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π IN THE MANDELBROT SET

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Abstract

The Mandelbrot set is arguably one of the most beautiful sets in mathematics. In 1991, Dave Boll discovered a surprising occurrence of the number π while exploring a seemingly unrelated property of the Mandelbrot set.[1] Boll's finding is easy to describe and understand, and yet it is not widely known — possibly because the result has not been rigorously shown. The purpose of this paper is to present and prove Boll's result.

1. INTRODUCTION

The Mandelbrot set $\mathcal{M}$ for

$$Q_c(z) = z^2 + c \tag{1}$$

(displayed in Fig. 1) is often defined to be the set of c-values for which the orbit of zero, $\{Q_c^n(0)\}|_{n=1}^{\infty}$, is bounded. It is easy to show (see Appendix 1) that if $|z| \geq |c|$ and $|z| > 2$ that the orbit $\{Q_c^n(z)\}|_{n=1}^{\infty}$ diverges. From this, we know that $\mathcal{M}$ is contained inside the disk $|c| \leq 2$. So, we can refine our definition of $\mathcal{M}$ as follows:

$$\mathcal{M} = \{c \in C : |Q_c^n(0)| \leq 2 \text{ for all } n = 1, 2, 3, \ldots\}\,. \tag{2}$$

The bands of shades or colors normally displayed outside $\mathcal{M}$ represent the smallest number of iterations N such that $\{|Q_c^n(0)|\}|_{n=1}^{N}$ exceeds 2. In what

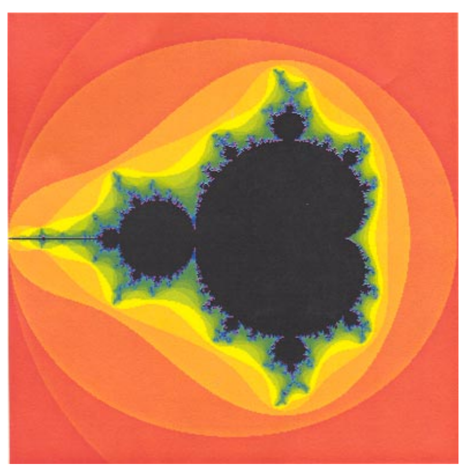


Fig. 1 The Mandelbrot set (shown in black) for $Q_c(z) = z^2 + c$ lies in the c–plane inside the disk $c \leq 2$. The Nth shaded band outside $\mathcal{M}$ shows the c values for which the first N iterations of the orbit, $\{Q_c^n(0)\}|_{n=1}^{N}$, escapes $c \leq 2$. The left side of the circle $c = 2$ is the outermost curve through the left middle of the image.

follows, we tell the story of an observation that weds the number π to the banded region outside the Mandelbrot set in a simple and yet remarkably beautiful way.

Our story began in 1991 when Dave Boll was a computer science graduate student at Colorado State University. Boll had many curiosities, among which was his fascination with computer-generated fractals. Boll was trying to convince himself that only the single point $c = (-0.75, 0)$ connects the cardioid and the disk to its left. He could not have been prepared for what he was to discover. Boll described his finding on his website[2] which we mostly follow here:

> "I was trying to verify that the neck of the set (which is at $(-0.75, 0)$) in the complex plane) is infinitely thin (it is). Accordingly, I was seeing how many iteration points of the form $(-0.75, \varepsilon)$ went through before escaping, with ε being a small number. Here's a table showing the number of iterations for various values of ε:

ε	# of iterations
1.0	3
0.1	33
0.01	315
0.001	3143
0.0001	31417
0.00001	314160
0.000001	3141593
0.0000001	31415928

> Does the product of ε and the number of iterations strike you as a suspicious number? It's pi, to with $+/-\varepsilon$. What the heck!

> Let's try it again, this time at the butt of the set. The butt of the set occurs at $(0.25, 0)$, and here's a table for points of the form $(0.25+\varepsilon, 0)$

ε	# of iterations
1.0	2
0.1	8
0.01	30
0.001	97
0.0001	312
0.00001	991
0.000001	3140
0.0000001	9933
0.00000001	31414
0.000000001	99344
0.0000000001	314157

> Again we get the same type of relationship, this time the # of iterations $*$ sqrt(ε) gives pi."

Boll's discovery has since been popularized, mostly thanks to its publication in *Chaos and Fractals: New Frontiers of Science*,[3] and also thanks to Gerald Edgar[4] at Ohio State University. However, as pointed out by Edgar, although the "π-result" for the sequence along the real axis was known at least as early as 1980,[5,6] only hueristic arguments had been provided for the observation. We now provide a proof for the "π-result" into the cusp of $\mathcal{M}$.

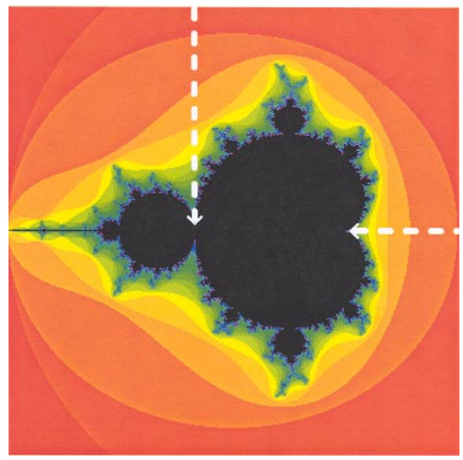


Fig. 2 The two routes to π discovered by Boll. The cusp route is $c = (0.25 + \varepsilon, 0)$ and the vertical route is $c = (-0.75, \varepsilon)$. Although not shown, $c = (-0.75, -\varepsilon)$ will work as well due to the symmetry of $\mathcal{M}$.

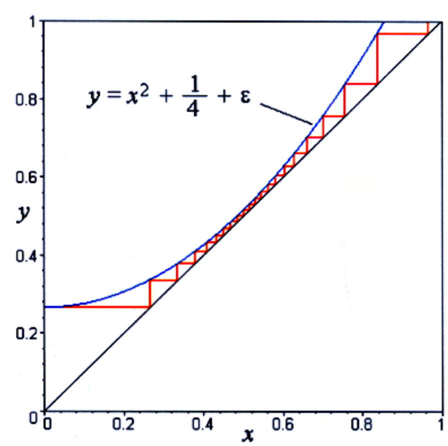


Fig. 3 Along the route $c = (0.25+\varepsilon, 0)$ into the cusp of $\mathcal{M}$, the orbit $\{Q_c^n(0)\}$ is real and is depicted by a web diagram. If $\varepsilon = 0$, the graph of $y = Q_c(x)$ and $y = x$ touch at the fixed point $x = 1/2$. If $\varepsilon > 0$ but small, there is no fixed point, but most iterations are close to $x = 1/2$.

2. MAIN THEOREM

The π-Theorem: Choose $\varepsilon > 0$ and let $N(\varepsilon)$ be the number of iterations required for the orbit of zero, under the map $Q_{1/4+\varepsilon}(x) = x^2 + 1/4 + \varepsilon$, to exceed 2, i.e.

$$N(\varepsilon) = \min_n Q^n_{1/4+\varepsilon}(0) > 2\,. \tag{3}$$

Then

$$\lim_{\varepsilon\to 0^+} \sqrt{\varepsilon}N(\varepsilon) = \pi\,. \tag{4}$$

Figure 3 shows that the majority of steps in the orbit of 0 are spent close to 1/2 and that the step-size goes to zero there as $\varepsilon \to 0^+$. This gives us reason to believe that the solution of the difference equation

$$x_{k+1} = x_k^2 + \frac{1}{4} + \varepsilon\,, \quad x_0 = 0 \tag{5}$$

or equivalently

$$x_{k+1} - x_k = \left(x_k - \frac{1}{2}\right)^2 + \varepsilon\,, \quad x_0 = 0 \tag{6}$$

may be well approximated by the differential equation

$$\frac{dx}{dt} = (x - 1/2)^2 + \varepsilon\,, \quad x(0) = 0 \tag{7}$$

for ε small and x near 1/2. Sure enough, as claimed by the following lemma, the differential equation for x near $x = 1/2$ yields the "π-result" analogous to Eq. (4).

Lemma 1. *If $\varepsilon > 0$, the time $T(\varepsilon)$ that it takes for the state variable $x(t)$ satisfying (7) to evolve to $x(T(\varepsilon)) = 2$ satisfies*

$$\lim_{\varepsilon\to 0^+} \sqrt{\varepsilon}T(\varepsilon) = \pi\,. \tag{8}$$

Proof. If we replace the initial condition in (7) with $x(0) = 1/2$, then

$$x(t) = \frac{1}{2} + \sqrt{\varepsilon}\tan\sqrt{\varepsilon}t\,. \tag{9}$$

To prove the lemma, we need to show that

$$x(T_-(\varepsilon)) = 0 \text{ and } x(T_+(\varepsilon)) = 2 \tag{10}$$

respectively imply

$$\lim_{\varepsilon\to 0^+} \sqrt{\varepsilon}T_-(\varepsilon) = -\frac{\pi}{2} \text{ and } \lim_{\varepsilon\to 0^+} \sqrt{\varepsilon}T_+(\varepsilon) = \frac{\pi}{2}\,. \tag{11}$$

The results in (11) follow immediately after writing (9) in its equivalent form

$$\sqrt{\varepsilon}t = \arctan\frac{x(t) - 1/2}{\sqrt{\varepsilon}} \tag{12}$$

and substituting the conditions (10) into (12). □

Remark 1. The terminal state 2 in (10) could be replaced with any number greater than 1/2. We use 2 because it is the radius of the smallest circle centered at the origin containing the Mandelbrot set $\mathcal{M}$.

What remains is to show that the solution of the differential Eq. (7) really does approximate the solution of the original difference Eq. (5). We begin by solving

$$x_{k+1} = x_k^2 + \frac{1}{4} + \varepsilon\,, \quad x_0 = 1/2\,. \tag{13}$$

The first four iterates give

$$x_1 = \frac{1}{2} + \varepsilon$$
$$x_2 = \frac{1}{2} + 2\varepsilon + \varepsilon^2$$
$$x_3 = \frac{1}{2} + 3\varepsilon + 5\varepsilon^2 + 4\varepsilon^3 + \varepsilon^4$$
$$x_4 = \frac{1}{2} + 4\varepsilon + 14\varepsilon^2 + 34\varepsilon^3 + 50\varepsilon^4 + \cdots$$

and we recognize a pattern on the lowest order terms in order to conjecture that

$$\begin{aligned} x_n &= \frac{1}{2} + n\varepsilon + \left(\sum_{i=0}^{n-1} i^2\right)\varepsilon^2 + \cdots \\ &= \frac{1}{2} + n\varepsilon + \left(\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n\right)\varepsilon^2 + \cdots\,. \end{aligned}$$

It is natural to Taylor expand the solution of the differential equation [given in Eq. (9)]

$$\begin{aligned} x(n) &= \frac{1}{2} + \sqrt{\varepsilon}\tan(\sqrt{\varepsilon}n) \\ &= \frac{1}{2} + n\varepsilon + \frac{1}{3}n^3\varepsilon^2 + \frac{2}{15}n^5\varepsilon^3 + \cdots \end{aligned} \tag{14}$$

in order to make comparisons with the solution of the difference equation. Since we seek to choose the largest possible integer n, we let $n = K/\sqrt{\varepsilon}$ (with

K less than but as close to $\pi/2$ as possible so that n is an integer) to get

$$x(n) = x(K/\sqrt{\varepsilon}) = \frac{1}{2} + \sqrt{\varepsilon}\left(K + \frac{1}{3}K^3 + \frac{2}{15}K^5 + \cdots\right) \quad (15)$$

which should be compared with the solution of the difference equation

$$x_n = \frac{1}{2} + \frac{K}{\sqrt{\varepsilon}}\varepsilon + \left(\frac{1}{3}\left(\frac{K}{\sqrt{\varepsilon}}\right)^3 - \frac{1}{2}\left(\frac{K}{\sqrt{\varepsilon}}\right)^2 + \frac{1}{6}\left(\frac{K}{\sqrt{\varepsilon}}\right)\right)\varepsilon^2 + \cdots = \frac{1}{2} + \sqrt{\varepsilon}\left(K + \frac{1}{3}K^3 + \frac{2}{15}K^5 + \cdots\right) + O(\varepsilon)\,. \quad (16)$$

Remark 2. We have chosen n so that $n\sqrt{\varepsilon} \approx \pi/2$ for ε small and so that $\lim_{\varepsilon\to 0^+} n\sqrt{\varepsilon} = \pi/2$.

Remark 3. The expression $(K + \frac{1}{3}K^3 + \frac{2}{15}K^5 + \cdots)$ from $x(n)$ in (15) is the tangent series. However, a similar expression from x_n in (16) is a partial sum of 2^{n-1} terms that we aim to show are the first 2^{n-1} nonzero terms of the tangent series.

Taylor's theorem says that

$$\tan K = K + \frac{K^3}{3} + \frac{2K^5}{15} + \cdots + a_{2^{n-1}}K^{2^n-1} + \frac{\tan^{(2^n)}\kappa}{(2^n)!}K^{2^n}$$

for $0 \le \kappa \le K$ where $\tan^{(j)} x$ denotes the jth derivative of $\tan x$, and a_j denotes the jth nonzero term in the tangent series. So, once we can show that (16) contains a partial sum for $\tan K$, Eq. (16) becomes

$$x_n = \frac{1}{2} + \sqrt{\varepsilon}\left(\tan K - \frac{\tan^{(2^{n-1}+1)}\kappa}{(2^{n-1}+1)!}K^{2^n}\right) + O(\varepsilon)\,.$$

Setting $x_n = x_{K/\sqrt{\varepsilon}} = 2$ in (16) implies

$$\frac{2 - 1/2}{\sqrt{\varepsilon}} = \tan K - \frac{\tan^{(2^n)}\kappa}{(2^n)!}K^{2^n} + O(\sqrt{\varepsilon}) \quad (17)$$

which should be contrasted with setting the solution of the differential equation $x(n) = x(K/\sqrt{\varepsilon}) = 2$ in (15) to get

$$\frac{2 - 1/2}{\sqrt{\varepsilon}} = \tan K\,. \quad (18)$$

Since the tangent series converges, we know that given any $\hat{\varepsilon} > 0$, there is a sufficiently large value of n for which the magnitude of the remainder term is less than $\hat{\varepsilon}$. Equivalently, there must be a positive exponent a so that the error term is less than $\hat{\varepsilon} = \varepsilon^a$ so that (17) gives

$$\frac{3}{2\sqrt{\varepsilon}} = \tan K - O(\varepsilon^a) + O(\sqrt{\varepsilon})\,, \quad a > 0$$

which leads to

$$\lim_{\varepsilon\to 0^+} n\sqrt{\varepsilon} = \lim_{\varepsilon\to 0^+} \tan^{-1}\left(\frac{3}{2\sqrt{\varepsilon}} + O(\varepsilon^a) - O(\sqrt{\varepsilon})\right) = \frac{\pi}{2}\,.$$

So, our goal is to show that the sum in (19) is really the partial sum for the tangent series. We'll also need to show a similar result for n negative. We begin with a lemma that describes the form of the solution x_n of the difference equation.

Lemma 2. *The solution x_n of the difference Eq.* (13) *is*

$$x_n = \sum_{j=0}^{2^{n-1}} p_j(n)\varepsilon^j\,, \quad n = 1, 2, 3, \ldots \quad (19)$$

where $p_0(n) = 1/2$ and $p_j(n)$ is a $2j-1$ degree polynomial in n (for $j = 1, 2, 3, \ldots$) that satisfies the recurrence relation

$$p_j(n+1) = p_j(n) + \sum_{i=1}^{j-1} p_i(n)p_{j-i}(n)$$
$$p_1(n) = n\,, \quad j = 2, 3, 4, \ldots\,. \quad (20)$$

Proof. Since $x_1 = 1/2 + \varepsilon$, we have $p_0(1) = 1/2$ and $p_1(1) = 1$ so (19) holds for $n = 1$. Now assume (19) is true for $n = m$. Then

$$x_{m+1} = (p_0(m) + p_1(m)\varepsilon + p_2(m)\varepsilon^2 + p_3(m)\varepsilon^3 + \cdots + p_{2^{m-1}}(m)\varepsilon^{2^{m-1}})^2 + \frac{1}{4} + \varepsilon \quad (21)$$

$$= \left[p_0^2(m) + \frac{1}{4}\right] + [2p_0(m)p_1(m) + 1]\varepsilon + \cdots + \left[\sum_{i=0}^{j} p_i(m)p_{j-i}(m)\right]\varepsilon^j + \cdots \quad (22)$$

for $j = 2, \ldots, 2^m$. We have now shown that (19) is true for all $n = 1, 2, 3 \ldots$. We must now prove the properties of the p_j's hold. Since our induction hypothesis includes $p_0(m) = 1/2$, (21) implies that $p_0(m+1) = (1/2)^2 + 1/4 = 1/2$ proving that $p_0(n) = 1/2$ for all $n = 1, 2, 3 \ldots$. Similarly, we assumed that $p_1(m) = m$, so (21) implies that $p_1(m+1) = 2 \cdot \frac{1}{2} \cdot m + 1$ proving that $p_1(n) = n$ for all $n = 1, 2, 3 \ldots$. Since $p_0(n) = 1/2$ for all n, the coefficient of ε^j in $x_{n+1} = x_n^2 + \frac{1}{4} + \varepsilon$ gives the recurrence relation

$$p_j(m+1) = p_j(m) + \sum_{i=1}^{j-1} p_i(m) p_{j-i}(m) \qquad (23)$$

directly. Finally, (23) is equivalent to

$$p_j(m+1) = \sum_{k=1}^{m} \sum_{i=1}^{j-1} p_i(k) p_{j-i}(k) \qquad (24)$$

since $p_j(1)$, the coefficient of ε^j in x_1, is zero. Since $p_i(m)$ and $p_{j-i}(m)$ are $2i-1$ and $2(j-i)-1$ degree polynomials, respectively, $p_i(m)p_{j-i}(m)$ is a $2j-2$ degree polynomial and so is $\sum_{i=1}^{j-1} p_i(k)p_{j-i}(k)$. It follows then that the right-hand-side of Eq. (24) is a $2j-1$ degree polynomial. □

We saw in Eq. (16) that the lower order terms in the polynomial coefficients $p_j(n)$ contribute to $O(\varepsilon)$ terms which ultimately have no effect on the result. What we need is the coefficient a_j of the highest degree term in $p_j(n)$.

Lemma 3. *Define a_j to be the coefficient of the highest degree term in $p_j(n)$. Then a_j satisfies the recurrence relation*

$$a_j = \frac{1}{2j-1} \sum_{i=1}^{j-1} a_i a_{j-i}\,, \quad a_1 = 1\,. \qquad (25)$$

Proof. By definition, $p_j(n) = a_j n^{2j-1} + c_j n^{2j-2} + \cdots$ where c_j is the constant coefficient of n^{2j-2}. From the recurrence relation (23) for $p_j(n)$, we have

$$\begin{aligned}
&(a_j n^{2j-1} + c_j n^{2j-2} + \cdots) \\
&\quad + \sum_{i=1}^{j-1} (a_i n^{2i-1} + c_i n^{2i-2} + \cdots) \\
&\quad \times (a_{j-i} n^{2(j-i)-1} + c_{j-i} n^{2(j-i)-2} + \cdots) \\
&= a_j (n+1)^{2j-1} + c_j (n+1)^{2j-2} + \cdots \\
&= a_j n^{2j-1} + (2j-1) a_j n^{2j-2} + c_j n^{2j-2} + \cdots .
\end{aligned}$$

Equating coefficients of n^{2j-2} gives

$$a_j = \frac{1}{2j-1} \sum_{i=1}^{j-1} a_i a_{j-i}$$

and $a_1 = 1$ since $b_1(n) = 1 \cdot n$. □

Next, we show that the a_j's are coefficients for the tangent series.

Lemma 4. *The nonzero coefficients of the tangent series are also generated by* (25).

Proof. We can write

$$\tan x = \sum_{k=1}^{\infty} c_k x^k\,, \quad -\pi/2 < x < \pi/2$$

since $\tan x$ is analytic on $-\pi/2 < x < \pi/2$. Futhermore, since $\tan x$ is odd, $c_k = 0$ for even k. We can differentiate the convergent power series and we have

$$\frac{d}{dx} \tan x = \sec^2 x = 1 + \tan^2 x$$

so that

$$\sum_{k=1}^{\infty} k c_k x^{k-1} = 1 + \left(\sum_{k=1}^{\infty} c_k x^k \right) \left(\sum_{k=1}^{\infty} c_k x^k \right). \qquad (26)$$

Equating coefficients of x^n for $n = 2j = 2, 4, 6, \ldots$ in (26) gives

$$(2j+1) c_{2j+1} = \sum_{\substack{m+l=2j \\ m,\, l \text{ odd}}} c_m c_l\,. \qquad (27)$$

If we then substitute $a_i = c_{2i-1}$, (27) becomes

$$(2j+1) a_{j+1} = \sum_{i=1}^{j} a_i a_{j+1-i}$$

which completes the proof after replacing j with $j-1$. □

Having just shown that the leading coefficients in $p_j(n)$ match the jth nonzero term in the tangent series, we have completed the proof that it is valid to replace the difference equation wth the differential equation for $1/2 \leq x_n, x(n) \leq 2$, $n \geq 0$. We next tackle the case of $0 \leq x_n, x(n) \leq$

$1/2$, $n \le 0$. Since we are restricted to $x \ge 0$, the map $f(x) = x^2 + 1/4 + \varepsilon$ has the inverse

$$f^{-1}(x) = \sqrt{x - 1/4 - \varepsilon}\,.$$

So, $x_{-n} = y_n$ for $n \ge 0$ if

$$y_{n+1} = \sqrt{y_n - 1/4 - \varepsilon}\,, \quad y_0 = 1/2\,. \tag{28}$$

The first few iterations are (expanded in ε)

$$\begin{aligned} y_1 &= \sqrt{1/4 - \varepsilon} \\ &= 1/2 - \varepsilon - \varepsilon^2 - 2\varepsilon^3 - \cdots \\ y_2 &= \sqrt{1/4 - 2\varepsilon - \cdots} \\ &= 1/2 - 2\varepsilon - 5\varepsilon^2 - 22\varepsilon^3 - \cdots \\ y_3 &= \sqrt{1/4 - 3\varepsilon - \cdots} \\ &= 1/2 - 3\varepsilon - 14\varepsilon^2 - 106\varepsilon^3 - \cdots \\ y_4 &= \sqrt{1/4 - 4\varepsilon - \cdots} \\ &= 1/2 - 4\varepsilon - 30\varepsilon^2 - 346\varepsilon^3 - \cdots\,. \end{aligned}$$

Conveniently, we see patterns similar to those found in the case where $x \ge 1/2$. We conjecture and prove the following.

Lemma 5. *The backwards solution of the difference equation $x_{n+1} = x_n^2 + 1/4 + \varepsilon$, $x_0 = 1/2$, $k = 0, -1, -2, \ldots$ (with $0 \le x_k < 1/2$) or equivalently the forward solution of $y_{n+1} = \sqrt{y_n - 1/4 - \varepsilon}, y_0 = 1/2$ by substituting $x_{-n} = y_n$ is*

$$y_n = \sum_{j=0}^{\infty} q_j(n)\varepsilon^j \tag{29}$$

where $q_0(n) = 1/2$ and $q_j(n)$ is a $2j-1$ degree polynomial in n (for $j = 1, 2, 3, \ldots$) that satisfies the recurrence relation

$$\begin{aligned} q_j(n+1) &= q_j(n) - \sum_{i=1}^{j-1} q_i(n)q_{j-i}(n) + O(n^{2j-3}) \\ q_1(n) &= -n\,, \quad j = 2, 3, 4, \ldots \end{aligned} \tag{30}$$

where $O(n^{2j-3})$ denotes a polynomial of degree at most $2j-3$. The careful reader has already noticed the similarities with Lemma 2.

Proof. We will find it more convenient to work with the difference equation

$$z_{n+1} = \sqrt{z_n - \varepsilon} - 1/4\,, \quad z_0 = 1/4 \tag{31}$$

which is equivalent to (28) via $y_k = z_k + 1/4$ so that (29) becomes

$$z_n = q_0(n) - \frac{1}{4} + \sum_{j=1}^{\infty} q_j(n)\varepsilon^j\,. \tag{32}$$

We first note that $q_0(n) = 1/2$ and $q_1(n) = -n$ follow trivially by mathematical induction upon substituting Eq. (32) into (31) and equating coefficients of ε^0 and ε^1 to get

$$q_0(n+1) = \sqrt{q_0(n) - \frac{1}{4}}\,, \quad q_0(1) = \frac{1}{2}$$

and

$$q_1(n+1) = q_1(n) - 1\,, \quad q_0(1) = -1$$

respectively.

Next, we seek a recurrence relation for $q_j(n)$. In order to do this, we must know what the first $j+1$ terms are in the Taylor series expansion for $f(\varepsilon) = \sqrt{z(\varepsilon) - \varepsilon} - 1/4$. We denote the jth derivative of f or z by $f^{(j)}$ or $z^{(j)}$, respectively.

Claim. For each $j = 2, 3, 4, \ldots$, the jth derivative of $f(\varepsilon)$ is

$$\begin{aligned} f^{(j)}(\varepsilon) &= \frac{-1}{4}(z-\varepsilon)^{-3/2}\left[\frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i} z^{(i)} z^{(j-i)}\right] \\ &\quad + \frac{1}{2}(z-\varepsilon)^{-1/2} z^{(j)} + \cdots \end{aligned} \tag{33}$$

where the terms denoted by "$\cdots$" are of the form

$$\alpha \prod_k [z^{(d_k)}]^{p_k} \tag{34}$$

where

$$\sum_k d_k p_k = j \text{ and } \sum_k p_k \ge 3 \text{ or} \tag{35}$$

$$\sum_k d_k p_k < j \tag{36}$$

and $\alpha = \gamma(z-\varepsilon)^{-r/2}$ for some constant γ and odd natural number r.

Remark. If we define $\sum_{i=1}^{0} x(i) = 0$, then the claim holds for $j = 1$ as well.

Proof of Claim. We first note that the terms of $f^{(j)}(\varepsilon)$ displayed in (33) do not satisfy condition

(35) or (36). The ith component of the first $j-1$ terms in (33) can be written

$$\frac{-1}{8}\binom{j}{i}(z-\varepsilon)^{-3/2}z^{(i)}z^{(j-i)} = \alpha\prod_{k=1}^{2}[z^{(d_k)}]^{p_k}$$

with $\alpha = \frac{-1}{8}\binom{j}{i}(z-\varepsilon)^{-3/2}$, $d_1 = i$, $d_2 = j-i$, and $p_1 = p_2 = 1$. So, $\sum_{k=1}^{2} p_k = 2 < 3$ and $\sum_{k=1}^{2} d_k p_k = j$ so that neither (35) nor (36) hold. The last term displayed in (33) term can be written

$$\frac{1}{2}(z-\varepsilon)^{-1/2}z^{(j)} = \alpha[z^{(d_1)}]^{p_1}$$

with $d_1 = j$ and $p_1 = 1$ so that $\sum_{k=1}^{1} p_k = 1 < 3$ and $\sum_{k=1}^{1} d_k p_k = j$.

We next show that $f^{(2)}(\varepsilon)$ satisfies the claim. Differentiating f gives

$$f^{(1)}(\varepsilon) = \frac{1}{2}(z-\varepsilon)^{-1/2}(z^{(1)}-1)$$

and

$$\begin{aligned} f^{(2)}(\varepsilon) &= \frac{-1}{4}(z-\varepsilon)^{-3/2}(z^{(1)}-1)^2 \\ &\quad + \frac{1}{2}(z-\varepsilon)^{-1/2}z^{(2)} \\ &= \frac{-1}{4}(z-\varepsilon)^{-3/2}[z^{(1)}]^2 + \frac{1}{2}(z-\varepsilon)^{-1/2}z^{(2)} \\ &\quad + \frac{1}{2}(z-\varepsilon)^{-3/2}z^{(1)} - \frac{1}{4}(z-\varepsilon)^{-3/2}. \end{aligned} \tag{37}$$

Taking $j = 2$ in the first displayed part of (33) gives the first term in (37) and taking $j = 2$ in the second displayed part of (33) gives the second term in (37). The last two terms in (37) are of the form (34) and both clearly satisfy condition (36).

We now proceed to assume the claim is true up to j and prove that it follows then for $j+1$. We differentiate (33) to get

$$\begin{aligned} f^{(j+1)}(\varepsilon) = {} & \frac{3}{8}(z-\varepsilon)^{-5/2}(z^{(1)}-1) \\ & \times\left[\frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i}z^{(i)}z^{(j-i)}\right] - \frac{1}{4}(z-\varepsilon)^{-3/2} \\ & \times\left[\frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i}[z^{(i+1)}z^{(j-i)} + z^{(i)}z^{(j+1-i)}]\right] \\ & - \frac{1}{4}(z-\varepsilon)^{-3/2}z^{(1)}z^{(j)} + \frac{1}{4}(z-\varepsilon)^{-3/2}z^{(j)} \\ & + \frac{1}{2}(z-\varepsilon)^{-1/2}z^{(j+1)} + \cdots. \end{aligned}$$

Before going further, we note that the first term can be written as

$$\begin{aligned} & \frac{3}{8}(z-\varepsilon)^{-5/2}(z^{(1)}-1)\left[\frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i}z^{(i)}z^{(j-i)}\right] \\ & = \sum_{i=1}^{j-1}\alpha_i\prod_{k=1}^{3}[z^{(d_k)}]^{p_k} - \sum_{i=1}^{j-1}\alpha_i\prod_{k=1}^{2}[z^{(d_k)}]^{p_k} \end{aligned}$$

where $\alpha_i = \frac{3}{16}(z-\varepsilon)^{-5/2}\binom{j}{i}$, $d_1 = i$, $d_2 = j-i$, $d_3 = 1$, $p_k = 1$ for $k = 1, 2, 3$ so that $\sum_{k=1}^{3} d_k p_k = j+1$ and $\sum_{k=1}^{2} p_k = 3$ (first part) $\sum_{k=1}^{2} d_k p_k = j < j+1$ (second part). Also, the fourth term fits the form

$$\frac{1}{4}(z-\varepsilon)^{-3/2}z^{(j)} = \alpha[z^{(d_1)}]^{p_1}$$

so that $\sum_{k=1}^{1} d_k p_k = j < j+1$. All that is left is

$$\begin{aligned} f^{(j+1)}(\varepsilon) = {} & -\frac{1}{4}(z-\varepsilon)^{-3/2} \\ & \times\left[\frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i}[z^{(i+1)}z^{(j-i)} + z^{(i)}z^{(j+1-i)}]\right] \\ & - \frac{1}{4}(z-\varepsilon)^{-3/2}z^{(1)}z^{(j)} \\ & + \frac{1}{2}(z-\varepsilon)^{-1/2}z^{(j+1)} + \cdots. \end{aligned}$$

We need, then, to show that

$$\begin{aligned} & \frac{1}{2}\sum_{i=1}^{j}\binom{j+1}{i}z^{(i)}z^{(j+1-i)} \\ & = \frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i}[z^{(i+1)}z^{(j-i)} + z^{(i)}z^{(j+1-i)}] \\ & \quad + z^{(1)}z^{(j)}. \end{aligned}$$

But

$$\begin{aligned} & \frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i}[z^{(i+1)}z^{(j-i)} + z^{(i)}z^{(j+1-i)}] + z^{(1)}z^{(j)} \\ & = \frac{1}{2}\left\{\sum_{i=2}^{j}\binom{j}{i-1}z^{(i)}z^{(j+1-i)} \right. \\ & \quad \left. + \sum_{i=1}^{j-1}\binom{j}{i}z^{(i)}z^{(j+1-i)}\right\} + z^{(1)}z^{(j)} \end{aligned}$$

$$= \frac{1}{2}\left\{\binom{j}{1} z^{(1)} z^{(j)} + \sum_{i=2}^{j-1}\left[\binom{j}{i-1} + \binom{j}{i}\right] \times z^{(i)} z^{(j+1-i)} + \binom{j}{j-1} z^{(j)} z^{(1)}\right\} + z^{(1)} z^{(j)}$$

$$= (j+1) z^{(1)} z^{(j)} + \frac{1}{2}\sum_{i=2}^{j-1}\binom{j+1}{i} \times z^{(i)} z^{(j+1-i)} = \frac{1}{2}\sum_{i=1}^{j}\binom{j+1}{i} z^{(i)} z^{(j+1-i)} .$$

We must now show that the derivative of terms originally of the form (34) retain the form (34) under differentiation. Let

$$g(\varepsilon) = \alpha \prod_{k=1}^{m} [z^{(d_k)}]^{p_k}$$

with properties for (34) satisfied along with conditions (35) and (36). Then

$$\frac{d}{d\varepsilon} g(\varepsilon) = \frac{d\alpha}{d\varepsilon}(z^{(1)} - 1) \prod_{k=1}^{m} [z^{(d_k)}]^{p_k} + \alpha \sum_{i=1}^{m} \prod_{i \neq k} [z^{(d_k)}]^{p_k} p_i [z^{(d_i)}]^{p_i - 1} z^{(d_i + 1)}$$

$$= \underbrace{-\frac{d\alpha}{d\varepsilon} \prod_{k=1}^{m} [z^{(d_k)}]^{p_k}}_{g_1} + \underbrace{\frac{d\alpha}{d\varepsilon} \prod_{k=1}^{m+1} [z^{(d_k)}]^{p_k}}_{g_2} + \underbrace{\alpha \sum_{i=1}^{m} \prod_{i \neq k} p_i [z^{(d_k)}]^{p_k} [z^{(d_i)}]^{p_i - 1} z^{(d_i+1)}}_{g_3} .$$

Clearly, piece g_1 satisfies whichever condition (35) or (36) that g did which immediately implies $\sum_{k=1}^{m} d_k p_k < j+1$. Piece g_2 has $\sum_{k=1}^{m+1} d_k p_k = \sum_{k=1}^{m} d_k p_k + 1$. If $\sum_{k=1}^{m} d_k p_k = j$, then $\sum_{k=1}^{m} p_k \geq 3$ (since (35) must hold) and we have $\sum_{k=1}^{m+1} d_k p_k = j+1$. If, on the other hand, $\sum_{k=1}^{m} d_k p_k < j$, then $\sum_{k=1}^{m+1} d_k p_k < j+1$. The ith term in g_3 has $\sum_{k \neq i} d_k p_k + d_i(p_i - 1) + d_i + 1 = \sum_{k=1}^{m} d_k p_k + 1$. So, as for g_2, g_3 still satisfies either condition (35) or (36) with j replaced by $j+1$. This completes the proof of the claim.

Recalling, now, that

$$z = \frac{1}{4} + q_1(n)\varepsilon + q_2(n)\varepsilon^2 + q_3(n)\varepsilon^3 + \cdots$$

we have

$$f^{(j)}(0) = \frac{-1}{4}\left(\frac{1}{4}\right)^{\frac{-3}{2}} \times \left[\frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i} i! q_i(n)(j-i)! q_{j-i}(n)\right] + \frac{1}{2}\left(\frac{1}{4}\right)^{\frac{-1}{2}} j! b_j(n) + g^{(j)}(0)$$

where

$$g^{(j)}(0) = \sum_i \alpha_i \prod_k [z^{(d_k)}]^{p_k}\Big|_{\varepsilon=0} = \sum_i \gamma_i \left(\frac{1}{4}\right)^{-r/2} \prod_k d_k! q_{d_k}^{p_k}(n) .$$

It follows then that for $j = 2, 3, \ldots$,

$$q_j(n+1) = \frac{f^{(j)}(0)}{j!} = -\sum_{i=1}^{j-1} q_i(n) q_{j-i}(n) + q_j(n) + \underbrace{\sum_i \frac{2^r \gamma_i}{j!} \prod_k d_k! q_{d_k}^{p_k}(n)}_{\text{remainder}} \tag{38}$$

where $q_1(n) = -n$. Since sums and products of polynomials are polynomials, $q_j(n)$ is a polynomial.

We now show that $q_j(n)$ is a $2j-1$ degree polynomial for $j = 1, 2, 3, \ldots$. Since it is true for $i = 1$, we now assume true for $i = 1, \ldots, j-1$. Using the notation $O(d)$ to denote a polynomial of degree d, we have

$$q_j(n+1) - q_j(n) = O(2i-1) \cdot O(2(j+1)-1) + O\left(\sum_k (2d_k - 1) p_k\right) = O(2j-2) + O\left(2\sum_k d_k p_k - \sum_k p_k\right) .$$

Conditions (35) and (36) imply that either $\sum_k d_k p_k = j$ and $\sum_k p_k \geq 3$ or $\sum_k d_k p_k < j$. Either way, $2\sum_k d_k p_k - \sum_k p_k \leq 2j-3$. So, the remainder term in (38) is at most a $2j-3$ degree polynomial and the difference $q_j(n+1) - q_j(n)$ is a $2j-2$ degree polynomial:

$$q_j(n+1) - q_j(n) = \beta_1 n^{2j-2} + \beta_2 n^{2j-3} + \cdots$$

or equivalently,

$$q_j(n+1) = \sum_{k=1}^{n} (\beta_1 k^{2j-2} + \beta_2 k^{2j-3} + \cdots)$$

which is a $2j-1$ degree polynomial. This completes the proof of Lemma 5. □

Finally, we use the fact that $q_j(n)$ is a $2j-1$ degree polynomial along with the fact that the remainder terms in (38) are at most a $2j-3$ degree polynomial to rewrite $q_j(n) - \sum_{i=1}^{j-1} q_i(n) q_{j-i}(n) + \cdots = q_j(n+1)$ in (38) as

$$\begin{aligned}
&(a_j n^{2j-1} + c_j n^{2j-2} + \cdots) \\
&\quad - \sum_{i=1}^{j-1} (a_i n^{2i-1} + c_i n^{2i-2} + \cdots) \\
&\quad \times (a_{j-i} n^{2(j-i)-1} + c_{j-i} n^{2(j-i)-2} + \cdots) + \underbrace{\cdots}_{O(2j-3)} \\
&= a_j (n+1)^{2j-1} + c_j (n+1)^{2j-2} + \cdots \\
&= a_j n^{2j-1} + (2j-1) a_j n^{2j-2} + c_j n^{2j-2} + \cdots .
\end{aligned}$$

Equating coefficients of n^{2j-2} gives

$$a_j = \frac{-1}{2j-1} \sum_{i=1}^{j-1} a_i a_{j-i}$$

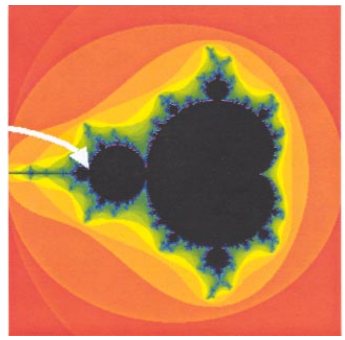
and we already know that $a_1 = -1$ since $q_1(n) = -1 \cdot n$. The proof of the "π-result" into the cusp of $\mathcal{M}$ now follows from the lemmas!

3. CONCLUSION

Rather than attempt to complete the proof of Boll's vertical route shown in Fig. 2, we do something much easier. We conjecture that there are infinitely many such routes at each of the infinitely many pinches of $\mathcal{M}$. In fact, in 1997, Jay Hill found that the parabolic route $c = (-1.25 - \varepsilon^2, \varepsilon)$ into the pinch at $(-1.25, 0)$ yields a "π-result".[4] We provide a table of our own experiment for this route in Fig. 4. We know of no other reported routes to date (except the obvious routes given by the symmetry of $\mathcal{M}$). Another open problem is to determine the function of ε that multiplies $N(\varepsilon)$. So far, we have

$$a\varepsilon^b N(\varepsilon) \to \pi \tag{39}$$

where we have seen $a = 1, 2$ and $b = 1, 1/2$. In general, should we expect (39) to hold for some rational values a and b? If so, what does the pinch location in $\mathcal{M}$ tell us about a and b?



$c = (-1.25 - \varepsilon^2, \varepsilon)$

ε	$N(\varepsilon)$	$2\varepsilon N(\varepsilon)$
0.1	18	3.6
0.01	159	3.18
0.001	1586	3.172
0.0001	15731	3.1462
0.00001	157085	3.1417
0.000001	1570800	3.1416
0.0000001	15707976	3.1415952

Fig. 4 The parabolic route $c = (-1.25 - \varepsilon^2, \varepsilon)$ into the point $(-1.25, 0)$ avoids crossing the decorations of $\mathcal{M}$ and produces another π-result. The symmetric route $c = (-1.25 - \varepsilon^2, -\varepsilon)$ gives the same result.

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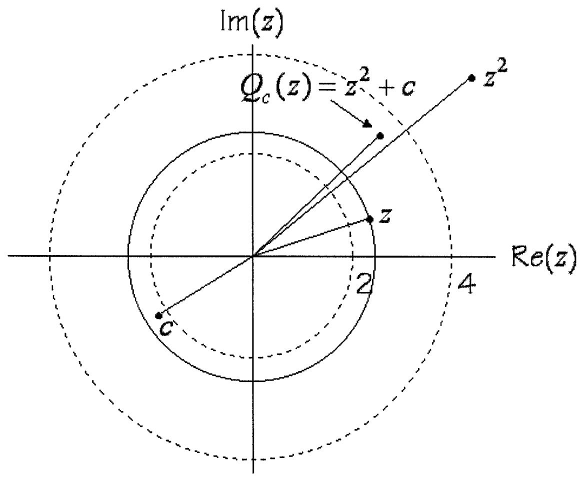

Fig. 5 If $|z| > 2$ and $|z| \geq |c|$, then $|Q_c(z)| > |z|$.

APPENDIX

Theorem (Escape Criterion): *If* $|z| > 2$ *and* $|z| \geq |c|$, *then* $Q_c(z) \to \infty$ *as* $n \to \infty$.

Picture Proof. Figure 5 gives the main idea of the proof: if $|z| > 2$, then $|Q_c(z)| > |z| > 2$. Taking $Q_c(z)$ to be our next z, the result follows. □

Proof. Let $|z_0| > 2$ and $|z_0| \geq |c|$. Then

$$\begin{aligned} |z_1| \stackrel{\text{def}}{=} |Q_c(z_0)| &= |z_0^2 + c| \\ &\geq |z_0|^2 - |c| \quad \text{(triangle inequality)} \\ &\geq |z_0|^2 - |z_0| \quad (|z_0| \geq |c|) \\ &= |z_0|(|z_0| - 1) \\ &= \lambda_0 |z_0| \end{aligned}$$

where $\lambda_0 = |z_0| - 1 > 1$. This implies that $|z_1| > |z_0| > 2$, so continuing we have

$$|z_2| \stackrel{\text{def}}{=} |Q_c(z_1)| \geq \lambda_1 |z_1| \geq \lambda_1 \lambda_0 |z_0| > \lambda_0^2 |z_0|$$

where we have defined $\lambda_n = |z_n| - 1$ so that clearly $\lambda_n > \lambda_m$ for $n > m$. Mathematical induction on n gives $|z_n| > \lambda_0^n |z_0|$. Taking $n \to \infty$ completes the proof. □