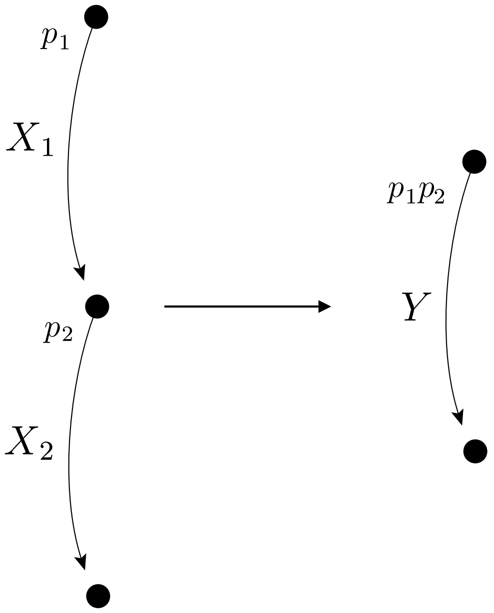
1. sequential aggregation

2. branching aggregation

3. cycle aggregation

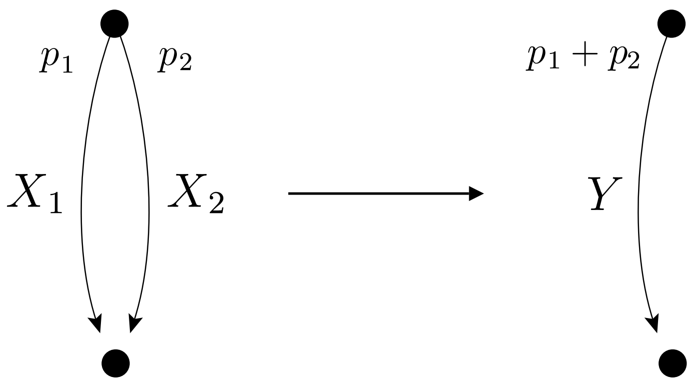
Sequential Aggregation

Using convolution to aggregate sequential transitions



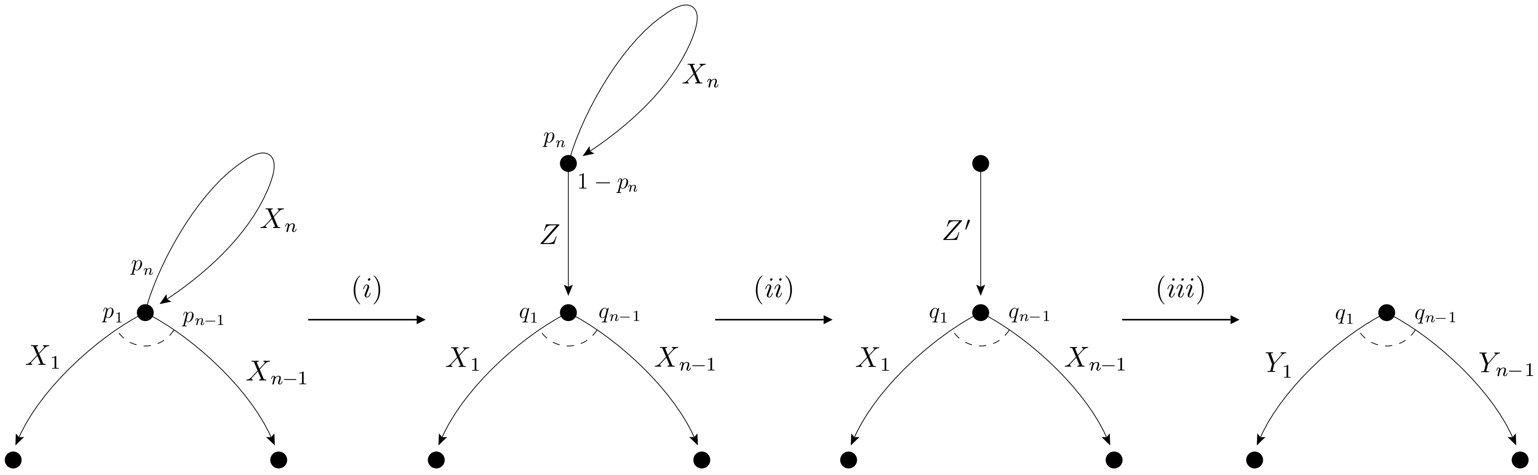
Branch Aggregation

Linear weighting of random variables to aggregate branches

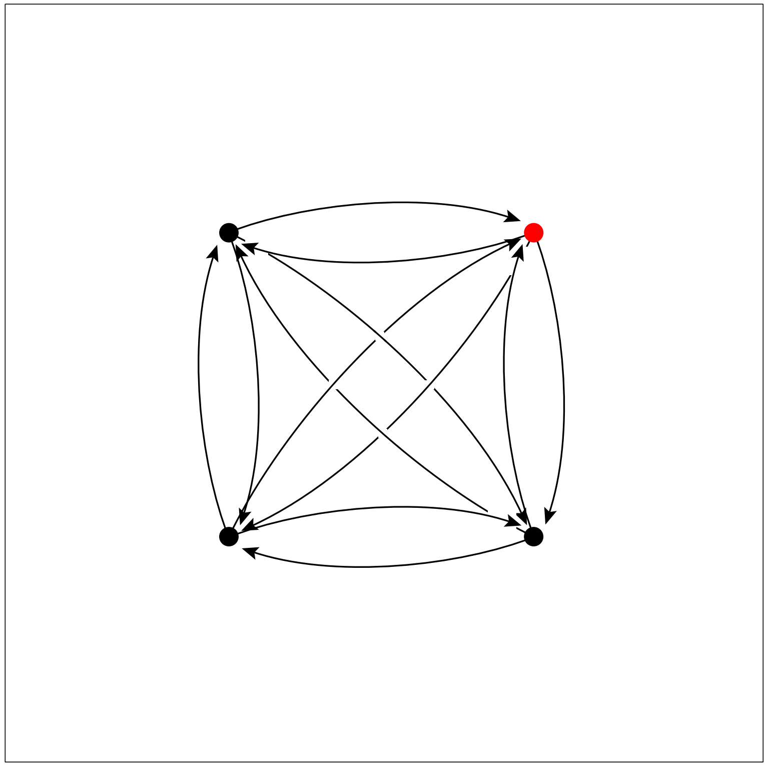


Cycle Aggregation

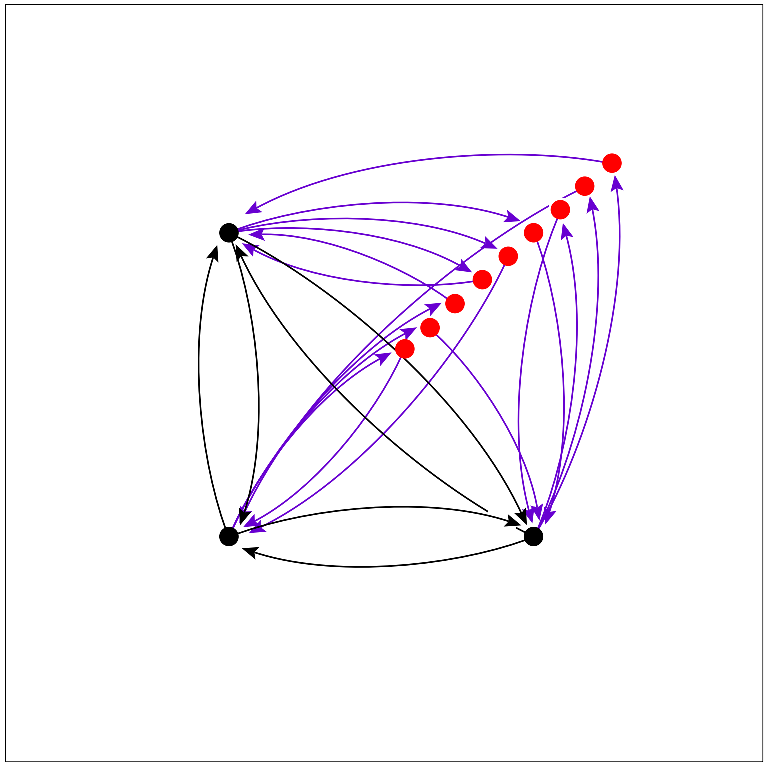
Cycle removal in an SMP



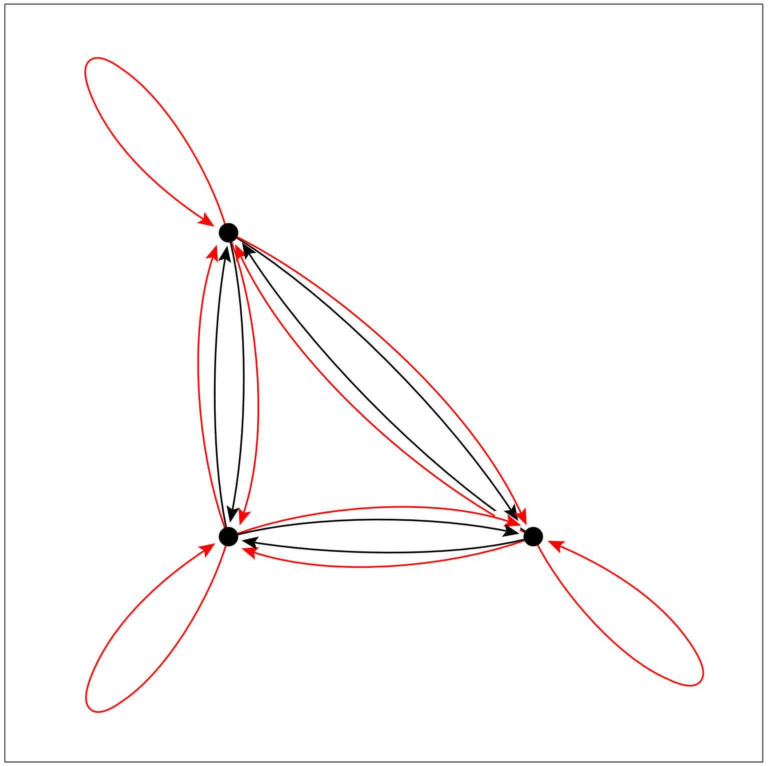
Semi-Markov Aggregation



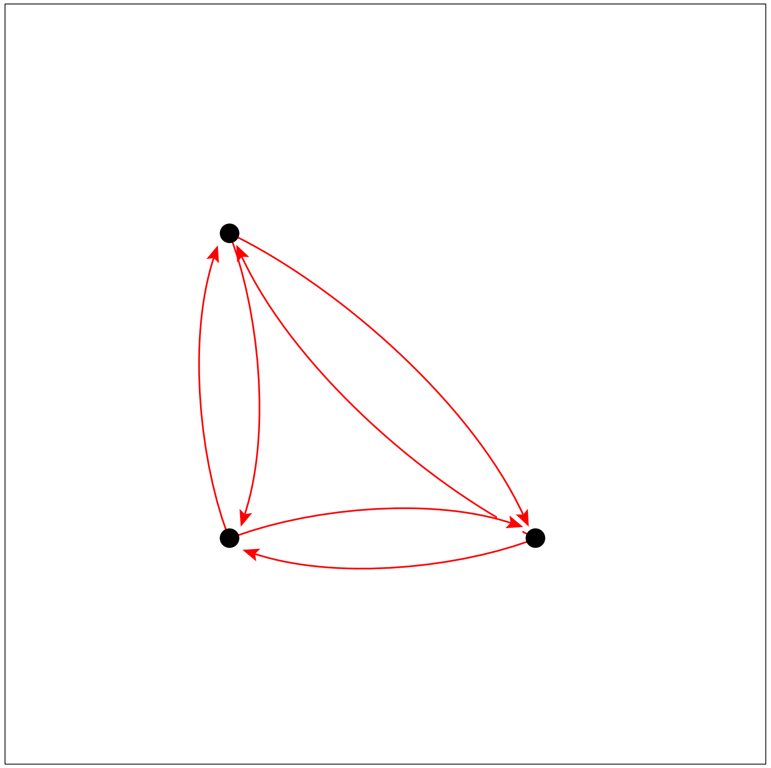
Semi-Markov Aggregation



Semi-Markov Aggregation

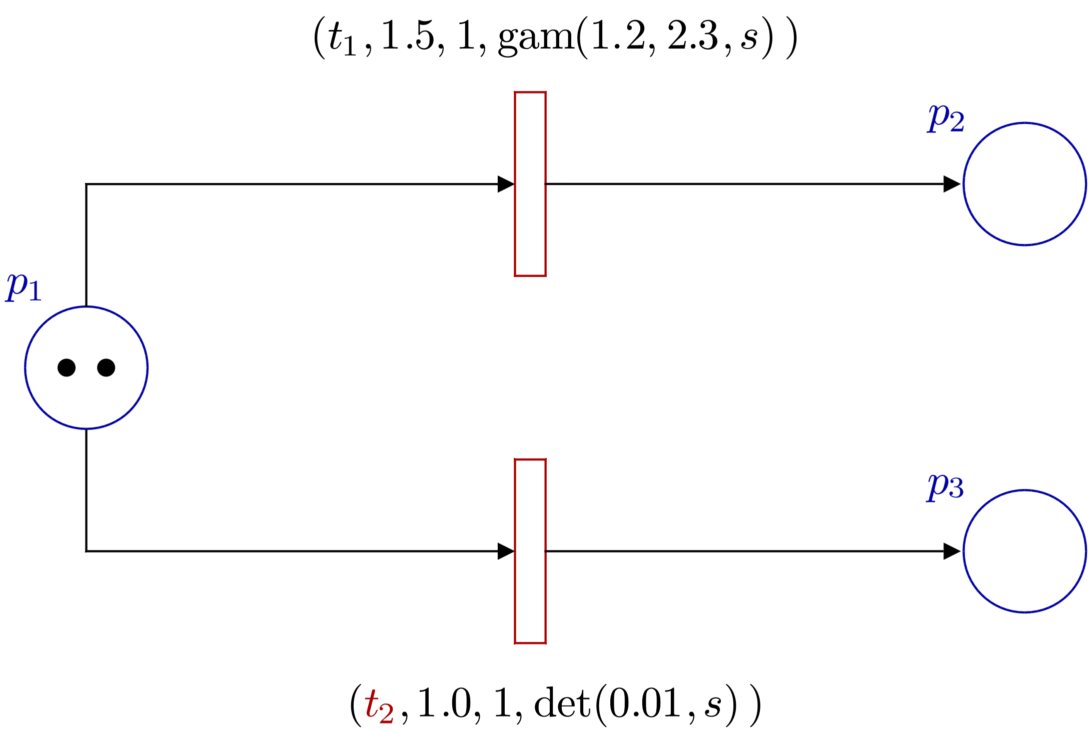


Semi-Markov Aggregation

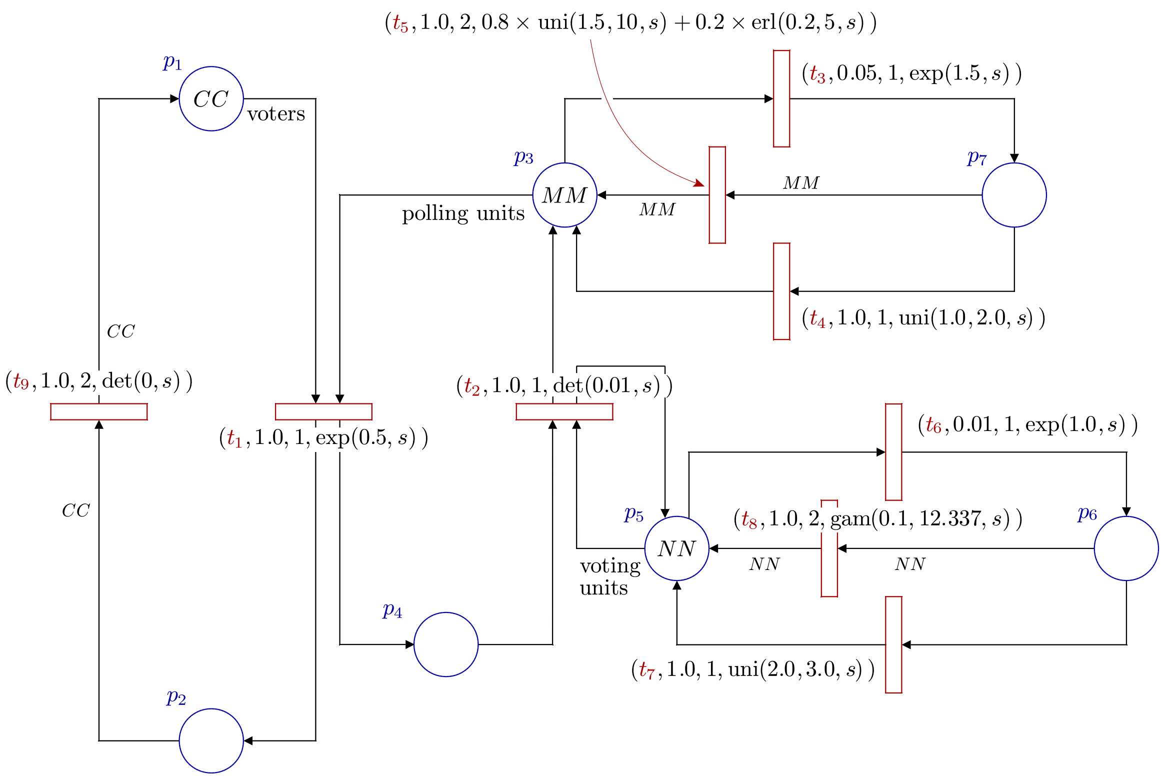


Worked Example

Use semi-Markov stochastic Petri nets to generate our SMP models

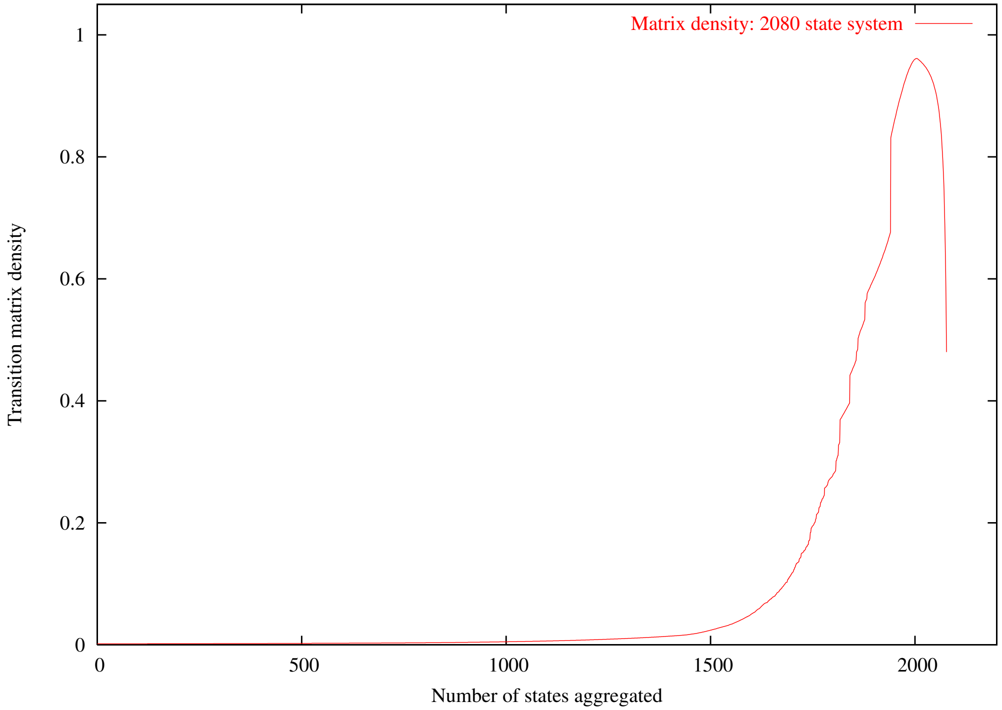


SM-SPN: Voting System



Matrix Density Explosion

Matrix density vs Number of states aggregated



Aggregation strategies

...or which state to aggregate first?

- Fewest paths
- Most successor states
- Most paths
- Hybrid policy

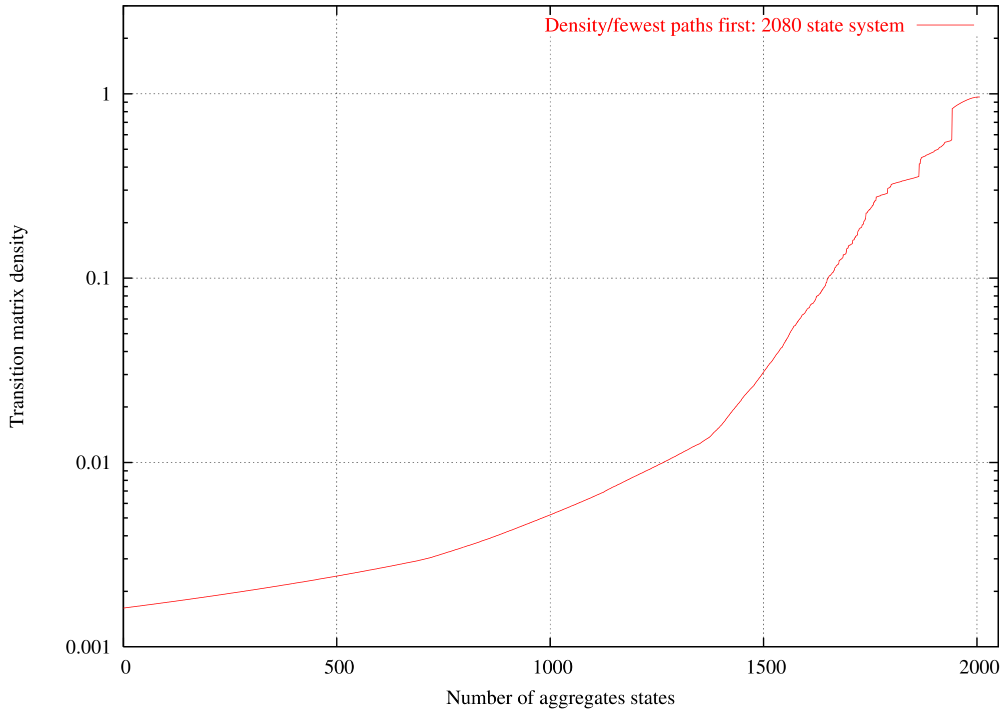
Aggregation Results

- How does matrix density vary with strategy?

- How does computational cost vary with strategy?

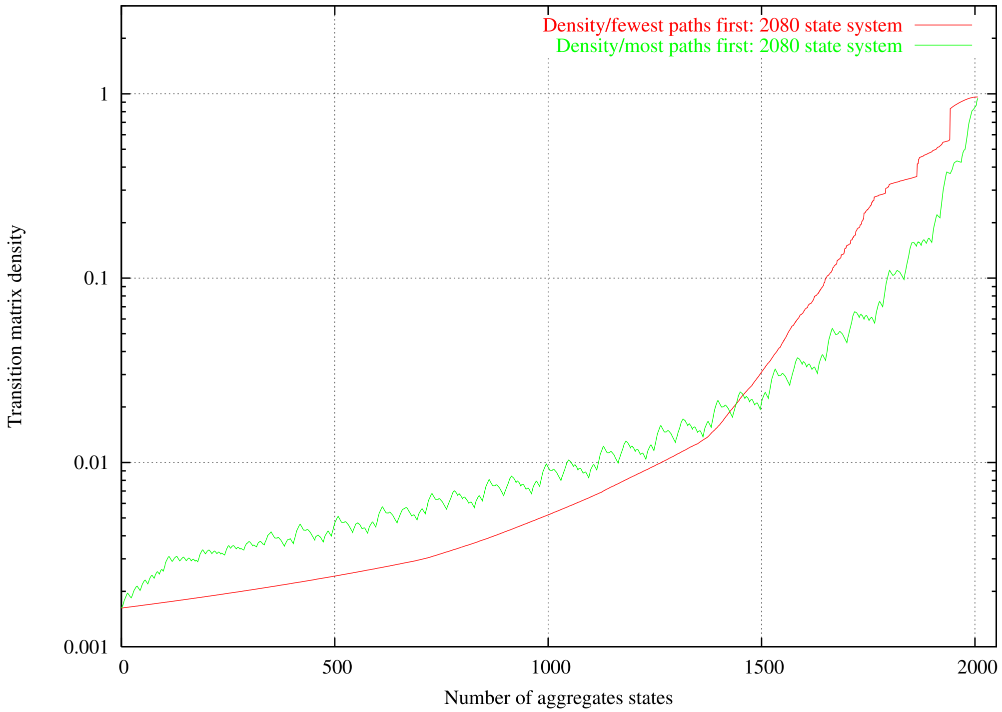
Matrix density

Logscale: Matrix density vs Number of states aggregated



Matrix density

Logscale: Matrix density vs Number of states aggregated



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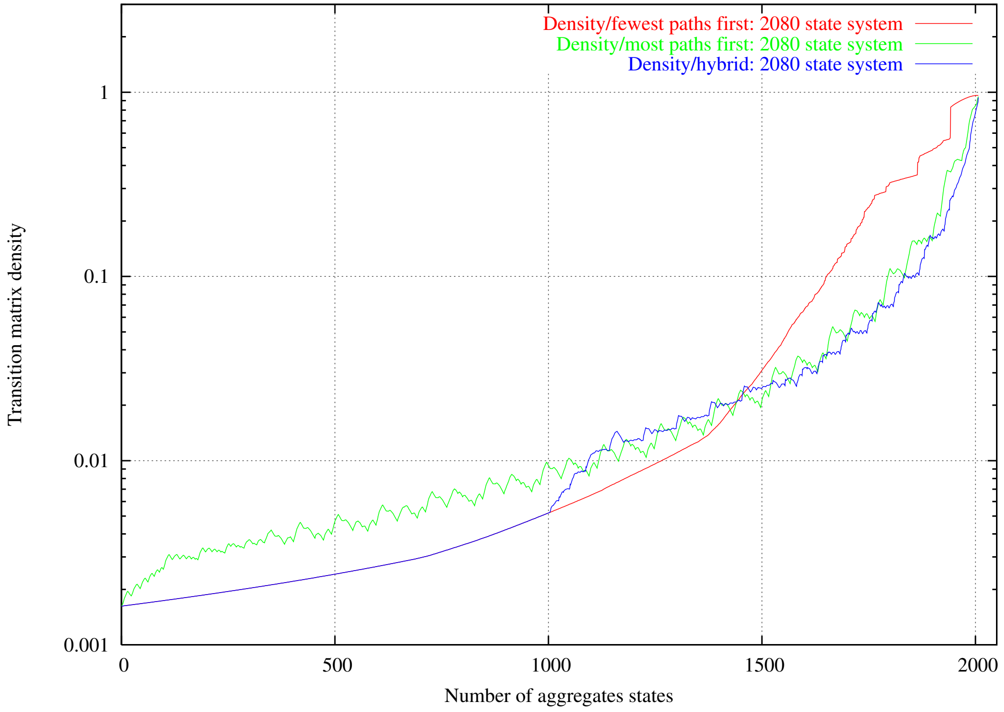
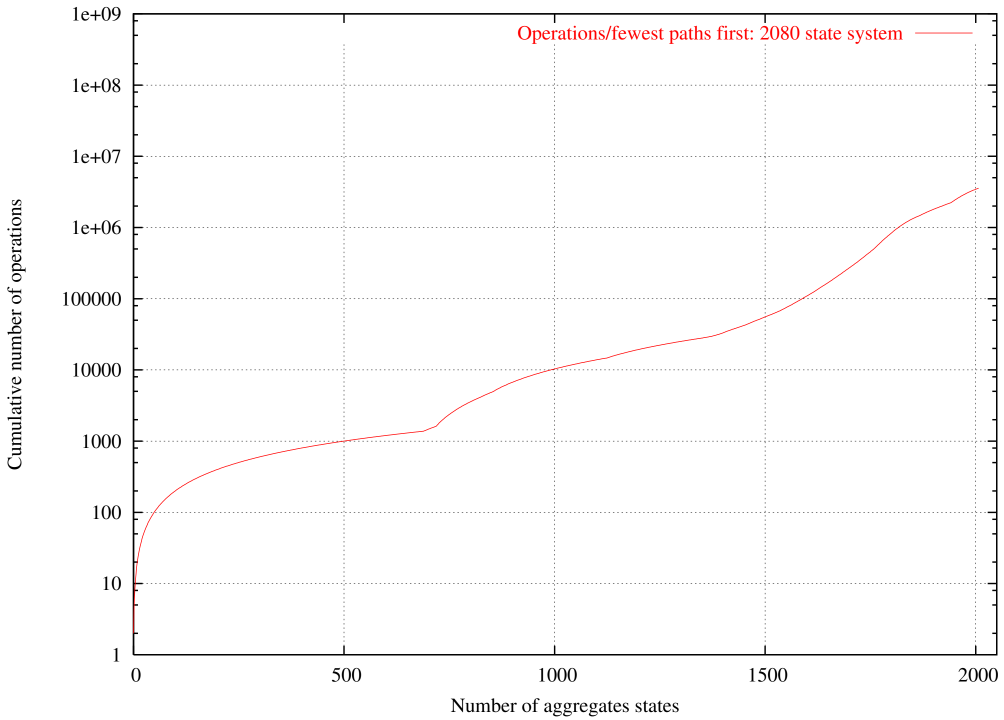


Aggregation operations

Logscale: Aggregation operations vs Number of states aggregated



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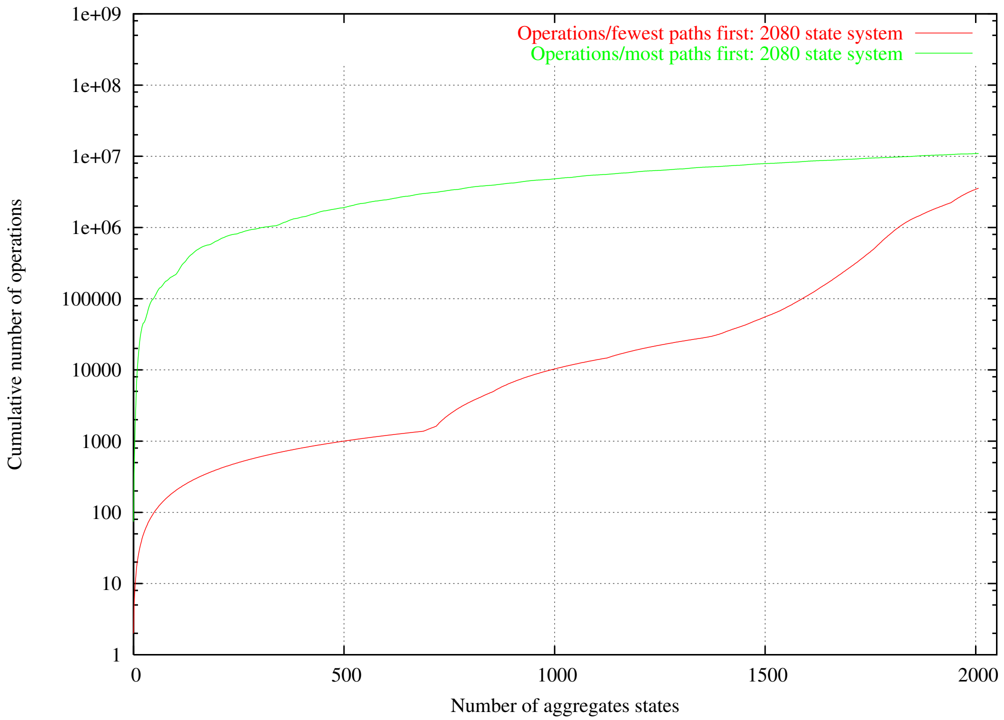

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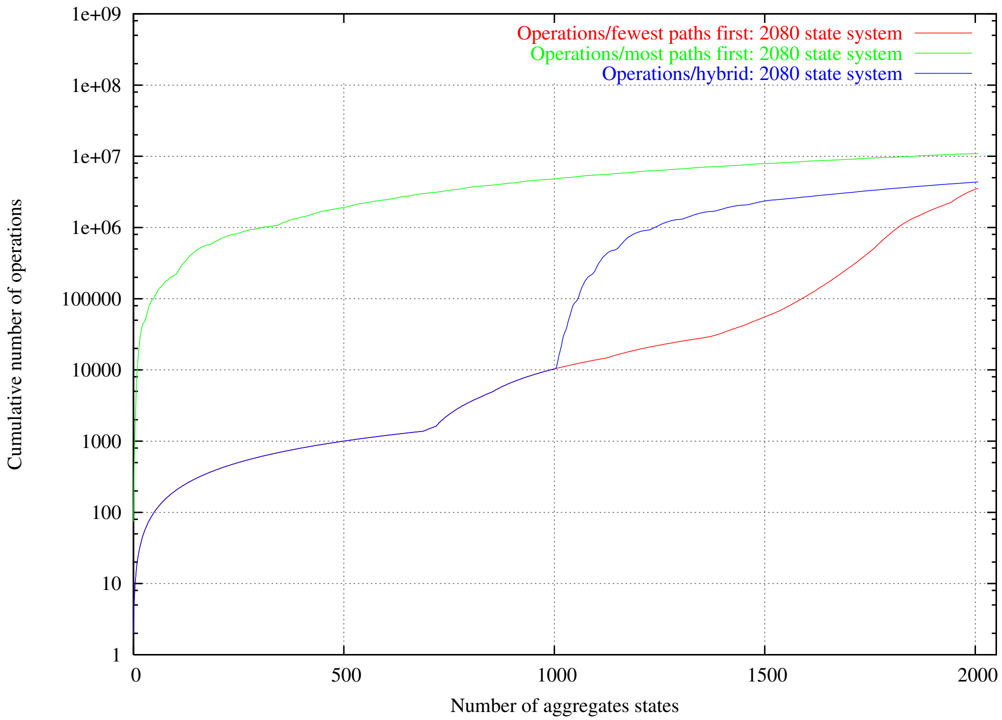


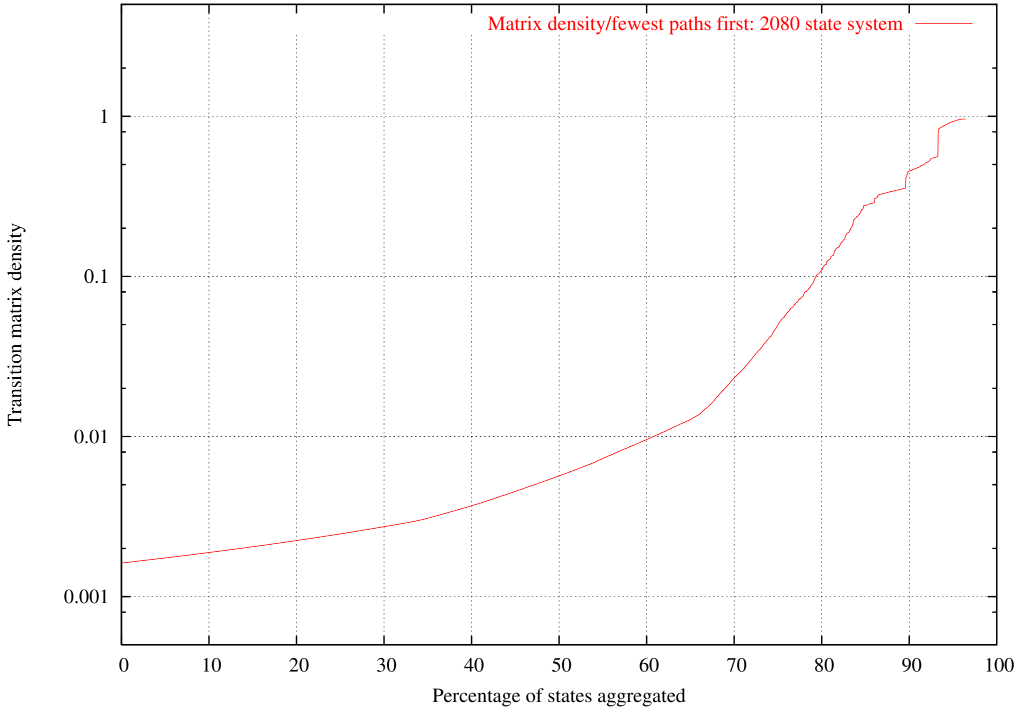
Aggregation Results

- How does matrix density vary with state space size?

- How does computational cost vary with state space size?

Matrix Density

Logscale: Matrix density vs Number of states aggregated



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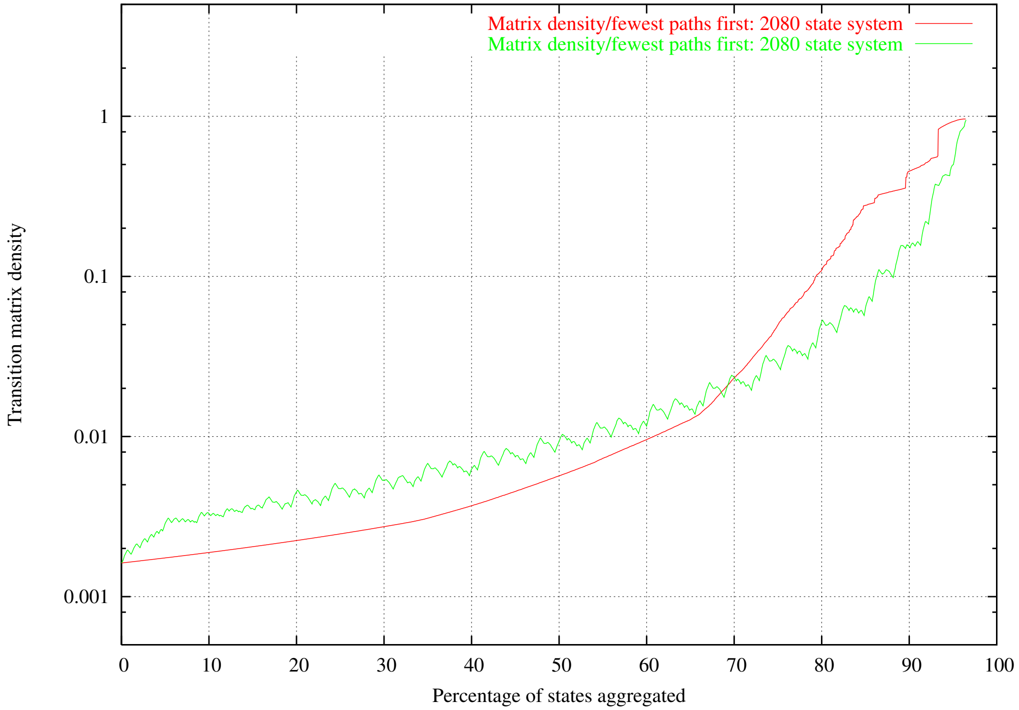

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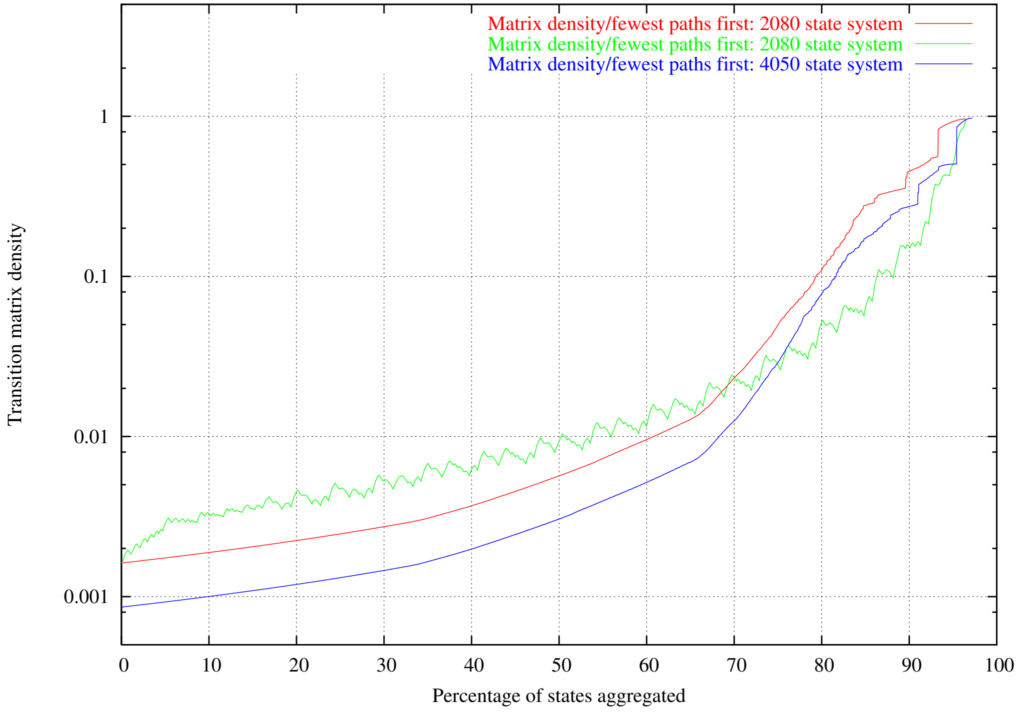

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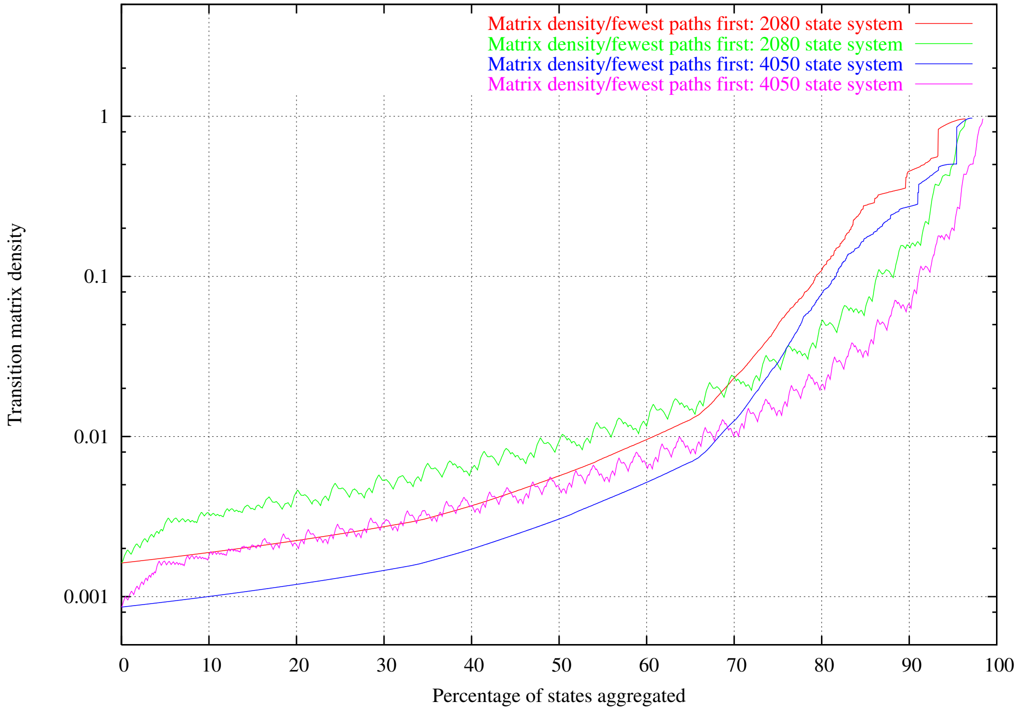
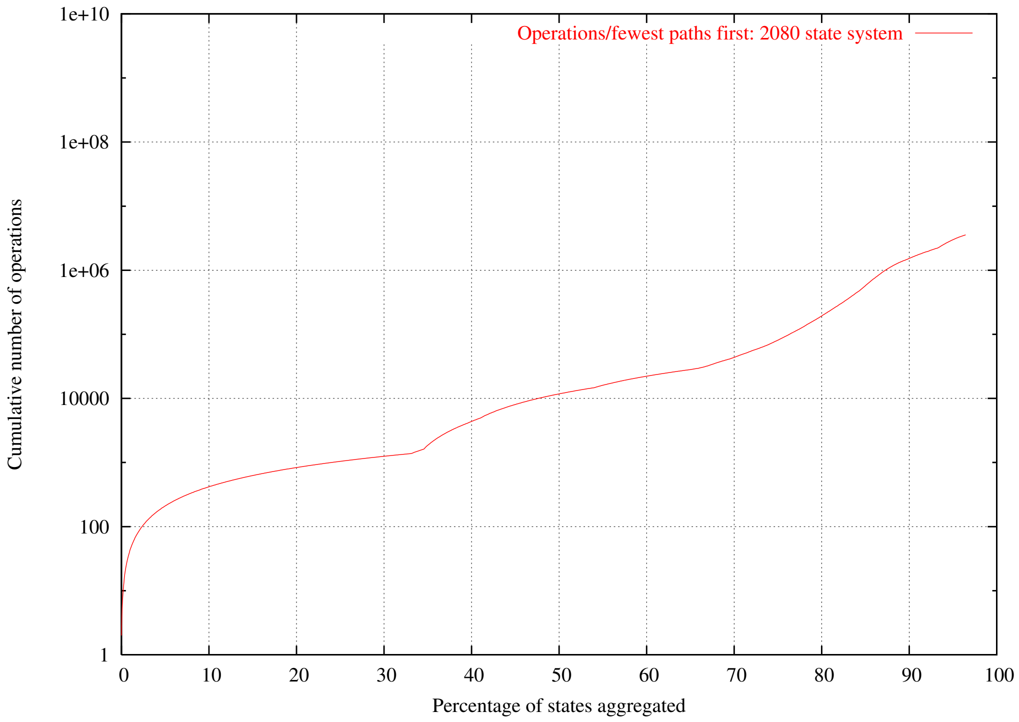


Aggregation operations

Logscale: Aggregation operations vs Number of states aggregated



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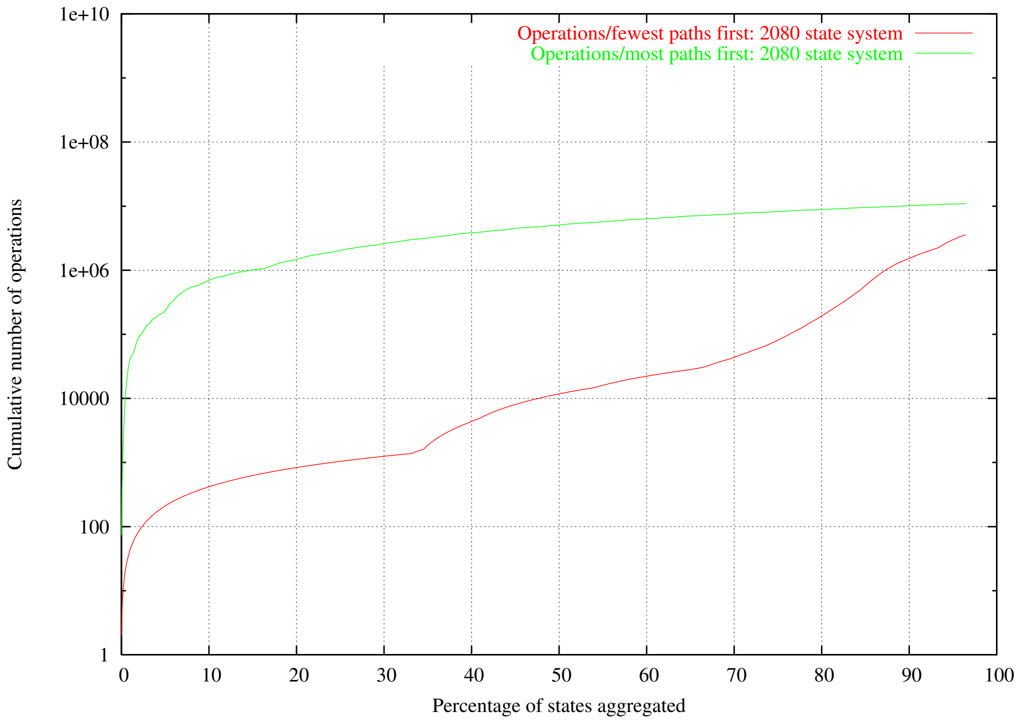

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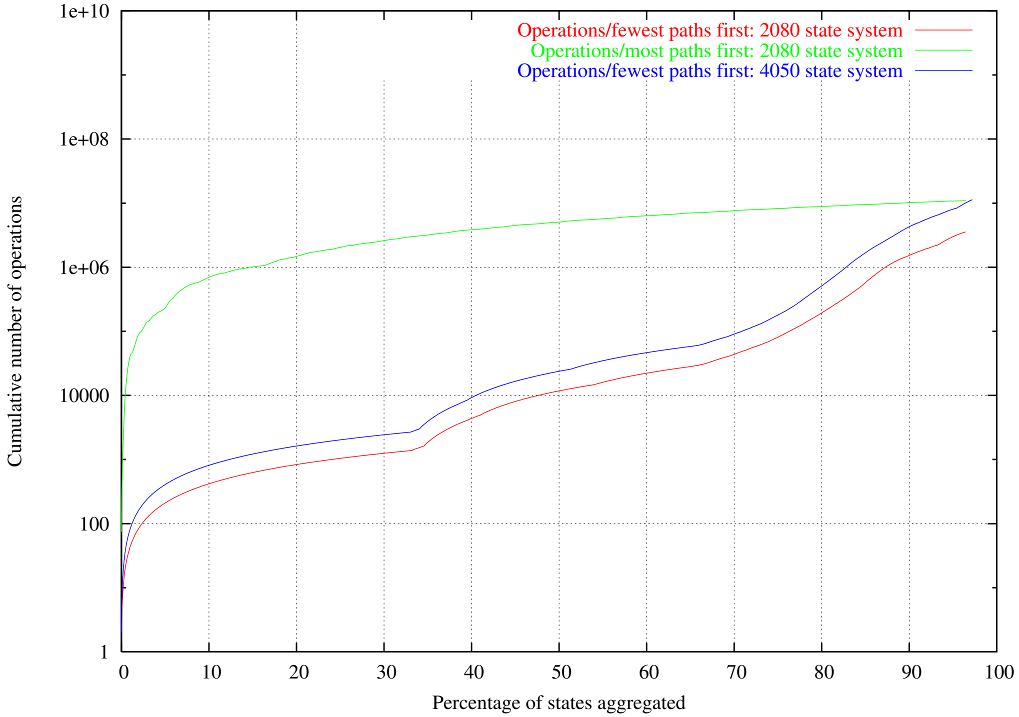

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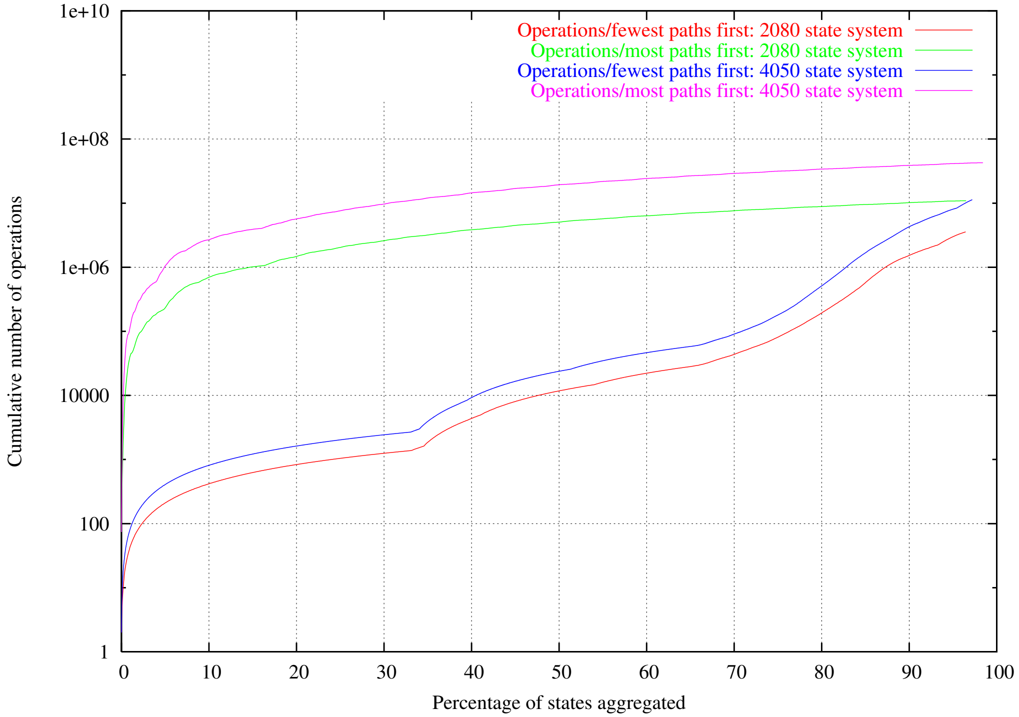
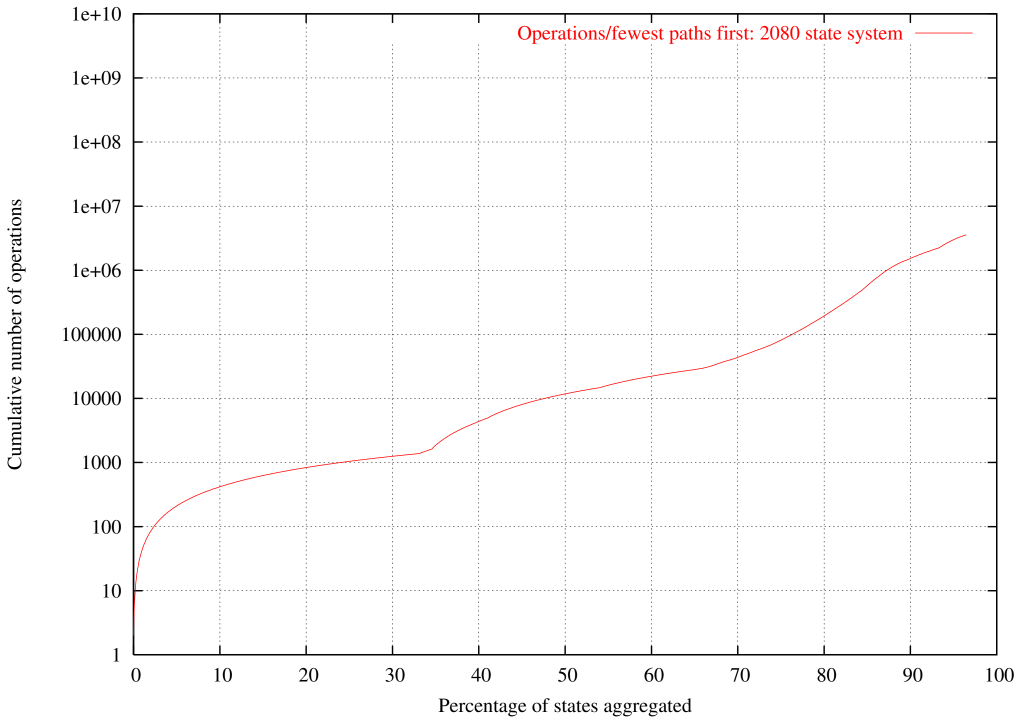


Aggregation Results

- Can we aggregate large systems with fewest-paths strategy?
- **YES**... but not all the way.

Aggregation operations

Logscale: Aggregation operations vs Number of states aggregated



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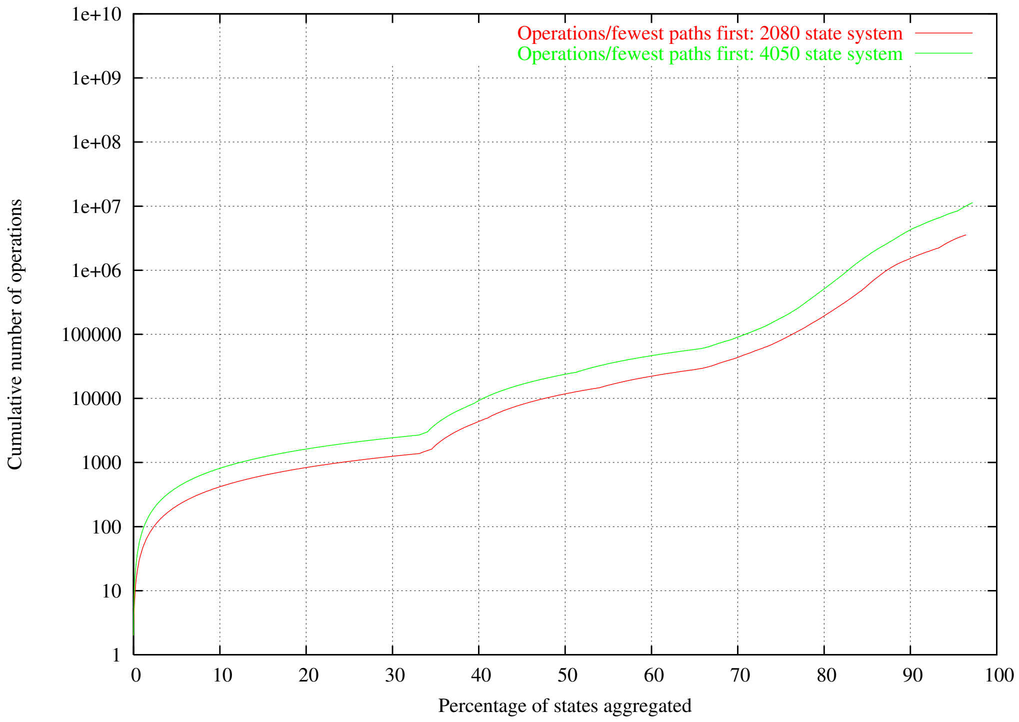

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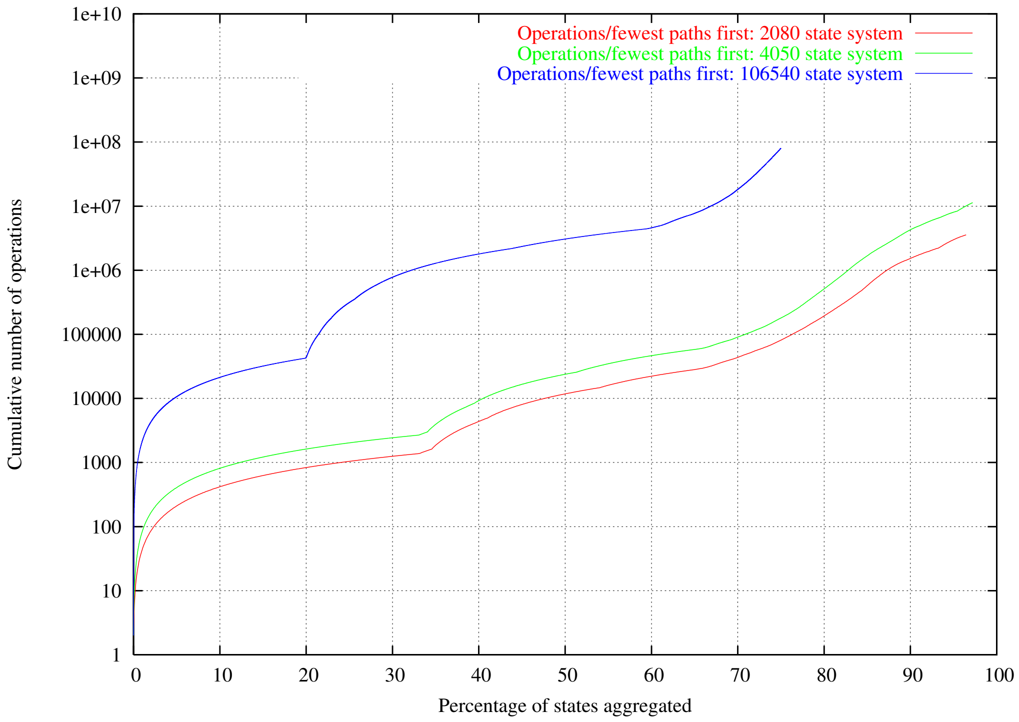

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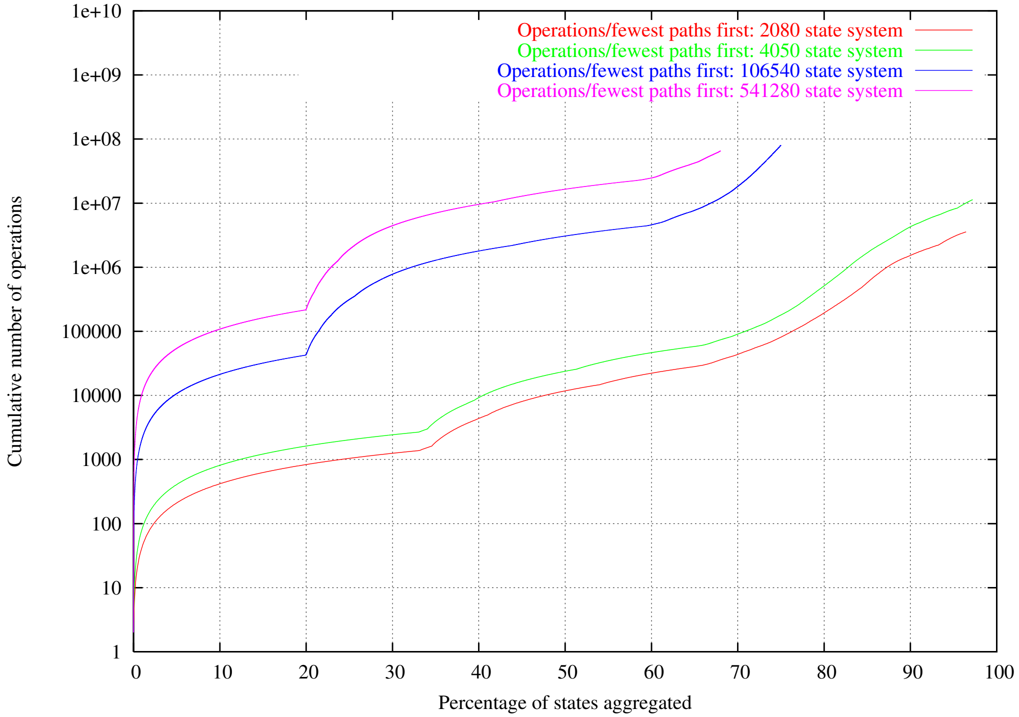
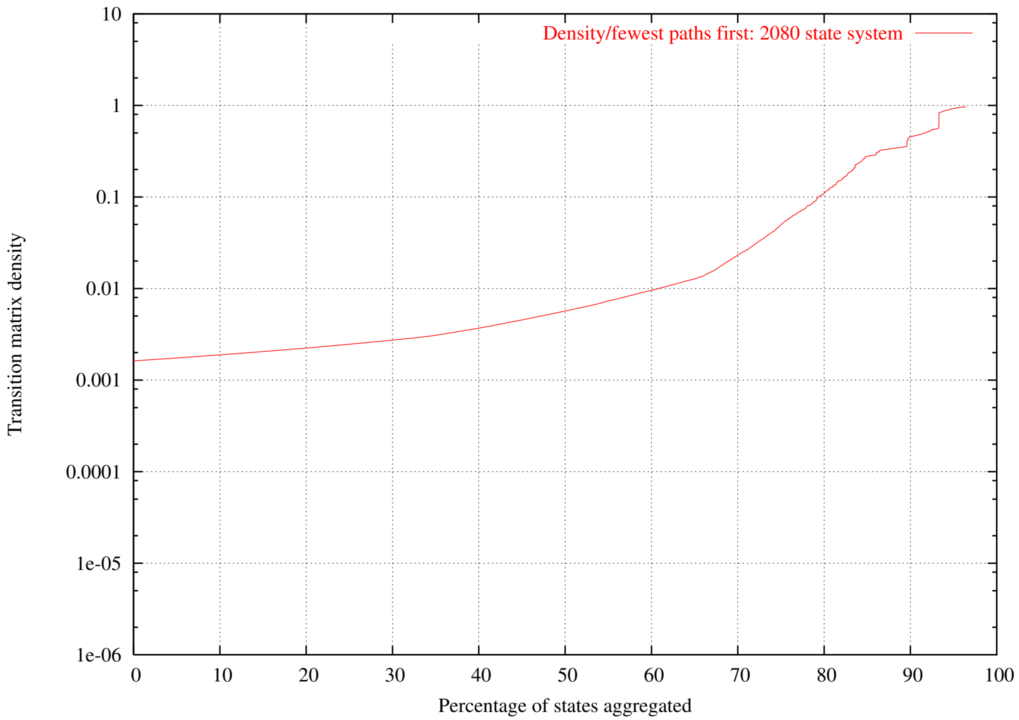


Matrix Density

Logscale: Matrix density vs Number of states aggregated



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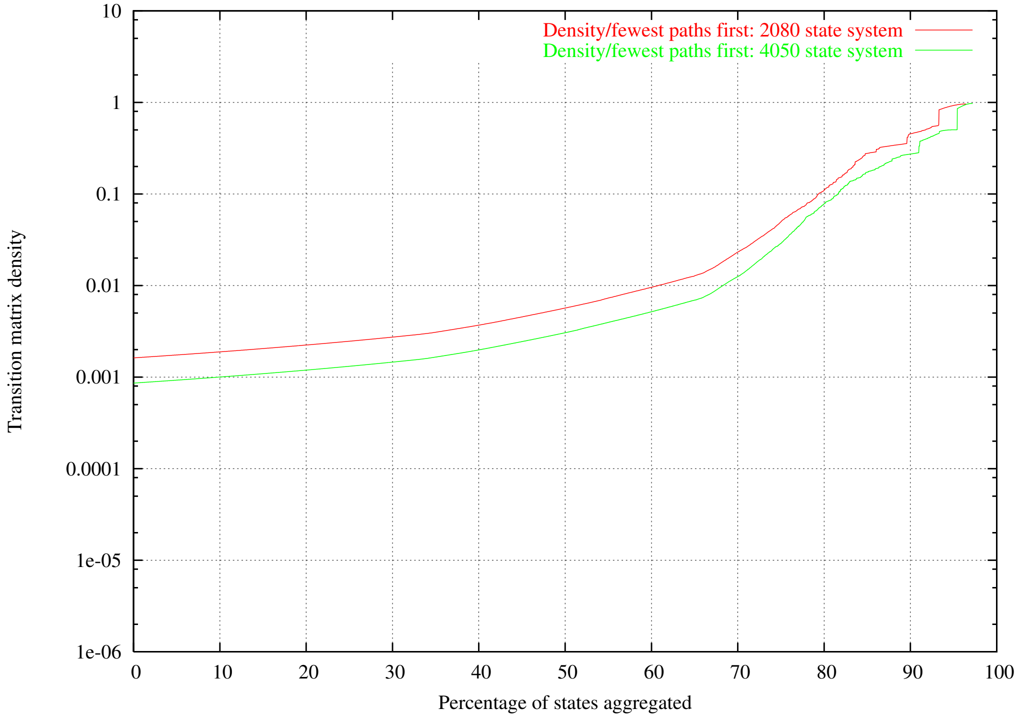

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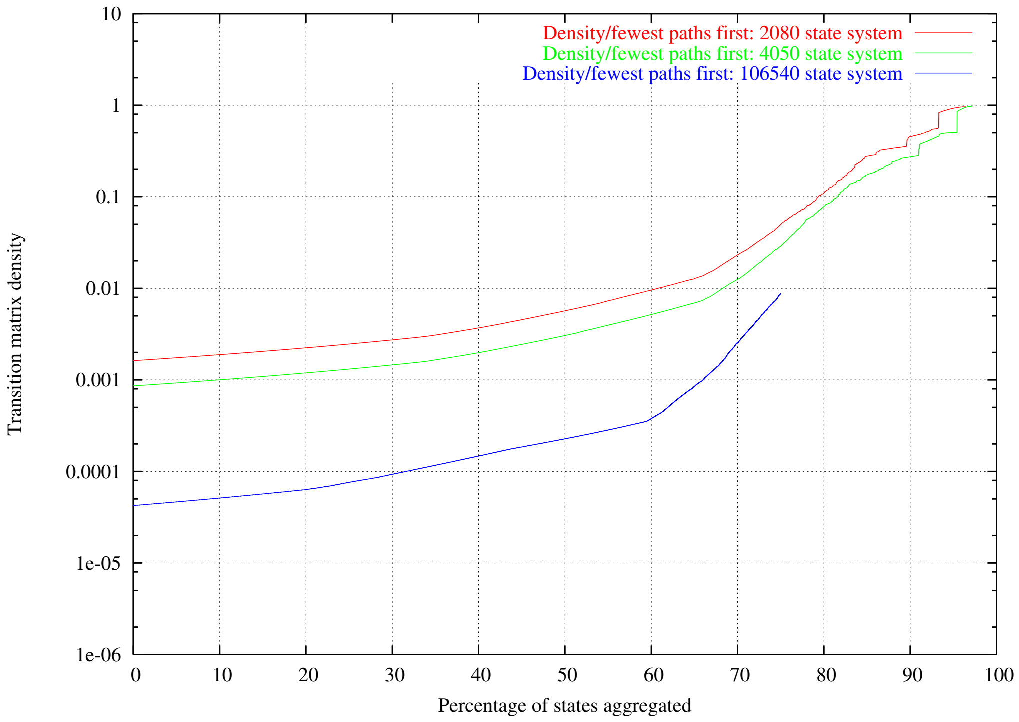

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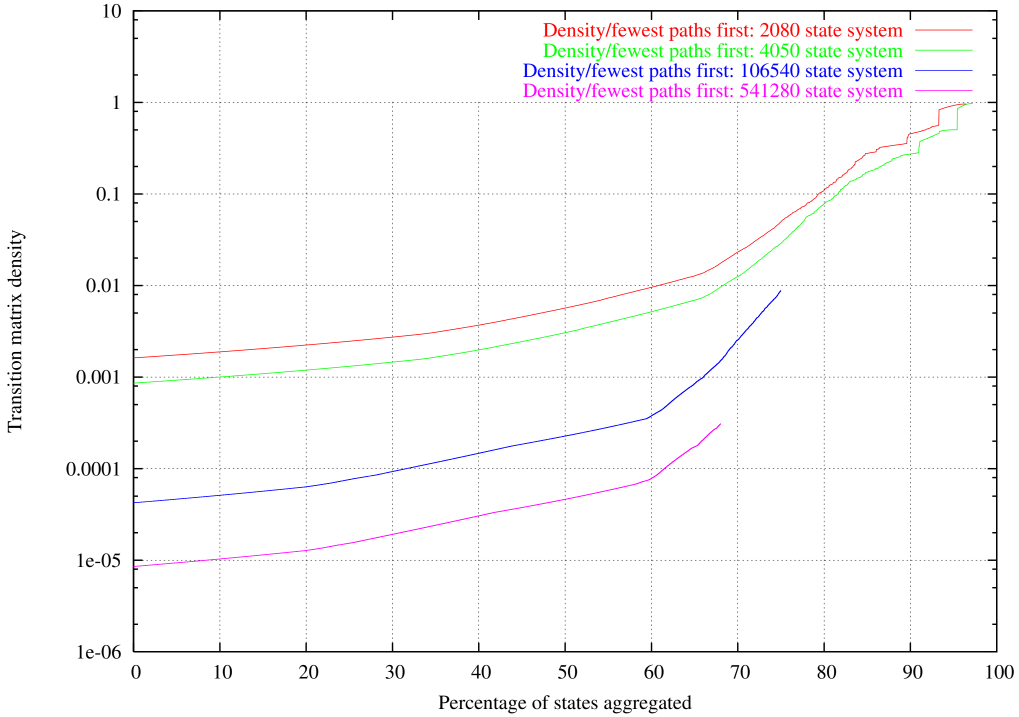

Conclusion

- SMP statewise aggregation is practical but...
  - only for 60-80% of the state space
  - have to be careful which order to aggregate the states

- Next step: divide-and-conquer aggregation