Mathematical Methods for Computer Science

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Methods Course Details

- Course title: Mathematical Methods
- Course lecturers:
  - Dr. J. Bradley (Weeks 2-5)
  - Prof. P. Harrison (Weeks 6-10)
- Course code: 145
- Lectures
  - Wednesdays: 11–12am, rm 308 (until 2nd November)
  - Thursdays: 10–11am, rm 308
  - Fridays: 11–12 noon, rm 308
- Tutorials
  - Thursdays: 11–12 noon OR Tuesdays 5–6pm
- Number of assessed sheets: 5 out of 8

Assessed Exercises

- Submission: through CATE
- https://sparrow.doc.ic.ac.uk/~cate/
- Assessed exercises (for 1st half of course):
  1. set 13 Oct; due 27 Oct
  2. set 19 Oct; due 3 Nov
  3. set 26 Oct; due 10 Nov

Recommended Books

You will find one of the following useful – no need to buy all of them:

Maths and Computer Science

- Why is Maths important to Computer Science?
- Maths underpins most computing concepts/applications, e.g.:
  - computer graphics and animation
  - stock market models
  - information search and retrieval
  - performance of integrated circuits
  - computer vision
  - neural computing
  - genetic algorithms

Highlighted Examples

- Search engines
  - Google and the PageRank algorithm
- Computer graphics
  - near photo realism from wireframe and vector representation

Searching with...

- How does Google know to put Imperial’s website top?

Searching for...
**The PageRank Algorithm**

- PageRank is based on the underlying web graph

**PageRank**

- So where's the Maths?
  - Web graph is represented as a matrix
  - Matrix is $9$ billion $\times$ $9$ billion in size
  - PageRank calculation is turned into an eigenvector calculation
  - Does it converge? How fast does it converge?

**Propagation of PageRank**

**Computer Graphics**

- Ray tracing with: POV-Ray 3.6
Key points of model are defined through vectors

- Vectors define position relative to an origin

How can we calculate light shading/shadow?

Used in (amongst others):

- Computational Techniques (2nd Year)
- Graphics (3rd Year)
- Computational Finance (3rd Year)
- Modelling and Simulation (3rd Year)
- Performance Analysis (3rd Year)
- Digital Libraries and Search Engines (3rd Year)
- Computer Vision (4th Year)
What is a vector?

- What is a vector?
- Useful vector tools:
  - Vector magnitude
  - Vector addition
  - Scalar multiplication
  - Dot product
  - Cross product
- Useful results – finding the intersection of:
  - a line with a line
  - a line with a plane
  - a plane with a plane

Vector Contents

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Vector Magnitude

- The size or magnitude of a vector \( \vec{p} = (p_1, p_2, p_3) \) is defined as its length:

\[
|\vec{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2} = \sqrt{\sum_{i=1}^{3} p_i^2}
\]

- e.g. \( \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \) – \( \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2} \)

- For an \( n \)-dimensional vector, \( \vec{p} = (p_1, p_2, \ldots, p_n) \), \( |\vec{p}| = \sqrt{\sum_{i=1}^{n} p_i^2} \)
**Vector Direction**

**Vector Angles**

- For a vector, \( \vec{p} = (p_1, p_2, p_3) \):
  - \( \cos(\theta_x) = p_1/|\vec{p}| \)
  - \( \cos(\theta_y) = p_2/|\vec{p}| \)
  - \( \cos(\theta_z) = p_3/|\vec{p}| \)

**Vector addition**

- Two vectors (of the same dimension) can be added together:
  - e.g. \[
  \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}
  \]
- So if \( \vec{p} = (p_1, p_2, p_3) \) and \( \vec{q} = (q_1, q_2, q_3) \) then:
  \[ \vec{p} + \vec{q} = (p_1 + q_1, p_2 + q_2, p_3 + q_3) \]
**Scalar Multiplication**

- A scalar is just a number, e.g. 3. Unlike a vector, it has no direction.
- Multiplication of a vector \( \vec{v} \) by a scalar \( \lambda \) means that each element of the vector is multiplied by the scalar.
- So if \( \vec{v} = (p_1, p_2, p_3) \) then:
  \[
  \lambda \vec{v} = (\lambda p_1, \lambda p_2, \lambda p_3)
  \]

**Vector notation**

- All vectors in 3D (or \( \mathbb{R}^3 \)) can be expressed as weighted sums of \( \vec{i}, \vec{j}, \vec{k} \).
- i.e. \( \vec{p} = (10, 5, 7) \equiv \begin{pmatrix} 10 \\ 5 \\ 7 \end{pmatrix} \equiv 10\vec{i} + 5\vec{j} + 7\vec{k} \)
- \( |p_1\vec{i} + p_2\vec{j} + p_3\vec{k}| = \sqrt{p_1^2 + p_2^2 + p_3^2} \)

**3D Unit vectors**

- We use \( \vec{i}, \vec{j}, \vec{k} \) to define the 3 unit vectors in 3 dimensions.
- They convey the basic directions along \( x, y \) and \( z \) axes.
- So: \( \vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \)
- All unit vectors have magnitude 1; i.e. \( |\vec{v}| = 1 \)

**Vector addition**

- We have \( \vec{a}, \vec{b}, \vec{c} \).
- \( \vec{a} + \vec{b} = \vec{c} \).
Dot Product

- Also known as: *scalar product*
- Used to determine how close 2 vectors are to being parallel/perpendicular
- The dot product of two vectors \( \vec{p} \) and \( \vec{q} \) is:
  \[ \vec{p} \cdot \vec{q} = |\vec{p}| |\vec{q}| \cos \theta \]
- where \( \theta \) is angle between the vectors \( \vec{p} \) and \( \vec{q} \)
- For \( \vec{p} = (p_1, p_2, p_3) \) and \( \vec{q} = (q_1, q_2, q_3) \) then:
  \[ \vec{p} \cdot \vec{q} = p_1q_1 + p_2q_2 + p_3q_3 \]

Properties of the Dot Product

- \( \vec{p} \cdot \vec{p} = |\vec{p}|^2 \)
- \( \vec{p} \cdot \vec{q} = 0 \) if \( \vec{p} \) and \( \vec{q} \) are perpendicular (at right angles)
- Commutative: \( \vec{p} \cdot \vec{q} = \vec{q} \cdot \vec{p} \)
- Linearity: \( \vec{p} \cdot (\lambda \vec{q}) = \lambda (\vec{p} \cdot \vec{q}) \)
- Distributive over addition:
  \[ \vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r} \]

Vector Projection

- \( \hat{n} \) is a unit vector, i.e. \( |\hat{n}| = 1 \)
- \( \hat{a} \cdot \hat{n} = |\hat{a}| \cos \theta \) represents the *amount* of \( \hat{a} \) that points in the \( \hat{n} \) direction

What can’t you do with a vector...

The following are classic mistakes — \( \vec{u} \) and \( \vec{v} \) are vectors, and \( \lambda \) is a scalar:
- **Don’t do it!**
  - Vector division: \( \vec{u} / \vec{v} \)
  - Divide a scalar by a vector: \( \lambda / \vec{a} \)
  - Add a scalar to a vector: \( \vec{a} + \lambda \)
  - Subtract a scalar from a vector: \( \vec{a} - \lambda \)
  - Cancel a vector in a dot product with vector:
    \[ \frac{1}{\vec{a} \cdot \vec{a}} \vec{a} - \frac{1}{\vec{a}} \]
Example: Rays of light

- A ray of light strikes a reflective surface...
- Question: in what direction does the reflection travel?

Problem: find \( \vec{r}' \), given \( \vec{s} \) and \( \vec{n} \)?

- angle of incidence = angle of reflection
  \[ -\vec{s} \cdot \vec{n} = \vec{r}' \cdot \vec{n} \]
- Also: \( \vec{r}' + (-\vec{s}) = \lambda \vec{n} \) thus \( \lambda \vec{n} - \vec{r}' - \vec{s} \)
- Taking the dot product of both sides:
  \[ \lambda |\vec{n}|^2 = \vec{r}' \cdot \vec{n} - \vec{s} \cdot \vec{n} \]
Rays of light

- But \( \hat{n} \) is a unit vector, so \( \hat{n}^2 = 1 \)
  \[ \Rightarrow \lambda = \vec{r} \cdot \hat{n} - \vec{s} \cdot \hat{n} \]
- ...and \( \vec{r} \cdot \hat{n} = -\vec{s} \cdot \hat{n} \)
  \[ \Rightarrow \lambda = -2\vec{s} \cdot \hat{n} \]

- Finally, we know that: \( \vec{r} + (-\vec{s}) = \lambda \hat{n} \)
  \[ \Rightarrow \vec{r} = \lambda \hat{n} + \vec{s} \]
  \[ \Rightarrow \vec{r} = \vec{s} - 2(\vec{s} \cdot \hat{n})\hat{n} \]

Equation of a line

- For a general point, \( \vec{r} \), on the line:
  \[ \vec{r} = \vec{a} + \lambda \vec{d} \]
- where: \( \vec{a} \) is a point on the line and \( \vec{d} \) is a vector parallel to the line

Equation of a plane

- Equation of a plane. For a general point, \( \vec{r} \), in the plane, \( \vec{r} \) has the property that:
  \[ \vec{r} \cdot \hat{n} = m \]
- where:
  - \( \hat{n} \) is the unit vector perpendicular to the plane
  - \( m \) is the distance from the plane to the origin (at its closest point)
**Equation of a plane**

\[
\vec{r} \cdot \vec{n} = m
\]

**How to solve Vector Problems**

1. **IMPORTANT**: Draw a diagram!
2. Write down the equations that you are given/apply to the situation
3. Write down what you are trying to find?
4. Try variable substitution
5. Try taking the dot product of one or more equations
   - What vector to dot with?

   Answer: if eqn (1) has term \( \vec{r} \) in and eqn (2) has term \( \vec{r} \cdot \vec{s} \) in: *dot eqn (1) with \( \vec{s} \).*

**Two intersecting lines**

- Application: projectile interception
- Problem — given two lines:
  - Line 1: \( \vec{r}_1 = \vec{a}_1 + t_1 \vec{d}_1 \)
  - Line 2: \( \vec{r}_2 = \vec{a}_2 + t_2 \vec{d}_2 \)
- Do they intersect? If so, at what point?
- This is the same problem as: find the values \( t_1 \) and \( t_2 \) at which \( \vec{r}_1 = \vec{r}_2 \) or:
  \[
  \vec{a}_1 + t_1 \vec{d}_1 = \vec{a}_2 + t_2 \vec{d}_2
  \]

**How to solve: 2 intersecting lines**

- Separate \( \vec{i}, \vec{j}, \vec{k} \) components of equation:
  \[
  \vec{a}_1 + t_1 \vec{d}_1 = \vec{a}_2 + t_2 \vec{d}_2
  \]
- ...to get 3 equations in \( t_1 \) and \( t_2 \)
- If the 3 equations:
  - contradict each other then the lines do not intersect
  - produce a single solution then the lines do intersect
  - are all the same (or multiples of each other) then the lines are identical (and always intersect)
Intersection of a line and plane

- Application: ray tracing, particle tracing, projectile tracking
- Problem — given one line/one plane:
  - Line: \( \vec{r} - \vec{a} + t\vec{d} \)
  - Plane: \( \vec{r} \cdot \vec{n} = s \)
- Take dot product of line equation with \( \vec{n} \) to get:
  \[ \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} + t(\vec{d} \cdot \vec{n}) \]

Example: intersecting planes

- Problem: find the line that represents the intersection of two planes

Intersection of a line and plane

- With \( \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} + t(\vec{d} \cdot \vec{n}) \) — what are we trying to find?
  - We are trying to find a specific value of \( t \) that corresponds to the point of intersection
- Since \( \vec{r} \cdot \vec{n} = s \) at intersection, we get:
  \[ t = \frac{s - \vec{a} \cdot \vec{n}}{\vec{d} \cdot \vec{n}} \]
- So using line equation we get our point of intersection, \( \vec{r}^i \):
  \[ \vec{r}^i = \vec{a} + \frac{s - \vec{a} \cdot \vec{n}}{\vec{d} \cdot \vec{n}} \vec{d} \]

Intersecting planes

- Application: edge detection
- Equations of planes:
  - Plane 1: \( \vec{r} \cdot \vec{n}_1 = s_1 \)
  - Plane 2: \( \vec{r} \cdot \vec{n}_2 = s_2 \)
- We want to find the line of intersection, i.e. find \( \vec{a} \) and \( \vec{d} \) in:
  \[ \vec{z} = \vec{a} + \lambda \vec{d} \]
- If \( \vec{z} = x\hat{i} + y\hat{j} + z\hat{k} \) is on the intersection line:
  \[ \Rightarrow \vec{z} \cdot \vec{n}_1 = s_1 \text{ and } \vec{z} \cdot \vec{n}_2 = s_2 \]
- Can use these two equations to generate equation of line
Example: Intersecting planes

- Equations of planes:
  - Plane 1: \[ \mathbf{p} \cdot (2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - 3 \]
  - Plane 2: \[ \mathbf{p} \cdot \mathbf{k} = 4 \]
- Pick point \( \mathbf{s} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \)
  - From plane 1: \( 2x - y + 2z = 3 \)
  - From plane 2: \( z = 4 \)
- We have two equations in 3 unknowns – not enough to solve the system
  - But... we can express all three variables in terms of one of the other variables

Cross Product

- Also known as: Vector Product
- Used to produce a 3rd vector that is perpendicular to the original two vectors
- Written as \( \mathbf{p} \times \mathbf{q} \) (or sometimes \( \mathbf{p} \land \mathbf{q} \))
- Formally: \( \mathbf{p} \times \mathbf{q} = (\mathbf{q} \cdot \mathbf{n})\mathbf{n} \)
  - where \( \mathbf{n} \) is the unit vector perpendicular to \( \mathbf{p} \) and \( \mathbf{q} \); \( \theta \) is the angle between \( \mathbf{p} \) and \( \mathbf{q} \)

Example: Intersecting planes

- From plane 1: \[ 2x - y + 2z = 3 \]
- From plane 2: \[ z = 4 \]
- Substituting (Eqn. 2) \( \rightarrow \) (Eqn. 1) gives:
  - \( 2x - y = 5 \)
- Also trivially: \( y - y \) and \( z - 4 \)
- Line: \( \mathbf{s} = -\frac{5}{2}\mathbf{i} + 4\mathbf{k} + y\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) \)
  - ...which is the equation of a line

Cross Product

- From definition: \[ |\mathbf{p} \times \mathbf{q}| = |\mathbf{p}| |\mathbf{q}| \sin \theta \]
- In coordinate form:
  \[ \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \]
  \[ \Rightarrow \mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \]
- Useful for: e.g. given 2 lines in a plane, write down the equation of the plane
Properties of Cross Product

- \( \vec{p} \times \vec{q} \) is itself a vector that is perpendicular to both \( \vec{p} \) and \( \vec{q} \), so:
  - \( \vec{p} \cdot (\vec{p} \times \vec{q}) = 0 \) and \( \vec{q} \cdot (\vec{p} \times \vec{q}) = 0 \)
- If \( \vec{p} \) is parallel to \( \vec{q} \) then \( \vec{p} \times \vec{q} = \vec{0} \)
  - where \( \vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k} \)
- NOT commutative: \( \vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \)
  - In fact: \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \)
- NOT associative: \( (\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c}) \)
- Left distributive: \( \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \)
- Right distributive: \( (\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a} \)

Final important vector product identity:

- \( \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \)
- which says that: \( \vec{a} \times (\vec{b} \times \vec{c}) = \lambda\vec{b} + \mu\vec{c} \)
  - i.e. the vector \( \vec{a} \times (\vec{b} \times \vec{c}) \) lies in the plane created by \( \vec{b} \) and \( \vec{c} \)

Matrices

- Used in (amongst others):
  - Computational Techniques (2nd Year)
  - Graphics (3rd Year)
  - Performance Analysis (3rd Year)
  - Digital Libraries and Search Engines (3rd Year)
  - Computing for Optimal Decisions (4th Year)
  - Quantum Computing (4th Year)
  - Computer Vision (4th Year)

Matrix Contents

- What is a Matrix?
- Useful Matrix tools:
  - Matrix addition
  - Matrix multiplication
  - Matrix transpose
  - Matrix determinant
  - Matrix inverse
  - Gaussian Elimination
  - Eigenvectors and eigenvalues
- Useful results:
  - solution of linear systems
  - Google's PageRank algorithm
**What is a Matrix?**

- A matrix is a 2 dimensional array of numbers
- Used to represent, for instance, a network:

  ![Matrix diagram](image)

  \[
  \begin{pmatrix}
  0 & 1 & 1 \\
  1 & 0 & 1 \\
  0 & 0 & 0 \\
  \end{pmatrix}
  \]

**Application: Markov Chains**

- Example: What is the probability that it will be sunny today given that it rained yesterday? (Answer: 0.25)

  ![Probability matrix](image)

- Example question: what is the probability that it's raining on Thursday given that it's sunny on Monday?

**Matrix Addition**

- In general matrices can have \( m \) rows and \( n \) columns – this would be an \( m \times n \) matrix. e.g. a \( 2 \times 3 \) matrix would look like:

  \[
  A = \begin{pmatrix}
  1 & 2 & 3 \\
  0 & -1 & 2 \\
  \end{pmatrix}
  \]

- Matrices with the same number of rows and columns can be added:

  \[
  \begin{pmatrix}
  1 & 2 & 3 \\
  0 & -1 & 2 \\
  \end{pmatrix} + \begin{pmatrix}
  3 & -1 & 0 \\
  2 & 2 & 1 \\
  \end{pmatrix} = \begin{pmatrix}
  4 & 1 & 3 \\
  2 & 1 & 3 \\
  \end{pmatrix}
  \]

**Scalar multiplication**

- As with vectors, multiplying by a scalar involves multiplying the individual elements by the scalar, e.g.:

  \[
  \lambda A = \lambda \begin{pmatrix}
  1 & 2 & 3 \\
  0 & -1 & 2 \\
  \end{pmatrix} = \begin{pmatrix}
  \lambda & 2\lambda & 3\lambda \\
  0 & -\lambda & 2\lambda \\
  \end{pmatrix}
  \]

- Now matrix subtraction is expressible as a matrix addition operation

  \[
  A - B = A + (-B) = A + (-1 \times B)
  \]
Matrix Identities

- An identity element is one that leaves any other element unchanged under a particular operation e.g. 1 is the identity in $5 \times 1 - 5$ under multiplication.
- There are two matrix identity elements: one for addition, 0, and one for multiplication, $I$.
- The zero matrix:
  
  $\begin{pmatrix}
    1 & 2 \\
    3 & -3
  \end{pmatrix} + \begin{pmatrix}
    0 & 0 \\
    0 & 0
  \end{pmatrix} = \begin{pmatrix}
    1 & 2 \\
    3 & -3
  \end{pmatrix}$
- In general: $A + 0 = A$ and $0 + A = A$

Matrix Multiplication

- The elements of a matrix, $A$, can be expressed as $a_{ij}$, so:
  
  $A = \begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
  \end{pmatrix}$
- Matrix multiplication can be defined so that, if $C = AB$ then:
  
  $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Matrix Identities

- For $2 \times 2$ matrices, the multiplicative identity, $I = \begin{pmatrix}
    1 & 0 \\
    0 & 1
  \end{pmatrix}$:
  
  $\begin{pmatrix}
    1 & 2 \\
    3 & -3
  \end{pmatrix} \times \begin{pmatrix}
    1 & 0 \\
    0 & 1
  \end{pmatrix} = \begin{pmatrix}
    1 & 2 \\
    3 & -3
  \end{pmatrix}$
- In general for square $(n \times n)$ matrices:
  
  $AI = A$ and $IA = A$

Matrix Multiplication

- Multiplication, $AB$, is only well defined if the number of columns of $A = $ the number of rows of $B$, i.e.
  
  - $A$ can be $m \times n$
  - $B$ has to be $n \times p$
  - the result, $AB$, is $m \times p$
- Example:
  
  $\begin{pmatrix}
    0 & 1 & 2 \\
    3 & 4 & 5
  \end{pmatrix} \begin{pmatrix}
    6 & 7 \\
    8 & 9 \\
    10 & 11
  \end{pmatrix} = \begin{pmatrix}
    0 \times 6 + 1 \times 8 + 2 \times 10 & 0 \times 7 + 1 \times 9 + 2 \times 11 \\
    3 \times 6 + 4 \times 8 + 5 \times 10 & 3 \times 7 + 4 \times 9 + 5 \times 11
  \end{pmatrix}$
Matrix Properties

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $\lambda A = A\lambda$
- $\lambda(A + B) = \lambda A + \lambda B$
- $(AB)C = A(BC)$
- $(A + B)C = AC + BC; C(A + B) = CA + CB$

But... $AB \neq BA$ i.e. matrix multiplication is NOT commutative

\[
\begin{pmatrix}
0 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\neq
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & -1
\end{pmatrix}
\]

Matrices in Graphics

- Matrix multiplication is a simple way to encode different transformations of objects in computer graphics, e.g.:
  - reflection
  - scaling
  - rotation
  - translation (requires $4 \times 4$ transformation matrix)

**Reflection**

The matrix which represents a reflection in the $x$-axis is:

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

This is applied to the coordinate matrix to give the coordinates of the reflected object:

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
5 & 9 & 8 \\
3 & 3 & 9
\end{pmatrix}
= \begin{pmatrix}
5 & 9 & 8 \\
-3 & -3 & -9
\end{pmatrix}
\]
**Scaling**

- Scaling matrix by factor of $\lambda$:
  \[
  \begin{pmatrix}
  \lambda & 0 \\
  0 & \lambda
  \end{pmatrix}
  \begin{pmatrix}
  1 \\
  2
  \end{pmatrix}
  =
  \begin{pmatrix}
  2\lambda \\
  2\lambda
  \end{pmatrix}
  \]

- Here triangle scaled by factor of 3

---

**Rotation**

- Rotation by angle $\theta$ about origin takes $\langle x, y \rangle \rightarrow \langle x', y' \rangle$

- Initially: $x = r \cos \psi$ and $y = r \sin \psi$

- After rotation: $x' = r \cos (\psi + \theta)$ and $y' = r \sin (\psi + \theta)$

---

**Rotation**

- Require matrix $R$ s.t.: $\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$

- Initially: $x = r \cos \psi$ and $y = r \sin \psi$

- Start with $x' = r \cos (\psi + \theta)$
  \[
  \Rightarrow x' = r \frac{\cos \psi \cos \theta - \sin \psi \sin \theta}{x} \\
  \Rightarrow x' = x \cos \theta - y \sin \theta
  \]

- Similarly: $y' = x \sin \theta + y \cos \theta$

- Thus $R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

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**3D Rotation**

- Anti-clockwise rotation of $\theta$ about $z$-axis:
  \[
  \begin{pmatrix}
  \cos \theta & -\sin \theta & 0 \\
  \sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
  \end{pmatrix}
  \]

- Anti-clockwise rotation of $\theta$ about $y$-axis:
  \[
  \begin{pmatrix}
  \cos \theta & 0 & \sin \theta \\
  0 & 1 & 0 \\
  -\sin \theta & 0 & \cos \theta
  \end{pmatrix}
  \]

- Anti-clockwise rotation of $\theta$ about $z$-axis:
  \[
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \cos \theta & -\sin \theta \\
  0 & \sin \theta & \cos \theta
  \end{pmatrix}
  \]}
**Transpose**

- For a matrix \( P \), the transpose of \( P \) is written \( P^T \) and is created by rewriting the \( i \)th row as the \( i \)th column.
- So for:
  \[
  P = \begin{pmatrix}
  1 & 3 & -2 \\
  2 & 5 & 0 \\
  -3 & -2 & 1
  \end{pmatrix} \Rightarrow P^T = \begin{pmatrix}
  1 & 2 & -3 \\
  3 & 5 & -2 \\
  -2 & 0 & 1
  \end{pmatrix}
  \]
- Note that taking the transpose leaves the leading diagonal, in this case \((1, 5, 1)\), unchanged.

**Application of Transpose**

- Main application: allows reversal of order of matrix multiplication.
- If \( AB = C \) then \( B^T A^T = C^T \)
- Example:
  \[
  \begin{pmatrix}
  1 & 2 \\
  3 & 4
  \end{pmatrix}
  \begin{pmatrix}
  5 & 6 \\
  7 & 8
  \end{pmatrix}
  = \begin{pmatrix}
  19 & 22 \\
  43 & 50
  \end{pmatrix}
  \]

**Matrix Determinant**

- The determinant of a matrix, \( P \):
  - represents the expansion factor that a \( P \) transformation applies to an object.
  - tells us if equations in \( P \bar{x} = \bar{b} \) are linearly dependent.
- If a square matrix has a determinant 0, then it is known as singular.
- The determinant of a \( 2 \times 2 \) matrix:
  \[
  \begin{vmatrix}
  a & b \\
  c & d
  \end{vmatrix} = ad - bc
  \]

**3 \times 3 Matrix Determinant**

- For a \( 3 \times 3 \) matrix:
  \[
  A = \begin{pmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3
  \end{pmatrix}
  \]
- ...the determinant can be calculated by:
  \[
  \begin{vmatrix}
  b_2 & b_3 \\
  c_2 & c_3
  \end{vmatrix} - a_2 \begin{vmatrix}
  b_1 & b_3 \\
  c_1 & c_3
  \end{vmatrix} + a_3 \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2
  \end{vmatrix}
  \]
  \[
  = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)
  \]
The Parity Matrix

- Before describing a general method for calculating the determinant, we require a parity matrix.
- For a $3 \times 3$ matrix this is:

$$\begin{pmatrix}
+1 & -1 & +1 \\
-1 & +1 & -1 \\
+1 & -1 & +1
\end{pmatrix}$$

- We will be picking pivot elements from our matrix $A$ which will end up being multiplied by $+1$ or $-1$ depending on where in the matrix the pivot element lies (e.g. $a_{12}$ maps to $-1$).

The general method...

- The $3 \times 3$ matrix determinant $|A|$ is calculated by:
  1. pick a row or column of $A$ as a pivot.
  2. for each element $x$ in the pivot, construct a $2 \times 2$ matrix, $B$, by removing the row and column which contain $x$.
  3. take the determinant of the $2 \times 2$ matrix, $B$.
  4. let $v$ = product of determinant of $B$ and $x$.
  5. let $u$ = product of $v$ with $+1$ or $-1$ (according to parity matrix rule – see previous slide).
  6. repeat from (2) for all the pivot elements $x$ and add the $u$-values to get the determinant.

Example

- Find determinant of:

$$A = \begin{pmatrix}
1 & 0 & -2 \\
4 & 2 & 3 \\
-2 & 5 & 1
\end{pmatrix}$$

- $|A| = +1 \times 1 \times \begin{vmatrix}
2 & 3 \\
5 & 1
\end{vmatrix} + (-1) \times 0 \times \begin{vmatrix}
4 & 3 \\
-2 & 1
\end{vmatrix} + 1 \times -2 \times \begin{vmatrix}
4 & 2 \\
-2 & 5
\end{vmatrix}$

$$= -13 + (-2 \times 24) - 61$$

Matrix Inverse

- The inverse of a matrix describes the reverse transformation that the original matrix described.

- A matrix, $A$, multiplied by its inverse, $A^{-1}$, gives the identity matrix, $I$.

- That is: $AA^{-1} = I$ and $A^{-1}A = I$.
Matrix Inverse Example

- The reflection matrix, \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)
- The transformation required to undo the reflection is another reflection.
- \( A \) is its own inverse \( \Rightarrow A = A^{-1} \) and:
  \[
  \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
  \]

2 \times 2 Matrix inverse

- As usual things are easier for 2 \times 2 matrices. For:
  \[
  A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
  \]
- The inverse exists only if \( |A| \neq 0 \) and:
  \[
  A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
  \]
- \( \Rightarrow \) if \( |A| = 0 \) then the inverse \( A^{-1} \) does not exist (very important: true for any \( n \times n \) matrix).

n \times n Matrix Inverse

- First we need to define \( C \), the cofactors matrix of a matrix, \( A \), to have elements \( c_{ij} = \pm \) minor of \( a_{ij} \), using the parity matrix as before to determine whether is gets multiplied by \( +1 \) or \( -1 \)
  - (The minor of an element is the determinant of the matrix formed by deleting the row/column containing that element, as before)
- Then the \( n \times n \) inverse of \( A \) is:
  \[
  A^{-1} = \frac{1}{|A|} C^T
  \]

Linear Systems

- Linear systems are used in all branches of science and scientific computing
- Example of a simple linear system:
  - If 3 PCs and 5 Macs emit 151W of heat in 1 room, and 6 PCs together with 2 Macs emit 142W in another. How much energy does a single PC or Mac emit?
  - Here we have: \( 3p + 5m = 151 \) and \( 6p + 2m = 142 \)
Linear Systems as Matrix Equations

Our PC/Mac example can be rewritten as a matrix/vector equation:
\[
\begin{pmatrix}
3 & 5 \\
6 & 2
\end{pmatrix}
\begin{pmatrix}
p \\
m
\end{pmatrix}
- \begin{pmatrix}
151 \\
142
\end{pmatrix}
\]

Then a solution can be gained from inverting the matrix, so:
\[
\begin{pmatrix}
p \\
m
\end{pmatrix}
- \begin{pmatrix}
3 & 5 \\
6 & 2
\end{pmatrix}^{-1}
\begin{pmatrix}
151 \\
142
\end{pmatrix}
\]

Gaussian Elimination

For larger \(n \times n\) matrix systems finding the inverse is a lot of work

A simpler way of solving such systems in one go is by Gaussian Elimination. We rewrite the previous model as:
\[
\begin{pmatrix}
3 & 5 \\
6 & 2
\end{pmatrix}
\begin{pmatrix}
p \\
m
\end{pmatrix}
- \begin{pmatrix}
151 \\
142
\end{pmatrix}
\rightarrow
\begin{pmatrix}
3 & 5 & 151 \\
6 & 2 & 142
\end{pmatrix}
\]

We can perform operations on this matrix:
- multiply/divide any row by a scalar
- add/subtract any row to/from another

Gaussian Elimination

Using just these operations we aim to turn:
\[
\begin{pmatrix}
3 & 5 & 151 \\
6 & 2 & 142
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & x \\
0 & 1 & y
\end{pmatrix}
\]

Why? ...because in the previous matrix notation, this means:
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
p \\
m
\end{pmatrix}
- \begin{pmatrix}
x \\
y
\end{pmatrix}
\]

So \(x\) and \(y\) are our solutions

Example Solution using GE

\((r1) \rightarrow 2 \times (r1):\)
\[
\begin{pmatrix}
3 & 5 & 151 \\
6 & 2 & 142
\end{pmatrix}
\rightarrow
\begin{pmatrix}
3 & 10 & 302 \\
6 & 2 & 142
\end{pmatrix}
\]

\((r2) \rightarrow (r2) - (r1):\)
\[
\begin{pmatrix}
3 & 10 & 302 \\
6 & 2 & 142
\end{pmatrix}
\rightarrow
\begin{pmatrix}
6 & 10 & 302 \\
0 & -8 & -160
\end{pmatrix}
\]

\((r2) \rightarrow (r2)/(-8):\)
\[
\begin{pmatrix}
6 & 10 & 302 \\
0 & -8 & -160
\end{pmatrix}
\rightarrow
\begin{pmatrix}
6 & 10 & 302 \\
0 & 1 & 20
\end{pmatrix}
\]
Example Solution using GE

1. \((r1) := (r1) - 10 \times (r2):\)
   
   \[
   \begin{pmatrix}
   6 & 10 & 102 \\
   0 & 1 & 20
   \end{pmatrix} \Rightarrow \begin{pmatrix}
   6 & 0 & 102 \\
   0 & 1 & 20
   \end{pmatrix}
   \]

2. \((r1) := (r1)/6:\)
   
   \[
   \begin{pmatrix}
   6 & 0 & 102 \\
   0 & 1 & 20
   \end{pmatrix} \Rightarrow \begin{pmatrix}
   1 & 0 & 17 \\
   0 & 1 & 20
   \end{pmatrix}
   \]

So we can say that our solution is \(p = 17\) and \(m = 20\)

Gaussian Elimination: 3 \times 3

1. \[
\begin{pmatrix}
  a & * & * \\
  * & * & * \\
  * & * & *
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 & * & * \\
  * & * & * \\
  * & * & *
\end{pmatrix}
\]

2. \[
\begin{pmatrix}
  0 & b & * \\
  0 & * & * \\
  0 & * & *
\end{pmatrix} \rightarrow \begin{pmatrix}
  0 & 1 & * \\
  0 & * & * \\
  0 & * & *
\end{pmatrix}
\]

3. \[
\begin{pmatrix}
  0 & 1 & * \\
  0 & 0 & c \\
  0 & 0 & *
\end{pmatrix} \rightarrow \begin{pmatrix}
  0 & 1 & * \\
  0 & 0 & 1 \\
  0 & 0 & *
\end{pmatrix}
\]

* represents an unknown entry

Linear Dependence

- System of \(n\) equations is linearly dependent:
  - if one or more of the equations can be formed from a linear sum of the remaining equations
  - For example – if our Mac/PC system were:
    1. \(3p + 5m = 151\) \((1)\)
    2. \(6p + 10m = 302\) \((2)\)
  - This is linearly dependent as:
    - \(\text{eqn}\ (2) - 2 \times \text{eqn}\ (1)\)
    - i.e. we get no extra information from eqn (2)
  - ...and there is no single solution for \(p\) and \(m\)
**Linear Dependence**

- If $P$ represents a matrix in $P\vec{x} = \vec{b}$ then the equations generated by $P\vec{x}$ are linearly dependent
  - iff $|P| = 0$ (i.e. $P$ is singular)
- The rank of the matrix $P$ represents the number of linearly independent equations in $P\vec{x}$
- We can use Gaussian elimination to calculate the rank of a matrix

**Calculating the Rank**

- If after doing GE, and getting to the stage where we have zeroes under the leading diagonal, we have:
  
  $$
  \begin{pmatrix}
  1 & * & * \\
  0 & 1 & * \\
  0 & 0 & *
  \end{pmatrix}
  $$
- Then we have a linearly dependent system where the number of independent equations or rank is 2

---

**Rank and Nullity**

- If we consider multiplication by a matrix $M$ as a function:
  - $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$
  - Input set is called the domain
  - Set of possible outputs is called the range
- The Rank is the dimension of the range (i.e. the dimension of right-hand sides, $\vec{b}$, that give systems, $M\vec{x} = \vec{b}$, that don’t contradict)
- The Nullity is the dimension of space (subset of the domain) that maps onto a single point in the range.
  (Alternatively, the dimension of the space which solves $M\vec{x} = \vec{0}$).

**Rank/Nullity theorem**

- If we consider multiplication by a matrix $M$ as a function:
  - $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$
  - If rank is calculated from number of linearly independent rows of $M$: nullity is number of dependent rows
  - We have the following theorem:
    
    Rank of $M$ + Nullity of $M = \dim(\text{Domain of } M)$
PageRank Algorithm

- Used by Google (and others?) to calculate a ranking vector for the whole web!
- Ranking vector is used to order search results returned from a user query
- PageRank of a webpage, $r_v$, is proportional to:

$$
sum \frac{\text{PageRank of } v}{\text{Number of links out of } v} \times \text{pages with links to } v
$$

- For a PageRank vector, $r'$, and a web graph matrix, $P$:

$$
P r' = \lambda r'
$$

PageRank and Eigenvectors

- PageRank vector is an eigenvector of the matrix which defines the web graph
- An eigenvector, $\tilde{r}$, of a matrix $A$ is a vector which satisfies the following equation:

$$
A \tilde{r} = \lambda \tilde{r} \quad (*)
$$

- where $\lambda$ is an eigenvalue of the matrix $A$
- If $A$ is an $n \times n$ matrix then there may be as many as $n$ possible interesting $\tilde{r}, \lambda$ eigenvector/eigenvalue pairs which solve equation $(*)$

Calculating the eigenvector

- From the definition $(*)$ of the eigenvector,

$$
A \tilde{r} = \lambda \tilde{r}
$$

$$
\Rightarrow A \tilde{r} - \lambda \tilde{r} = 0
$$

$$
\Rightarrow (A - \lambda I) \tilde{r} = 0
$$

- Let $M$ be the matrix $A - \lambda I$ then if $|M| \neq 0$ then:

$$
\tilde{r} = M^{-1} \tilde{0}
$$

- This means that any interesting solutions of $(*)$ must occur when $|A - \lambda I| = 0$ thus:

$$
|A - \lambda I| = 0
$$

Eigenvector Example

- Find eigenvectors and eigenvalues of

$$
A = \begin{pmatrix}
4 & 1 \\
2 & 3
\end{pmatrix}
$$

- Using $|A - \lambda I| = 0$, we get:

$$
\begin{vmatrix}
4 - \lambda & 1 \\
2 & 3 - \lambda
\end{vmatrix} = 0
$$

$$
\Rightarrow \begin{vmatrix}
1 - \lambda & 0 \\
0 & 1 - \lambda
\end{vmatrix} = 0
$$

$$
\Rightarrow \begin{vmatrix}
4 - \lambda & 1 \\
2 & 3 - \lambda
\end{vmatrix} = 0
$$
Eigenvector Example

Thus by definition of a $2 \times 2$ determinant, we get:

$(4 - \lambda)(3 - \lambda) - 2 - 0$

This is just a quadratic equation in $\lambda$ which will give us two possible eigenvalues

$\lambda^2 - 7\lambda + 10 - 0$

$\Rightarrow (\lambda - 5)(\lambda - 2) - 0$

$\Rightarrow \lambda = 5 \text{ or } 2$

We have two eigenvalues and there will be one eigenvector solution for $\lambda = 5$ and another for $\lambda = 2$

Finding Eigenvectors

Given an eigenvalue, we now use equation (*) in order to find the eigenvectors. Thus $A\vec{v} = \lambda \vec{v}$ and $\lambda = 5$ gives:

\[
\begin{pmatrix}
4 & 1 \\
2 & 3
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
= 5
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
\]

\[
\begin{pmatrix}
4 & 1 \\
2 & 3
\end{pmatrix} - 5I
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
= \vec{0}
\]

\[
\begin{pmatrix}
-1 & 1 \\
2 & -2
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
= \begin{pmatrix}0 \\
0\end{pmatrix}
\]

Finding Eigenvectors

This gives us two equations in $v_1$ and $v_2$:

$-v_1 + v_2 = 0 \quad (1.a)$

$2v_1 - 2v_2 = 0 \quad (1.b)$

These are linearly dependent: which means that equation (1.b) is a multiple of equation (1.a) and vice versa

$(1.b) = -2 \times (1.a)$

This is expected in situations where $|M| = 0$ in $M\vec{v} = \vec{0}$

Eqn. (1.a) or (1.b) $\Rightarrow v_1 = v_2$

First Eigenvector

$v_1 - v_2$ gives us the $\lambda = 5$ eigenvector:

\[
\begin{pmatrix}
v_1 \\
v_1
\end{pmatrix}
= v_1 \begin{pmatrix}1 \\
1\end{pmatrix}
\]

We can ignore the scalar multiplier and use the remaining vector as the eigenvector

Checking with equation (*) gives:

\[
\begin{pmatrix}
4 & 1 \\
2 & 3
\end{pmatrix}
\begin{pmatrix}1 \\
1\end{pmatrix}
= 5 \begin{pmatrix}1 \\
1\end{pmatrix}
\]
**Second Eigenvector**

- For $A \vec{v} - \lambda \vec{v}$ and $\lambda = 2$: 
  \[
  \begin{pmatrix}
  2 & 1 \\
  2 & 1
  \end{pmatrix}
  \begin{pmatrix}
  v_1 \\
  v_2
  \end{pmatrix} = 
  \begin{pmatrix}
  0 \\
  0
  \end{pmatrix}
  \]
  \[
  2v_1 + v_2 = 0 \text{ (and } 2v_1 + v_2 = 0) \\
  v_2 = -2v_1
  \]
  - Thus second eigenvector is $\vec{v} = v_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
  - ...or just $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

---

**Differential Equations: Contents**

- What are differential equations used for?
  - Useful differential equation solutions:
    - 1st order, constant coefficient
    - 1st order, variable coefficient
    - 2nd order, constant coefficient
    - Coupled ODEs, 1st order, constant coefficient
    - Useful for:
      - Performance modelling (3rd year)
      - Simulation and modelling (3rd year)

---

**Differential Equations: Background**

- Used to model how systems evolve over time:
  - e.g. computer systems, biological systems, chemical systems
- Terminology:
  - Ordinary differential equations (ODEs) are first order if they contain a $\frac{dy}{dx}$ term but no higher derivatives
  - ODEs are second order if they contain a $\frac{d^2y}{dx^2}$ term but no higher derivatives

---

**Ordinary Differential Equations**

- First order, constant coefficients:
  - For example, $2 \frac{dy}{dx} + y = 0 \quad \text{(1)}$
  - Try: $y = e^{mx}$
  \[
  2me^{mx} + e^{mx} = 0 \\
  e^{mx} (2m + 1) = 0 \\
  e^{mx} = 0 \text{ or } m = -\frac{1}{2}
  \]
  - $e^{mx} \neq 0$ for any $x, m$. Therefore $m = -\frac{1}{2}$
  - General solution to (1):
  \[
  y = Ae^{\frac{-x}{2}}
  \]
Ordinary Differential Equations

- First order, variable coefficients of type:
  \[ \frac{dy}{dx} + f(x)y = g(x) \]
  
- Use integrating factor (IF): \( e^{\int f(x) \, dx} \)
  
  - For example: \( \frac{dy}{dx} + 2xy - x = 0 \)  
  
  - Multiply throughout by IF: \( e^{\int 2x \, dx} = e^{x^2} \)
    
    \[ e^{x^2} \frac{dy}{dx} + 2xe^{x^2}y - xe^{x^2} = 0 \]
    
    \[ \frac{d}{dx}(e^{x^2}y) - xe^{x^2} = 0 \]
    
    \[ e^{x^2}y - \frac{1}{2}e^{x^2} + C \]
    
    So, \( y = Ce^{-x^2} + \frac{1}{2} \)

Ordinary Differential Equations

- Second order, constant coefficients:
  
  - For example, \( \frac{d^2y}{dx^2} + \frac{5}{2} \frac{dy}{dx} + 6y = 0 \)  
  
  - Try: \( y = e^{mx} \)
    
    \[ m^2e^{mx} + 5me^{mx} + 6e^{mx} = 0 \]
    
    \[ e^{mx}(m^2 + 5m + 6) = 0 \]
    
    \[ e^{mx}(m + 3)(m + 2) = 0 \]
    
    \[ m = -3, -2 \]
    
    i.e. two possible solutions
    
    - General solution to \( (*) \):
      
      \[ y = A e^{-2x} + Be^{-3x} \]

Ordinary Differential Equations

- Second order, constant coefficients (repeated root):
  
  - For example, \( \frac{d^2y}{dx^2} - \frac{6}{5} \frac{dy}{dx} + 9y = 0 \)  
  
  - Try: \( y = e^{mx} \)
    
    \[ m^2e^{mx} - 6me^{mx} + 9e^{mx} = 0 \]
    
    \[ e^{mx}(m^2 - 6m + 9) = 0 \]
    
    \[ e^{mx}(m - 3)^2 = 0 \]
    
    \[ m = 3 \) (twice)
    
    - General solution to \( (*) \) for repeated roots:
      
      \[ y = (Ax + B)e^{3x} \]
Applications: Coupled ODEs

- Coupled ODEs are used to model massive state-space physical and computer systems
- Coupled Ordinary Differential Equations are used to model:
  - chemical reactions and concentrations
  - biological systems
  - epidemics and viral infection spread
  - large state-space computer systems (e.g. distributed publish-subscribe systems)

**Coupled ODEs**

- Coupled ODEs are of the form:
  \[
  \begin{align*}
  \frac{dy_1}{dx} &= a_{11}y_1 + a_{12}y_2 \\
  \frac{dy_2}{dx} &= a_{21}y_1 + a_{22}y_2 
  \end{align*}
  \]

- If we let \( \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \), we can rewrite this as:
  \[
  \begin{pmatrix}
  \frac{dy_1}{dx} \\
  \frac{dy_2}{dx}
  \end{pmatrix} =
  \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
  \end{pmatrix}
  \begin{pmatrix}
  y_1 \\
  y_2
  \end{pmatrix}
  \quad \text{or}
  \quad \frac{d\vec{y}}{dx} = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
  \end{pmatrix} \vec{y}
  \]

**Coupled ODE solutions**

- For coupled ODE of type: \( \frac{d\vec{y}}{dx} = A\vec{y} \) \( (*) \)
- Try \( \vec{y} = \vec{v}e^{\lambda x} \) so \( \frac{d\vec{v}}{dx} = \lambda \vec{v}e^{\lambda x} \)
- But also \( \frac{d\vec{y}}{dx} = A\vec{y} \), so \( A\vec{v}e^{\lambda x} = \lambda \vec{v}e^{\lambda x} \)
- Now solution of \( (*) \) can be derived from an eigenvector solution of \( A\vec{v} = \lambda \vec{v} \)
- For \( n \) eigenvectors \( \vec{v}_1, \ldots, \vec{v}_n \) and corresp. eigenvalues \( \lambda_1, \ldots, \lambda_n \): general solution of \( (*) \) is \( \vec{y} = B_1\vec{v}_1e^{\lambda_1x} + \cdots + B_n\vec{v}_n e^{\lambda_nx} \)

**Coupled ODEs: Example**

- Example coupled ODEs:
  \[
  \begin{align*}
  \frac{dy_1}{dx} &= 2y_1 + 8y_2 \\
  \frac{dy_2}{dx} &= 5y_1 + 5y_2 
  \end{align*}
  \]
- So \( \frac{d\vec{y}}{dx} = \begin{pmatrix} 2 & 8 \\ 5 & 5 \end{pmatrix} \vec{y} \)
- Require to find eigenvectors/values of
  \[
  A = \begin{pmatrix} 2 & 8 \\ 5 & 5 \end{pmatrix}
  \]
**Coupled ODEs: Example**

- Eigenvalues of $A$: $\begin{pmatrix} 2 - \lambda & 8 \\ 5 & 5 - \lambda \end{pmatrix}$
  
  $\lambda^2 - 7\lambda - 30 = (\lambda - 10)(\lambda + 3) = 0$

- Thus eigenvalues $\lambda = 10, -3$

- Giving:
  
  $\lambda_1 = 10, \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -3, \bar{v}_2 = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$

- Solution of ODEs:
  
  $\vec{y} = B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10x} + B_2 \begin{pmatrix} 8 \\ -5 \end{pmatrix} e^{-3x}$

**Partial Derivatives**

- Used in (amongst others):
  
  - Computational Techniques (2nd Year)
  - Optimisation (3rd Year)
  - Computational Finance (3rd Year)

**Differentiation Contents**

- What is a (partial) differentiation used for?

- Useful (partial) differentiation tools:
  
  - Differentiation from first principles
  - Partial derivative chain rule
  - Derivatives of a parametric function
  - Multiple partial derivatives

**Optimisation**

- Example: look to find best predicted gain in portfolio given different possible shareholdings in portfolio
Differentiation

- Gradient on a curve $f(x)$ is approximately:
  \[
  \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}
  \]

Definition of derivative

- The derivative at a point $x$ is defined by:
  \[
  \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
  \]

- Take $f(x) = x^n$
  - We want to show that:
    \[
    \frac{df}{dx} = nx^{n-1}
    \]

Derivative of $x^n$

- \[
  \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
  \]
- \[
  = \lim_{\Delta x \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k} \Delta x^k}{\Delta x}
  \]
- \[
  = \lim_{\Delta x \to 0} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \Delta x^{k-1}
  \]
- \[
  = \lim_{\Delta x \to 0} \left( \binom{n}{0} x^n + \sum_{k=1}^{n} \binom{n}{k} x^{n-k} \Delta x^{k-1} \right)
  \]
- \[
  = \frac{n!}{(n-1)!} x^{n-1} \quad \text{as} \quad \Delta x \to 0
  \]

Partial Differentiation

- Ordinary differentiation $\frac{df}{dx}$ applies to functions of one variable i.e. $f \equiv f(x)$
- What if function $f$ depends on one or more variables e.g. $f \equiv f(x_1, x_2)$
- Finding the derivative involves finding the gradient of the function by varying one variable and keeping the others constant
- For example for $f(x, y) = x^2 y + xy^3$; partial derivatives are written:
  \[
  \frac{\partial f}{\partial x} = 2xy + y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + 3xy^2
  \]
**Partial Derivative: example**

- $f(x, y) = x^2 + y^2$

**Extended Chain Rule**

- If $f$ is a function of $x$ and $y$ where $x$ and $y$ are themselves functions of $s$ and $t$ then:
  \[ \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \]
  \[ \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \]

- which can be expressed as a matrix equation:
  \[
  \begin{pmatrix}
  \frac{\partial f}{\partial s} \\
  \frac{\partial f}{\partial t}
  \end{pmatrix}
  =
  \begin{pmatrix}
  \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}
  \end{pmatrix}
  \begin{pmatrix}
  \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\
  \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t}
  \end{pmatrix}
  \]

- Useful for changes of variable e.g. to polar coordinates

**Further Examples**

- $f(x, y) = (x + 2y^3)^2$
  \[ \Rightarrow \frac{\partial f}{\partial x} = 2(x + 2y^3) \frac{\partial}{\partial x} (x + 2y^3) - 2(x + 2y^3) \]
- If $x$ and $y$ are themselves functions of $t$ then
  \[ \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \]
- So if $f(x, y) = x^2 + 2y$ where $x = \sin t$ and $y = \cos t$ then:
  \[ \frac{df}{dt} = 2x \cos t - 2\sin t - 2\sin t (\cos t - 1) \]
Jacobian

- The modulus of this matrix is called the Jacobian:
  \[
  J = \begin{vmatrix}
  \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
  \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
  \end{vmatrix}
  \]
- Just as when performing a substitution on the integral:
  \[
  \int f(x) \, dx
  \]
  we would use: \( du \equiv \frac{\partial f(x)}{\partial x} \, dx \)
- So if converting between multiple variables in an integration, we would use \( du \equiv J \, dx \).

Formal Definition

- Similar to ordinary derivative. For a two variable function \( f(x, y) \):
  \[
  \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}
  \]
  and in the \( y \)-direction:
  \[
  \frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}
  \]

Further Notation

- Multiple partial derivatives (as for ordinary derivatives) are expressed:
  \- \( \frac{\partial^2 f}{\partial x^2} \) is the second partial derivative of \( f \)
  \- \( \frac{\partial^2 f}{\partial y^2} \) is the \( n \)-th partial derivative of \( f \)
  \- \( \frac{\partial^2 f}{\partial x \partial y} \) is the partial derivative obtained by first partial differentiating by \( y \) and then \( x \)
  \- \( \frac{\partial^2 f}{\partial y \partial x} \) is the partial derivative obtained by first partial differentiating by \( x \) and then \( y \)
- If \( f(x, y) \) is a \textit{nice} function then: \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \)