Hypergraph-based

Parallel Computation of Passage Times in Large Semi-Markov Models

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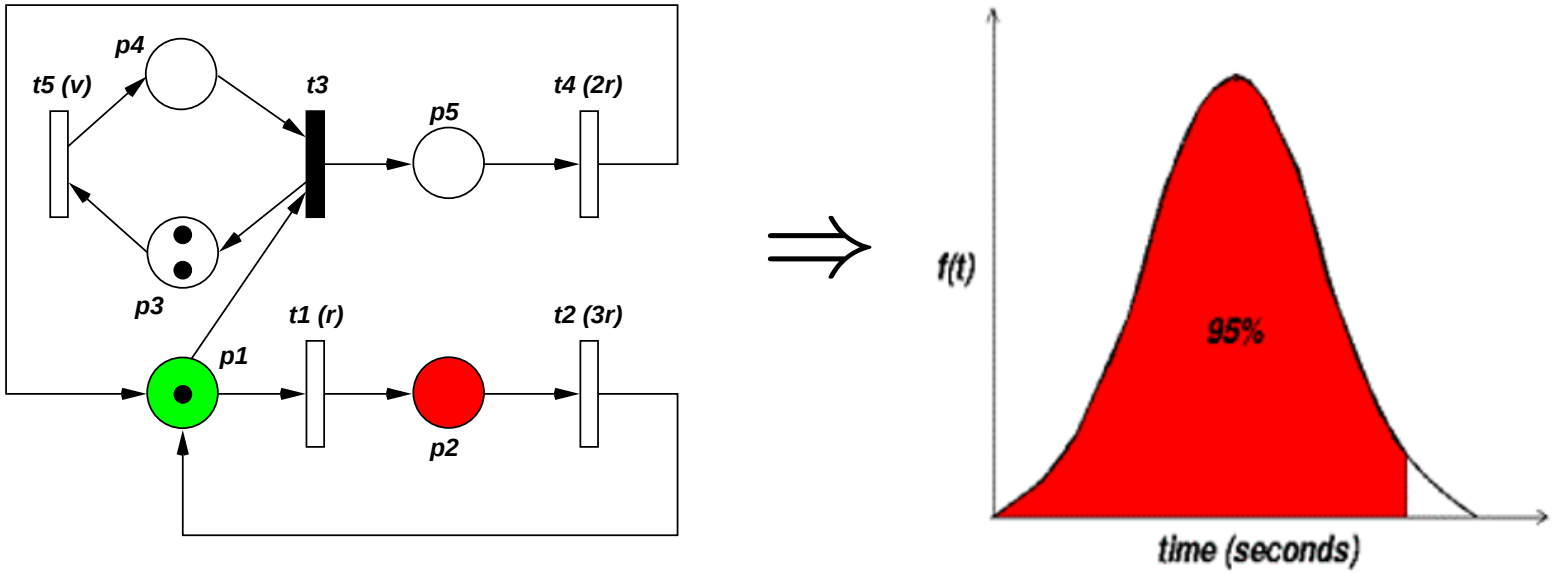
Motivation

- Complex concurrent systems are often modelled as Markov and semi-Markov chains
- Traditional steady-state analysis can compute performance measures such as:
  - system availability/throughput
- Require techniques to compute response-time measures for:
  - QoS metrics in benchmarks/Service Level Agreements

Aims



Semi-Markov Processes

- A more expressive generalisation of Markov processes
- An N-state SMP is characterised by:
  - an $N \times N$ one-step probability matrix P
  - an $N \times N$ matrix H where $h_{ij}(t)$ is the distribution function of the sojourn time of the process in state i, given that its going to state j
- We define the kernel $R(i, j, t)$ of an SMP as:

$$R(i, j, t) = p_{ij} H_{ij}(t)$$

First Passage Times

- First passage time from state i to target states $\vec{j}$ in an SMP:

$$P_{i\vec{j}} = \inf\{u > 0 : Z(u) \in \vec{j}, N(u) > 0, Z(0) = i\}$$

  - $Z(t)$ is the state of the SMP at time t
  - $N(u)$ is the number of state transitions by time u

- Laplace transform of $P_{i\vec{j}}$ density can be computed as:

$$L_{i\vec{j}}(s) = \sum_{k \notin \vec{j}} r^*_{ik}(s) L_{k\vec{j}}(s) + \sum_{k \in \vec{j}} r^*_{ik}(s)$$

  where $r^*_{ik}(s)$ is the LST of $R(i, k, t)$

Iterative Algorithm I

- Generates successively better approximations to the passage time quantity $P_{i\vec{j}}$
- $P^{(r)}_{i\vec{j}}$ is the conditional passage time for the system to reach any state in $\vec{j}$ in r transitions:

$$P^{(r)}_{i\vec{j}} = \inf\{u > 0 : Z(u) \in \vec{j}, 0 < N(u) \leq r, Z(0) = i\}$$
$$P^{(r)}_{i\vec{j}} \to P_{i\vec{j}} \text{ as } r \to \infty$$

- The second property ensures convergence of the algorithm

Iterative Algorithm II

- Calculate Laplace transform of rth iteration, $P_{i\vec{j}}^{(r)}$:

$$L_{i\vec{j}}^{(r)}(s) = \sum_{m=1}^{r} \text{contribution from } m\text{th transition matrix}$$

$$= \sum_{m=1}^{r} UU'^{m-1}\ \tilde{e}_{\vec{j}}$$

where matrix U has elements $u_{pq} = r_{pq}^{*}(s)$ and U' has states in $\vec{j}$ made absorbing

Iterative Algorithm III

- $L^{(r)}_{\vec{i}\vec{j}}(s)$ can be generalised to multiple source states, $\vec{i}$, and calculated in one go:

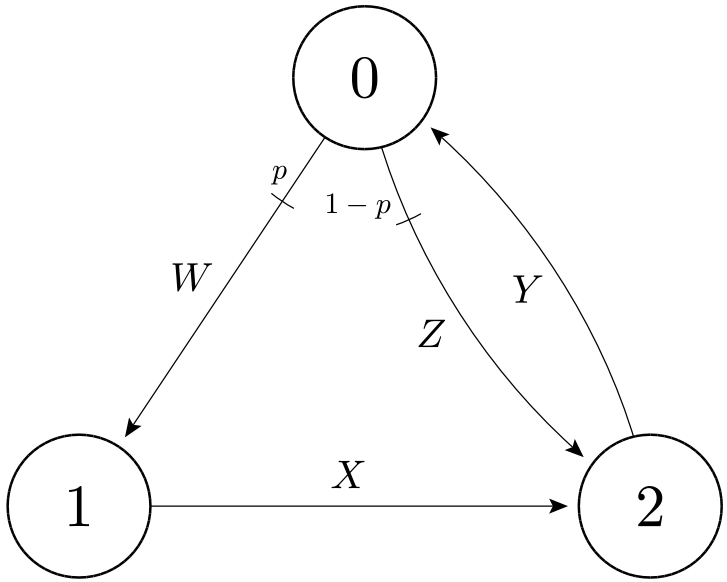
$$L^{(r)}_{\vec{i}\vec{j}}(s) = \sum_{k=0}^{r-1} \tilde{\alpha} U U'^{k} \; \tilde{e}_{\vec{j}}$$

  where $\tilde{\alpha}$ is a vector of weights for the source states in $\vec{i}$, e.g.:

$$\alpha_k = \begin{cases} \pi_k / \sum_{j \in \vec{i}} \pi_j & \text{if } k \in \vec{i} \\ 0 & \text{otherwise} \end{cases}$$

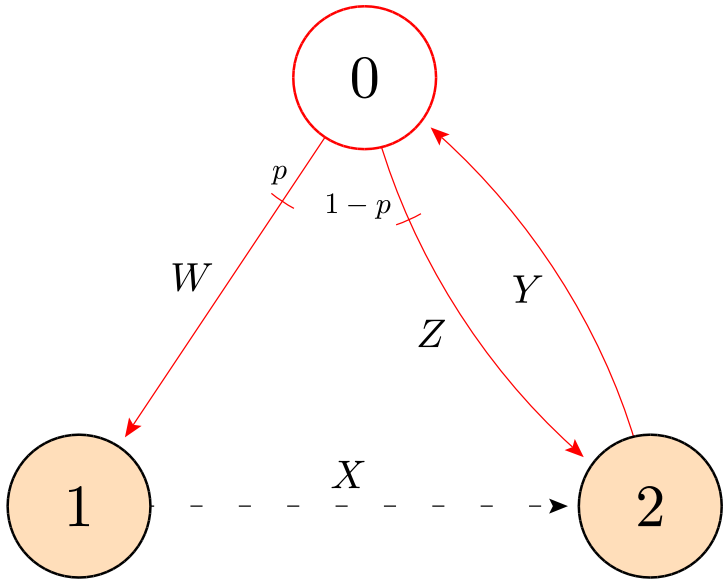
- Compute this efficiently using sparse matrix-vector multiplications and a vector accumulator

Passage Time Example



Distributions: $W \sim \Gamma(\alpha, \lambda)$, $Z \sim \text{Hyper}\left(\frac{2}{3}, \frac{1}{3}; \lambda, \mu\right)$, $X \sim \text{U}\left[\frac{1}{\lambda}, \frac{1}{\mu}\right]$, $Y \sim \delta\left[\frac{1}{\mu}\right]$

Passage Time: state 2 to 1



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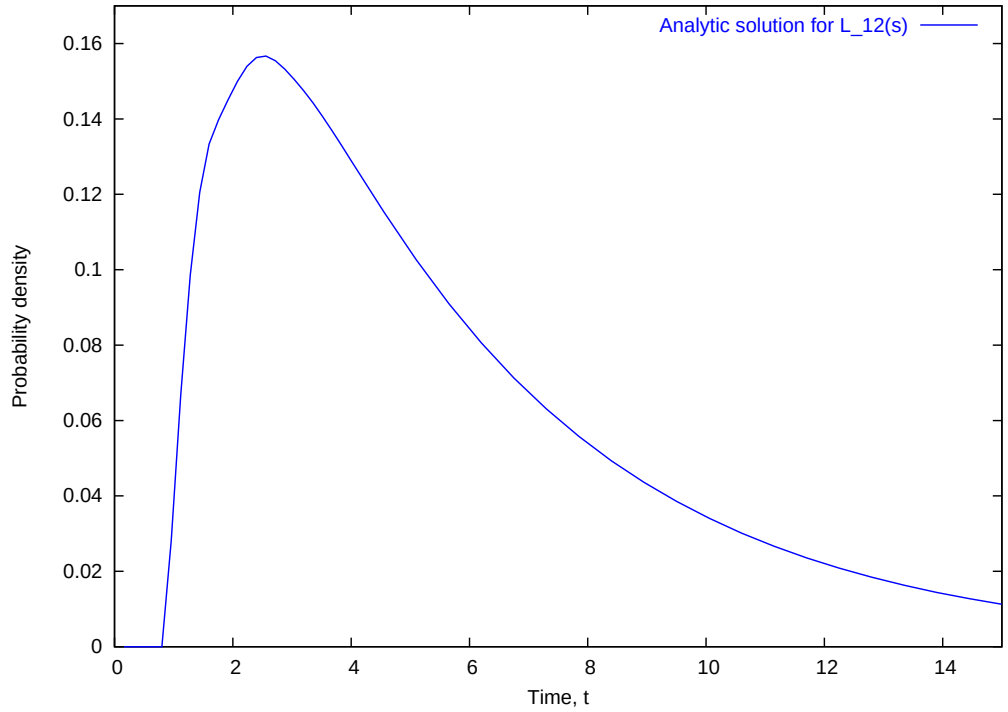
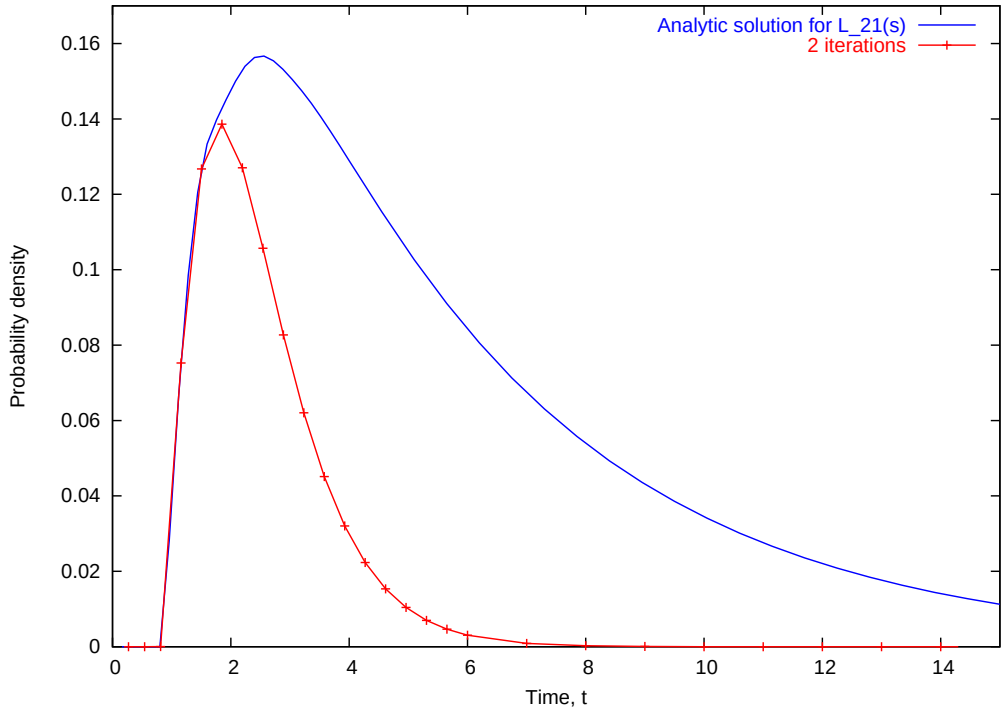
Passage Time: state 2 to 1

Analytic solution



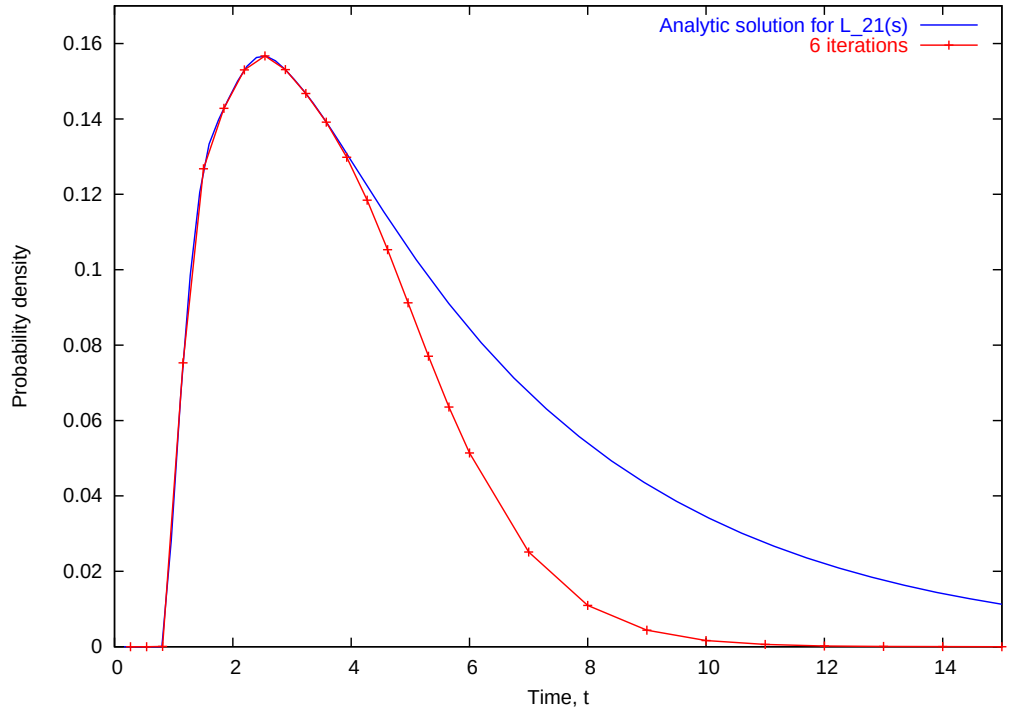
Passage Time: state 2 to 1

Iterations of algorithm: 2



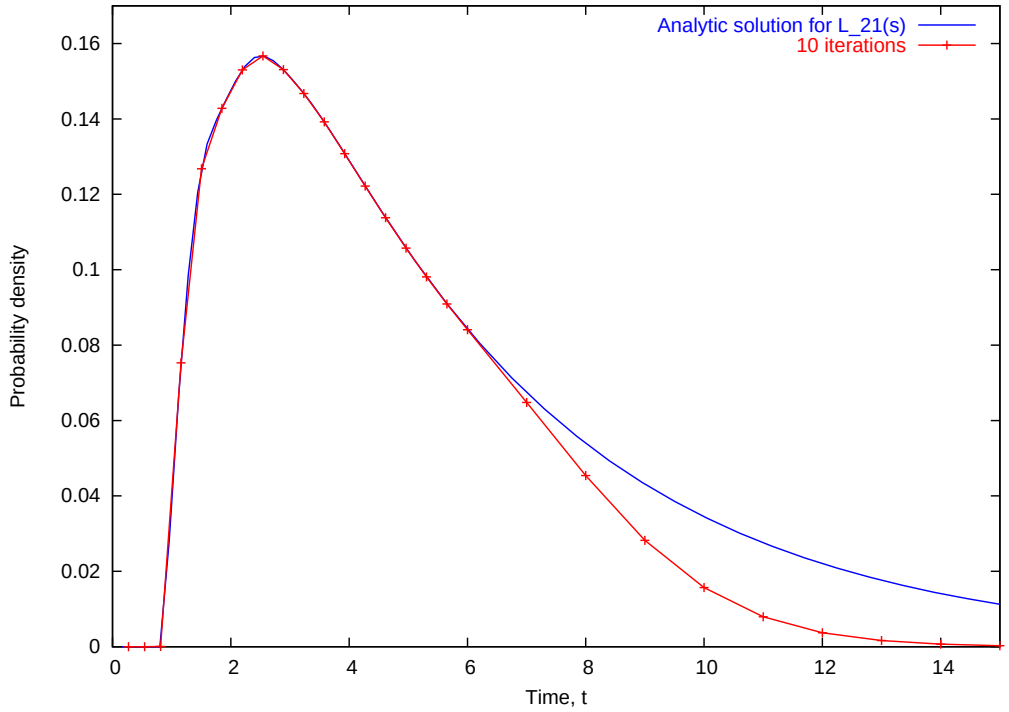
Passage Time: state 2 to 1

Iterations of algorithm: 6



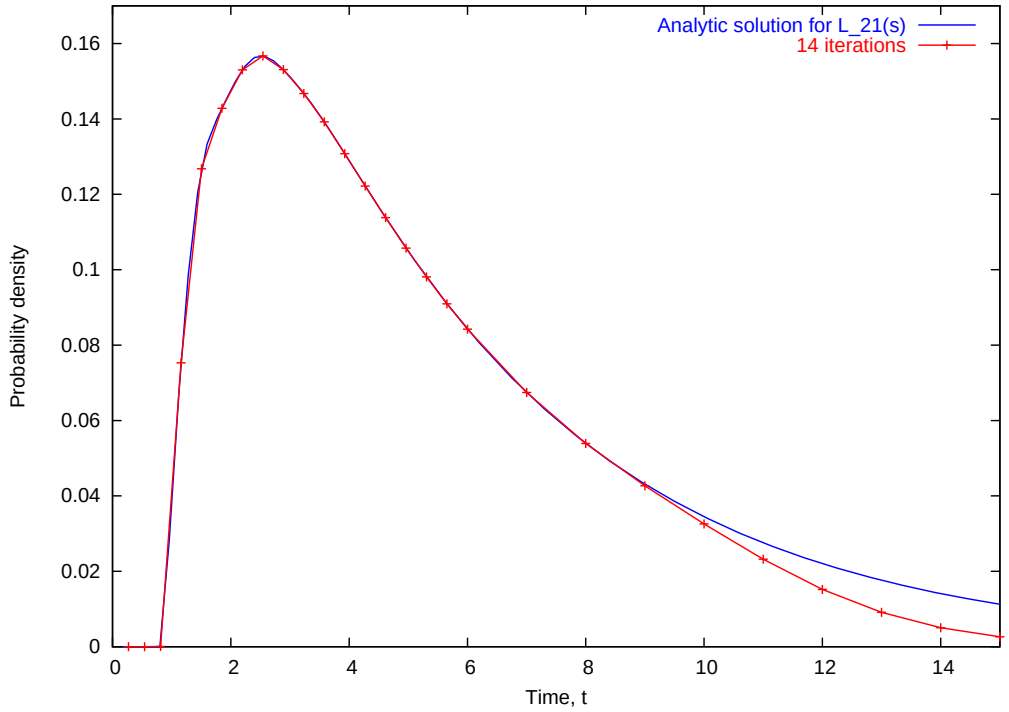
Passage Time: state 2 to 1

Iterations of algorithm: 10



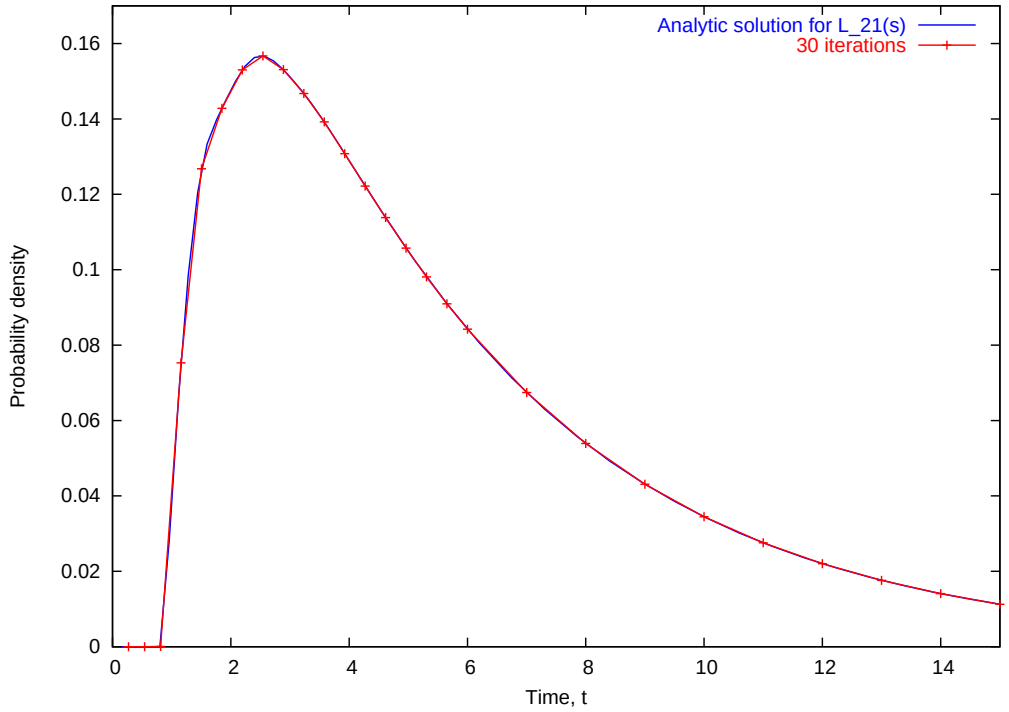
Passage Time: state 2 to 1

Iterations of algorithm: 14



Passage Time: state 2 to 1

Iterations of algorithm: 30



Truncation Error I

- The error for truncating the $L^{(r)}_{\overrightarrow{ij}}(s)$ sum at the rth term is

$$\sum_{i=r+1}^{\infty} \tilde{\alpha} U U'^{i} \simeq \frac{\lambda_1}{1-\lambda_1} \tilde{\alpha} U U'^{r}$$

  where λ_1 is dominant eigenvalue of U'

- Theoretical convergence condition is to find the minimum r such that:

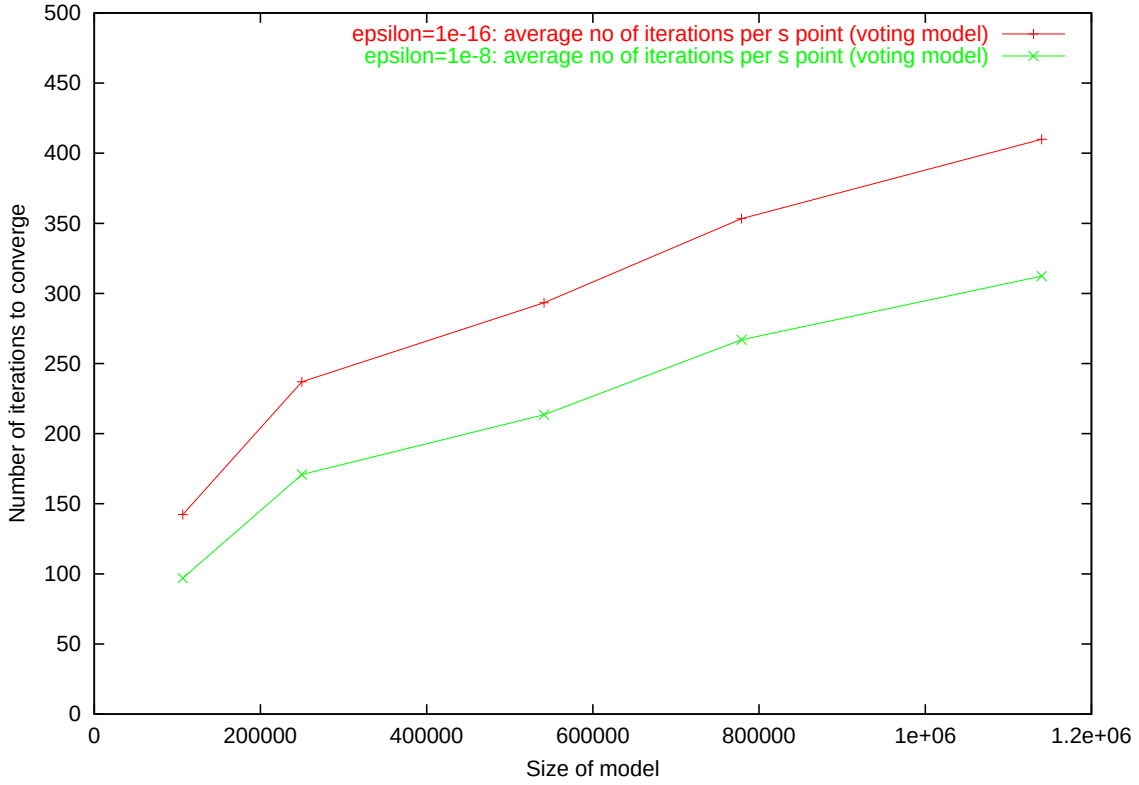
$$\left| \frac{\lambda_1}{1-\lambda_1} \tilde{\alpha} U U'^{r} \right| < \epsilon$$

Truncation Error II

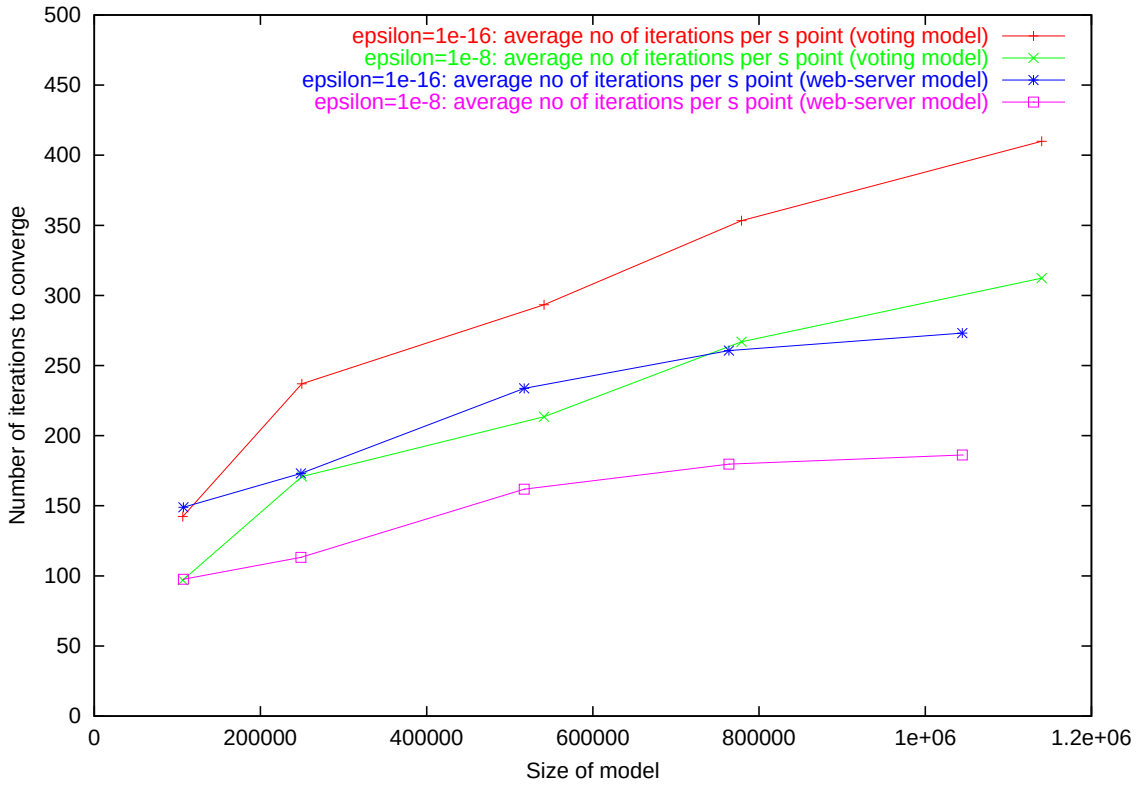
- Practical convergence test (if λ_1 is hard to find) is to compare the difference between successive iterations:

  $$|\mathrm{Re}(L_{\vec{i}\vec{j}}^{(r+1)}(s) - L_{\vec{i}\vec{j}}^{(r)}(s)| < \epsilon \quad \text{and} \quad |\mathrm{Im}(L_{\vec{i}\vec{j}}^{(r+1)}(s) - L_{\vec{i}\vec{j}}^{(r)}(s)| < \epsilon$$

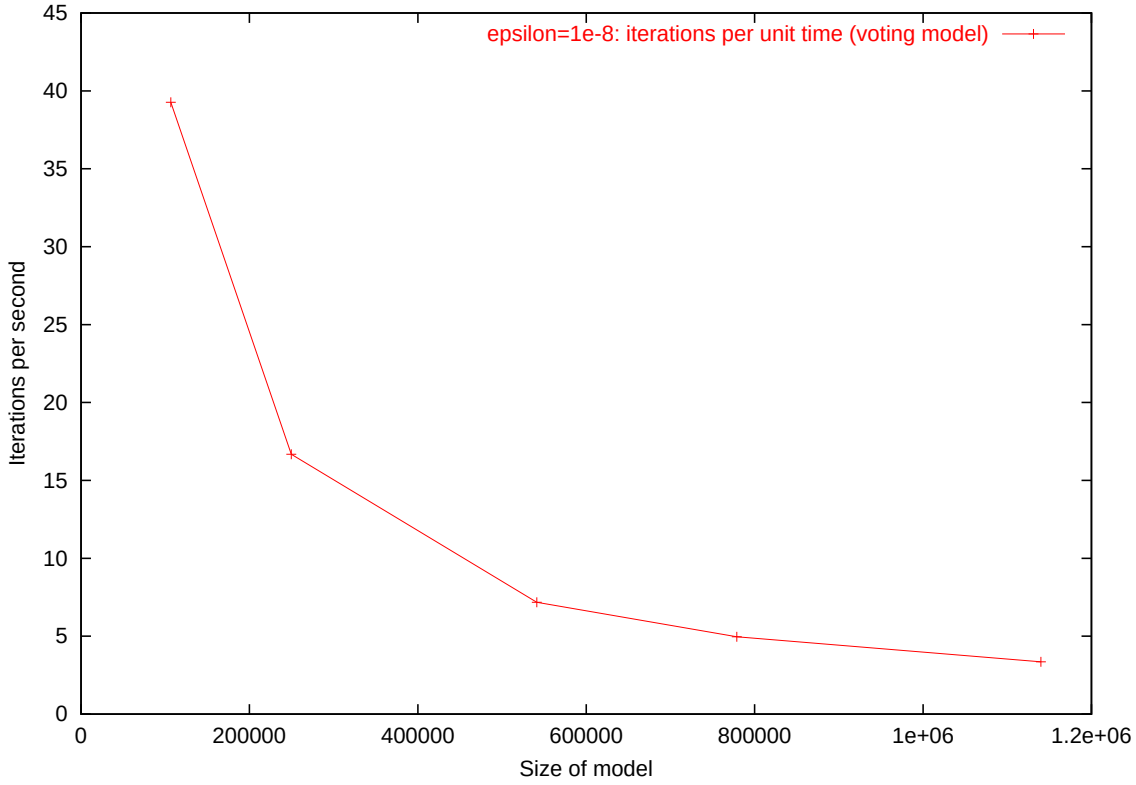
Convergence Results I



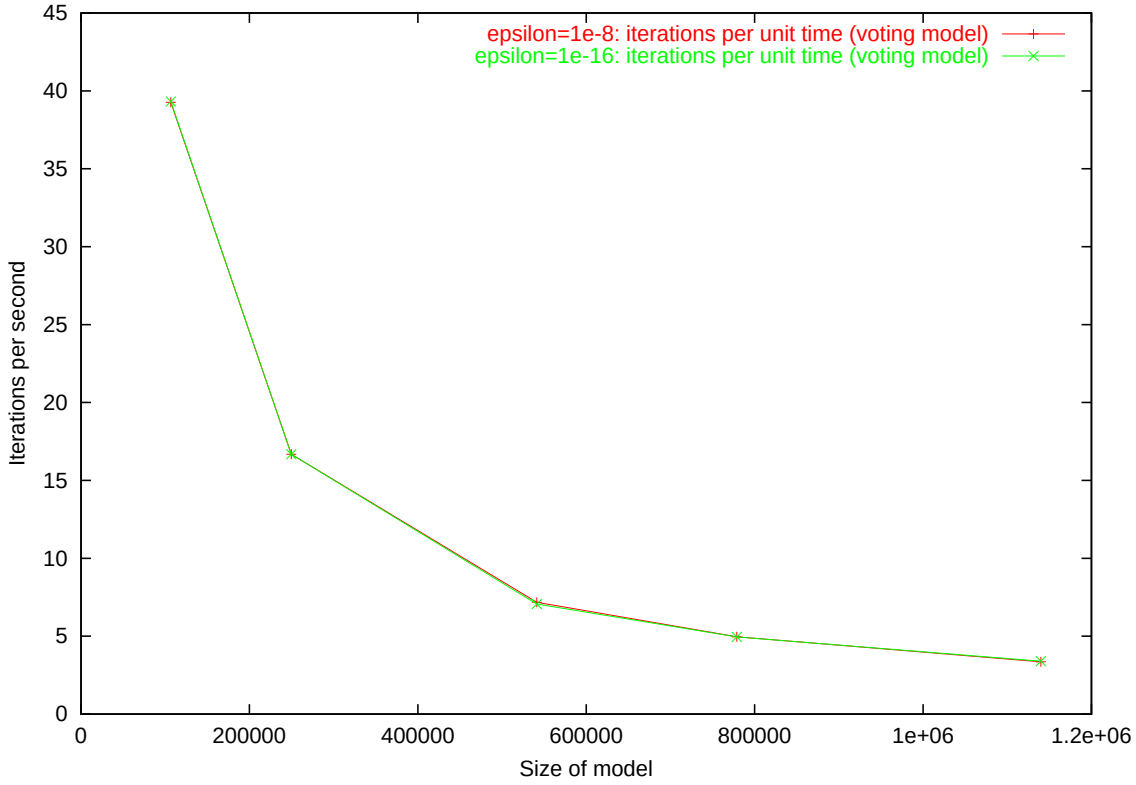
Convergence Results I



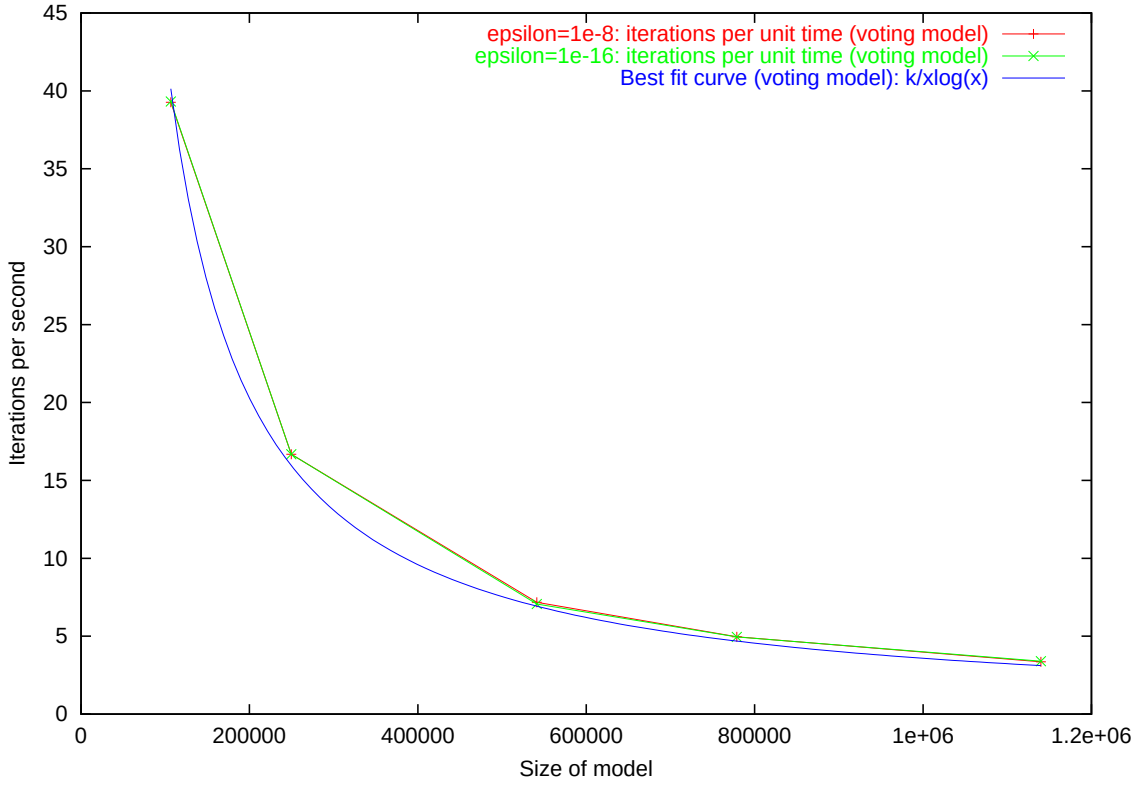
Convergence Results II



Convergence Results II



Convergence Results II



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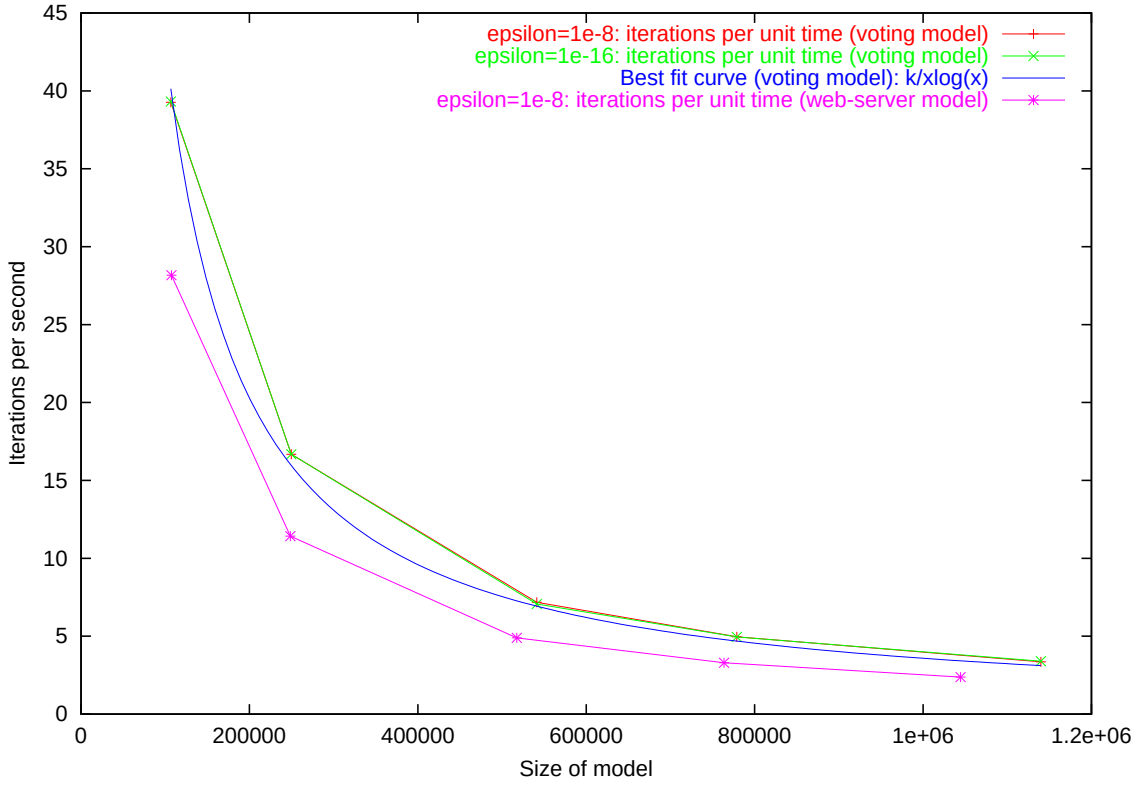
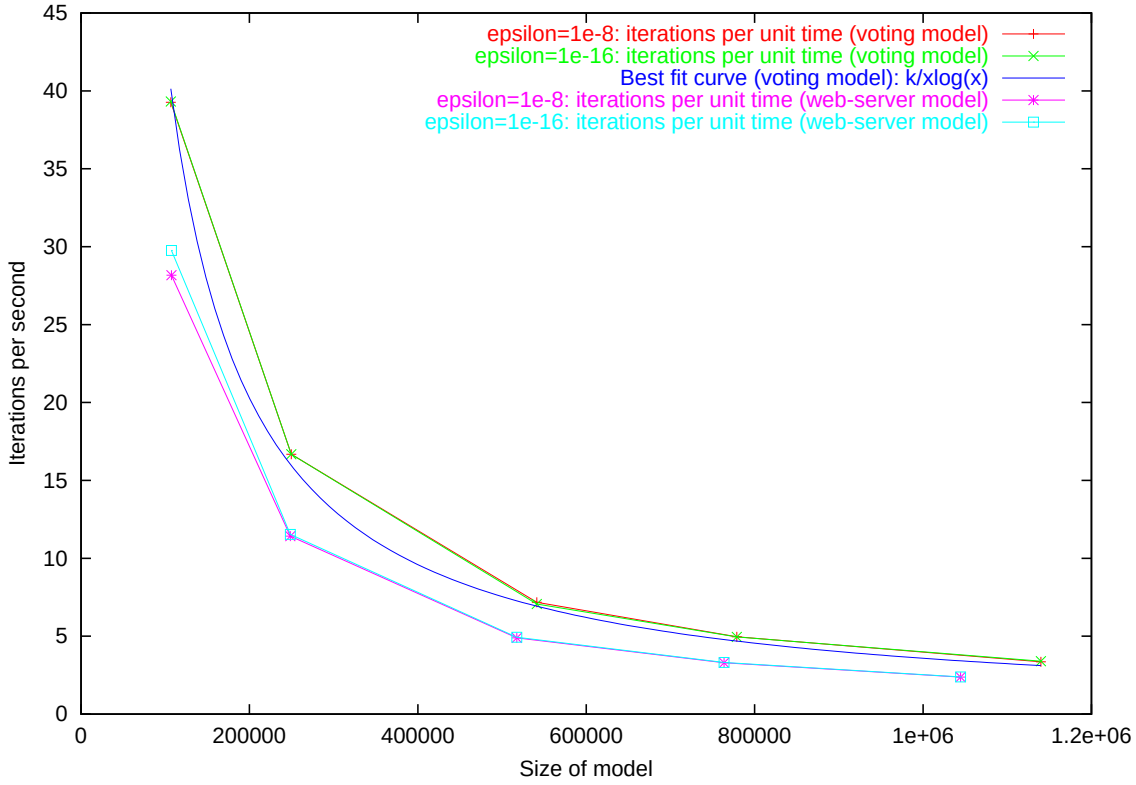


Convergence Results II



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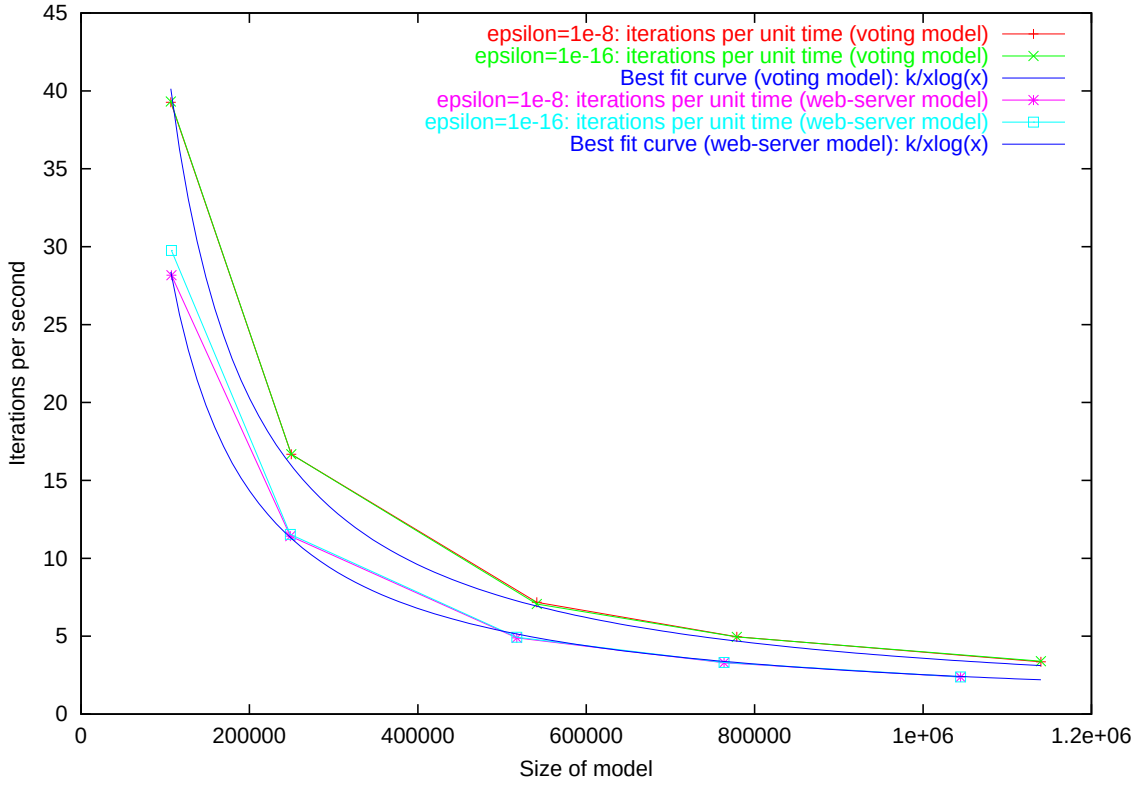
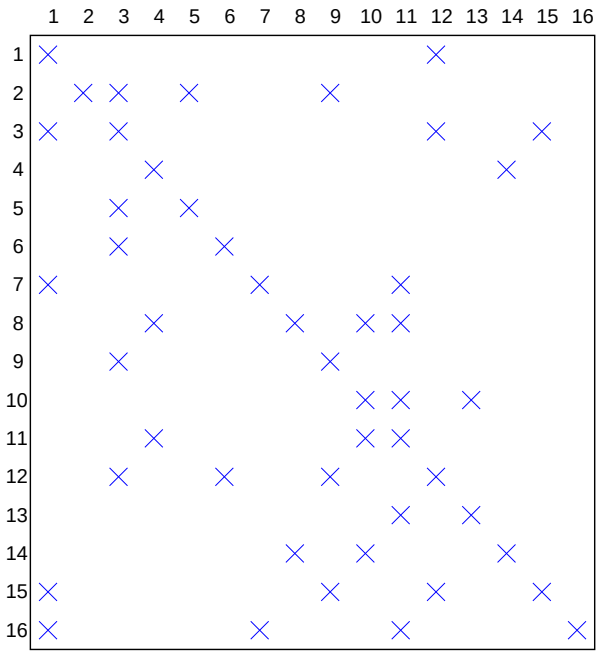
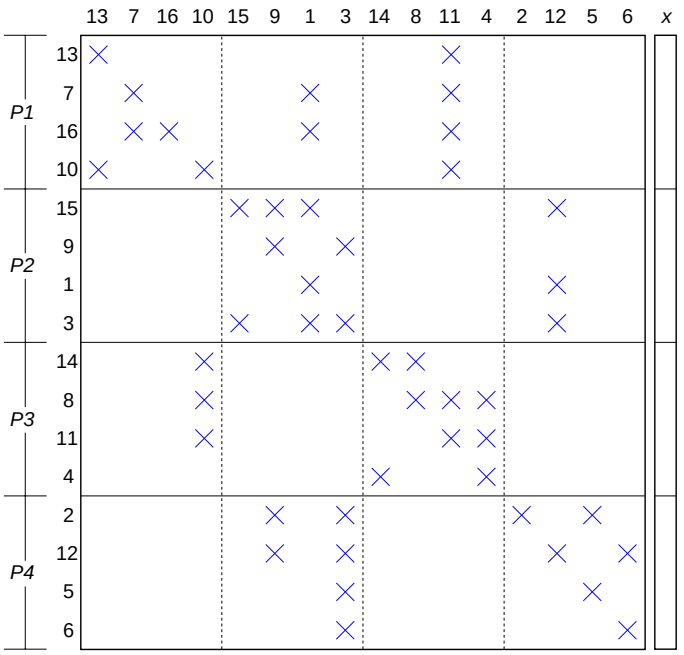


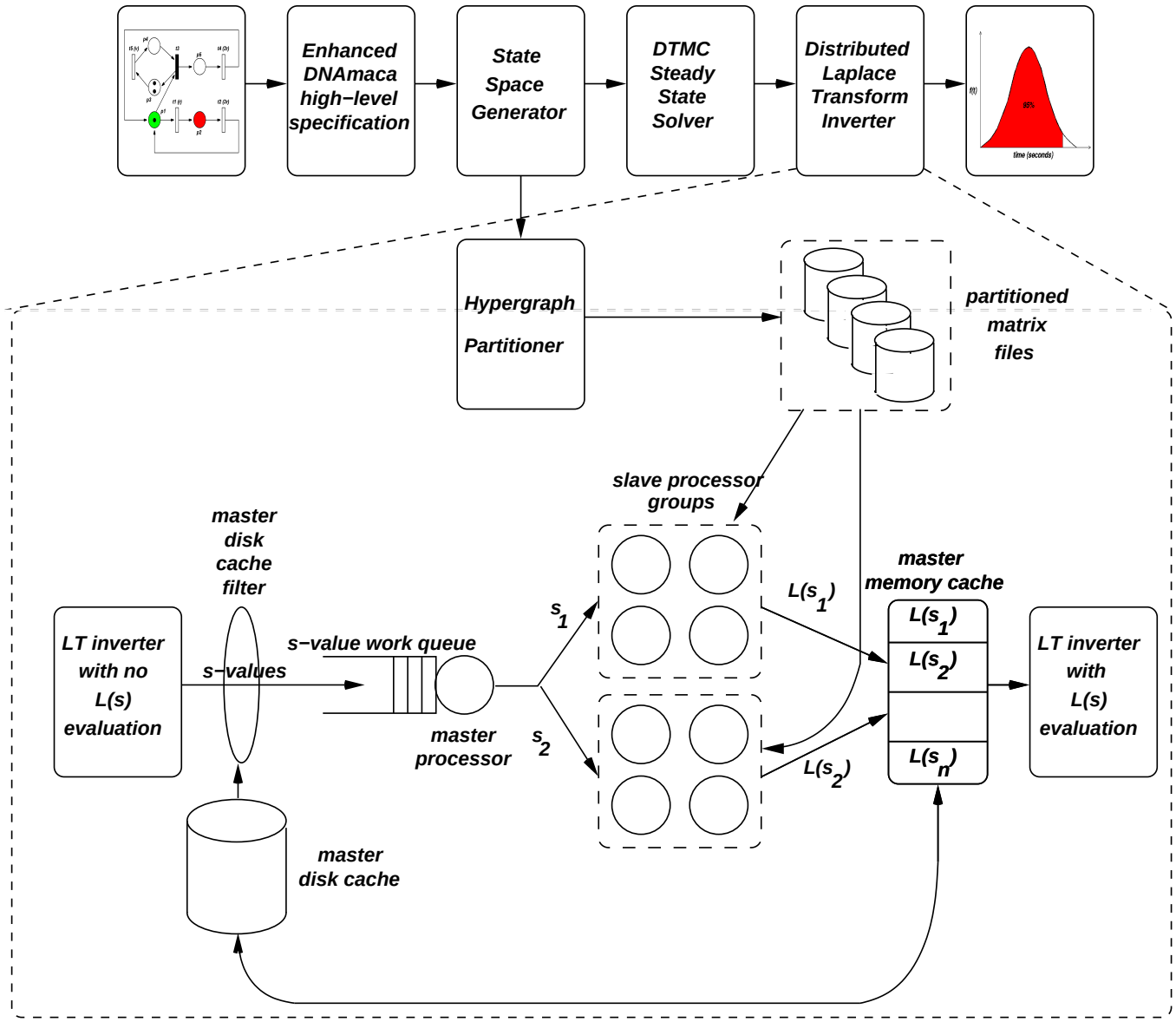
Data Partitioning Techniques

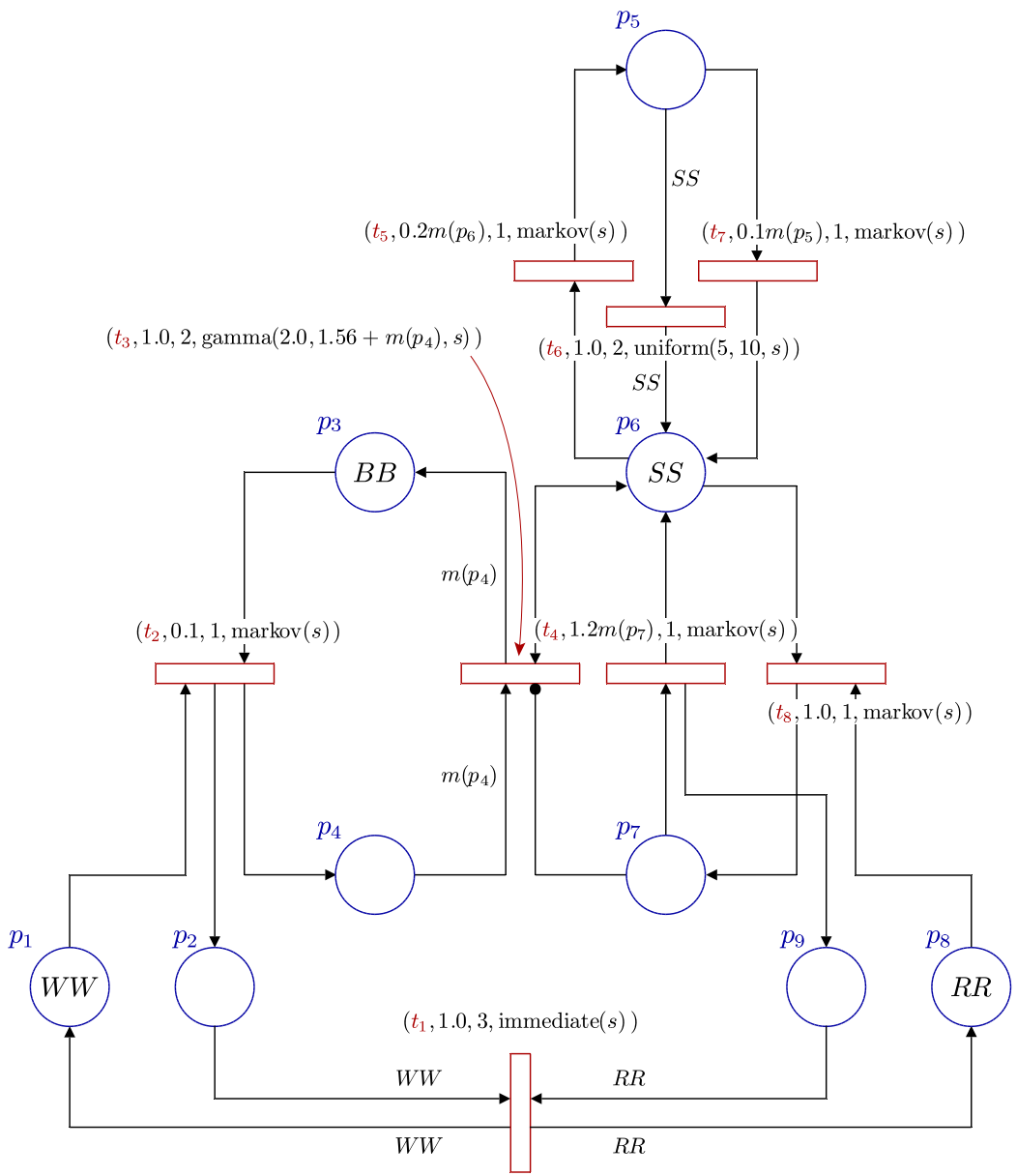
- For large models, we need to take advantage of the combined compute power and memory capacity of a network of workstations
- Main opportunity for parallelisation is in sparse matrix-vector multiplications
- Need to minimise the communication between processors while maintaining load balance

Hypergraph Partitioning

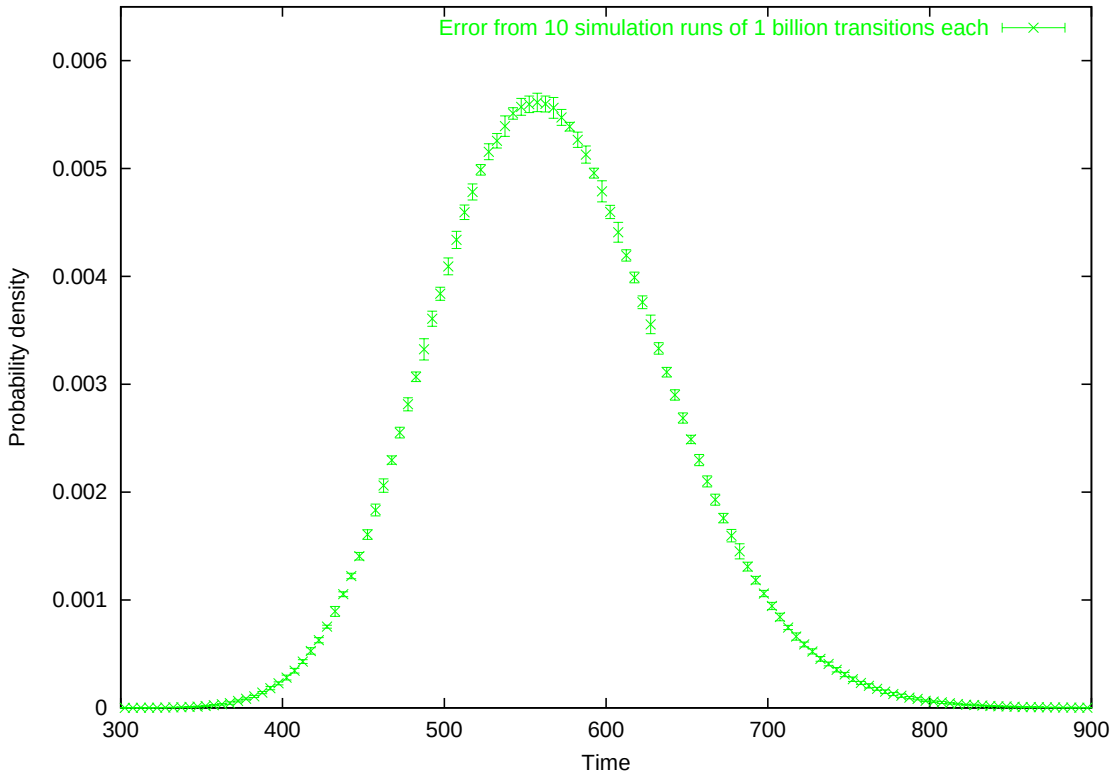





Enhanced DNAmaca high−level specification
State Space Generator
DTMC Steady State Solver
Distributed Laplace Transform Inverter
f(t)
95%
time (seconds)
Hypergraph Partitioner
partitioned matrix files
slave processor groups
master disk cache filter
master memory cache
LT inverter with no L(s) evaluation
s−values
s−value work queue
master processor
s_1
s_2
$L(s_1)$
$L(s_2)$
$L(s_n)$
LT inverter with L(s) evaluation
master disk cache


p_5
SS
$(t_5, 0.2m(p_6), 1, \text{markov}(s))$
$(t_7, 0.1m(p_5), 1, \text{markov}(s))$
$(t_3, 1.0, 2, \text{gamma}(2.0, 1.56 + m(p_4), s))$
$(t_6, 1.0, 2, \text{uniform}(5, 10, s))$
SS
p_3
p_6
BB
SS
$m(p_4)$
$(t_2, 0.1, 1, \text{markov}(s))$
$(t_4, 1.2m(p_7), 1, \text{markov}(s))$
$(t_8, 1.0, 1, \text{markov}(s))$
$m(p_4)$
p_4
p_7
p_1
p_2
p_9
p_8
WW
RR
$(t_1, 1.0, 3, \text{immediate}(s))$
WW
RR
WW
RR

Numerical Results I



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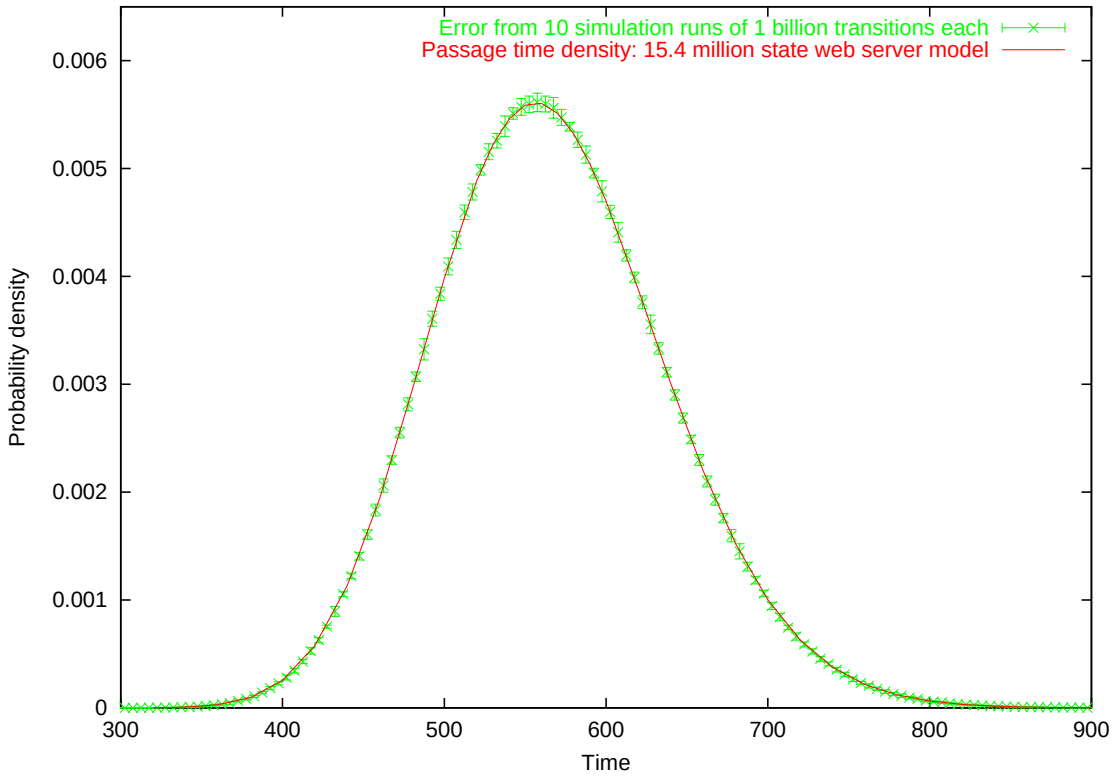
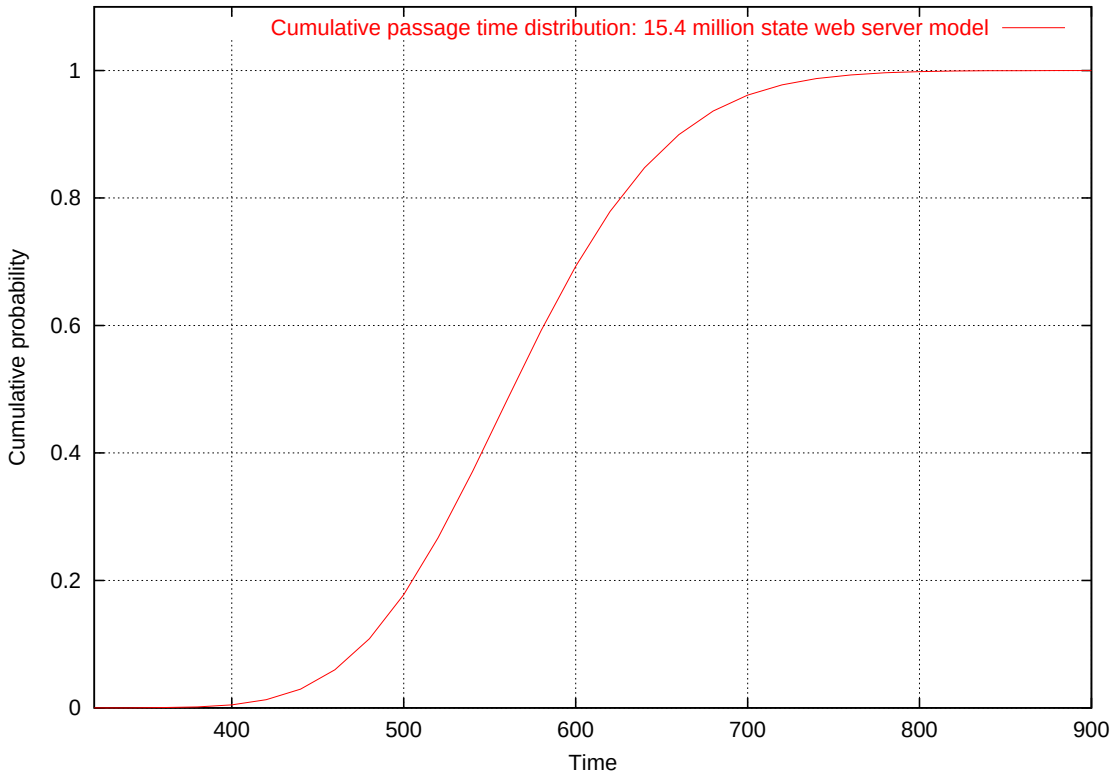
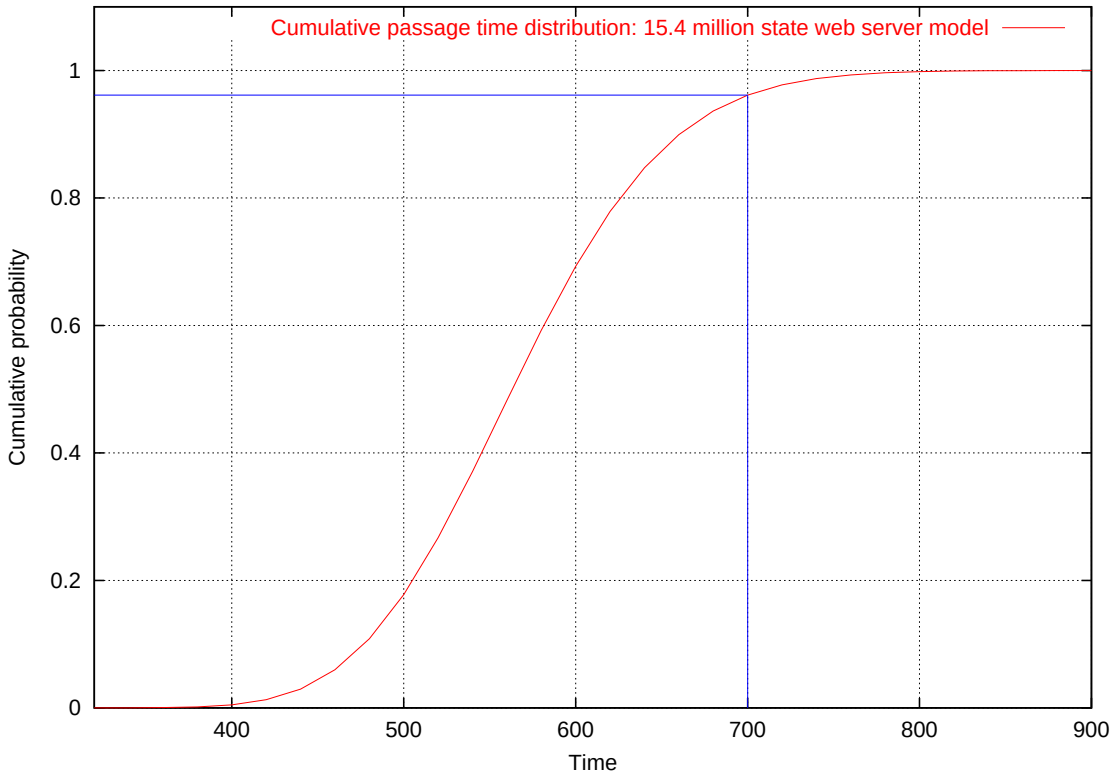


Numerical Results II



Numerical Results II



Scalability Results

p	Time (s)	Speedup	Efficiency
1	3968.07	1.00	1.000
2	2199.98	1.80	0.902
4	1122.97	3.53	0.883
8	594.07	6.68	0.835
16	320.19	12.39	0.775
32	188.14	21.09	0.659

Scalability Results

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1	3968.07	1.00	1.000
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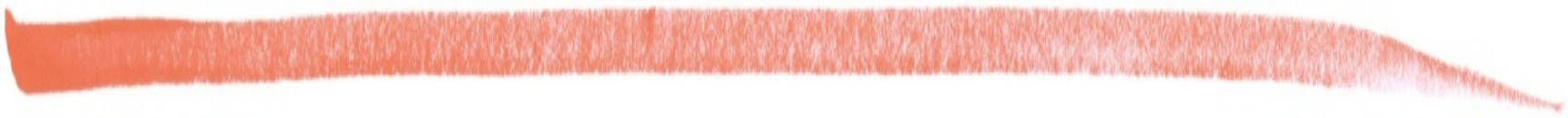
4	1122.97	3.53	0.883
8	594.07	6.68	0.835
16	320.19	12.39	0.775
32	188.14	21.09	0.659

p	Time (s)	Speedup	Efficiency
32×1	150.13	26.43	0.830
16×2	159.55	24.87	0.777
8×4	162.13	24.47	0.765
4×8	165.24	24.01	0.750
16×2	173.76	22.84	0.714
1×32	188.14	21.09	0.659

Conclusions

- We have developed a parallel iterative algorithm for computing passage time densities in large semi-Markov models
- We have examined the truncation error of our algorithm and observed a practical complexity of better than $O(N^2 \log N)$
- Key to its scalability is the use of hypergraph partitioning
- The method has been validated against simulation and found to be extremely accurate



Convergence of Algorithm

- Iterative algorithm approximates $L_{\vec{ij}}(s)$ with a geometric sum:

$$\lim_{r\to\infty} \sum_{i=0}^{r} \tilde{\alpha} U U'^{i} = \tilde{\alpha} U (I - U')^{-1}$$

- Converges iff $|\lambda_1| < 1$, where λ_1 is dominant eigenvalue of U'

- We know this is true for U' since:

$$P_{\vec{ij}}^{(r)} \to P_{\vec{ij}} \text{ as } r \to \infty$$