Differential Equations: Contents

- What are differential equations used for?
- Useful differential equation solutions:
  - 1st order, constant coefficient
  - 1st order, variable coefficient
  - 2nd order, constant coefficient
  - Coupled ODEs, 1st order, constant coefficient
- Useful for:
  - Performance modelling (3rd year)
  - Simulation and modelling (3rd year)

Differential Equations: Background

- Used to model how systems evolve over time:
  - e.g. computer systems, biological systems, chemical systems
- Terminology:
  - Ordinary differential equations (ODEs) are first order if they contain a $\frac{dy}{dx}$ term but no higher derivatives
  - ODEs are second order if they contain a $\frac{d^2y}{dx^2}$ term but no higher derivatives

Ordinary Differential Equations

- First order, constant coefficients:
  - For example, $2\frac{dy}{dx} + y = 0$ (*)
  - Try: $y = e^{mx}$
    $\Rightarrow 2me^{mx} + e^{mx} = 0$
    $\Rightarrow e^{mx}(2m + 1) = 0$
    $\Rightarrow e^{mx} = 0$ or $m = -\frac{1}{2}$
  - $e^{mx} \neq 0$ for any $x, m$. Therefore $m = -\frac{1}{2}$
  - General solution to (*):
    $$y = Ae^{-\frac{1}{2}x}$$
Ordinary Differential Equations

- First order, variable coefficients of type:
  \[ \frac{dy}{dx} + f(x)y = g(x) \]

- Use integrating factor (IF): \( e^{\int f(x) \, dx} \)
  - For example: \( \frac{dy}{dx} + 2xy = x (\ast) \)
  - Multiply throughout by IF: \( e^{\int 2x \, dx} = e^{x^2} \)
  \[ e^{x^2} \frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2} \]
  \[ \frac{d}{dx}(e^{x^2}y) = xe^{x^2} \]
  \[ e^{x^2}y = \frac{1}{2}e^{x^2} + C \]
  - So, \( y = Ce^{-x^2} + \frac{1}{2} \)

Ordinary Differential Equations

- Second order, constant coefficients:
  - For example, \( \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 (\ast) \)
  - Try: \( y = e^{mx} \)
    \[ m^2e^{mx} + 5me^{mx} + 6e^{mx} = 0 \]
    \[ e^{mx}(m^2 + 5m + 6) = 0 \]
    \[ e^{mx}(m + 3)(m + 2) = 0 \]
    \( m = -3, -2 \)
    - i.e. two possible solutions
  - General solution to (\ast):
    \[ y = Ae^{-2x} + Be^{-3x} \]

Ordinary Differential Equations

- Second order, constant coefficients (repeated root):
  - For example, \( \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 (\ast) \)
  - Try: \( y = e^{mx} \)
    \[ m^2e^{mx} - 6me^{mx} + 9e^{mx} = 0 \]
    \[ e^{mx}(m^2 - 6m + 9) = 0 \]
    \[ e^{mx}(m - 3)^2 = 0 \]
    \( m = 3 \text{ (twice)} \)
  - General solution to (\ast) for repeated roots:
    \[ y = (Ax + B)e^{3x} \]
Applications: Coupled ODEs

- Coupled ODEs are used to model massive state-space physical and computer systems.
- Coupled Ordinary Differential Equations are used to model:
  - chemical reactions and concentrations
  - biological systems
  - epidemics and viral infection spread
  - large state-space computer systems (e.g. distributed publish-subscribe systems)

Coupled ODEs

- Coupled ODEs are of the form:
  \[
  \begin{align*}
  \frac{dy_1}{dx} &= ay_1 + by_2 \\
  \frac{dy_2}{dx} &= cy_1 + dy_2
  \end{align*}
  \]

- If we let \( \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \), we can rewrite this as:
  \[
  \frac{d\vec{y}}{dx} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{y} \quad \text{or} \quad \frac{d\vec{y}}{dx} = \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
  \]

Coupled ODE solutions

- For coupled ODE of type: \( \frac{d\vec{y}}{dx} = A\vec{y} \) \((*)\)
- Try \( \vec{y} = \vec{v}e^{\lambda x} \) so, \( \frac{d\vec{y}}{dx} = \lambda \vec{v}e^{\lambda x} \)
- But also \( \frac{d\vec{y}}{dx} = A\vec{y} \), so \( A\vec{v}e^{\lambda x} = \lambda \vec{v}e^{\lambda x} \)
- Now solution of \((*)\) can be derived from an eigenvector solution of \( A\vec{v} = \lambda \vec{v} \)
- For \( n \) eigenvectors \( \vec{v}_1, \ldots, \vec{v}_n \) and corresponding eigenvalues \( \lambda_1, \ldots, \lambda_n \), the general solution of \((*)\) is \( \vec{y} = B_1\vec{v}_1e^{\lambda_1 x} + \cdots + B_n\vec{v}_n e^{\lambda_n x} \)

Coupled ODEs: Example

- Example coupled ODEs:
  \[
  \begin{align*}
  \frac{dy_1}{dx} &= 2y_1 + 8y_2 \\
  \frac{dy_2}{dx} &= 5y_1 + 5y_2
  \end{align*}
  \]
- So \( \frac{d\vec{y}}{dx} = \begin{pmatrix} 2 & 8 \\ 5 & 5 \end{pmatrix} \vec{y} \)
- Require to find eigenvectors/values of \( A = \begin{pmatrix} 2 & 8 \\ 5 & 5 \end{pmatrix} \)
**Coupled ODEs: Example**

- Eigenvalues of $A$:
  
  $\begin{vmatrix} 2 - \lambda & 8 \\ 5 & 5 - \lambda \end{vmatrix} = 0$

  $\lambda^2 - 7\lambda - 30 = (\lambda - 10)(\lambda + 3) = 0$

- Thus eigenvalues $\lambda = 10, -3$

- Giving:
  
  $\lambda_1 = 10, \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -3, \vec{v}_2 = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$

- Solution of ODEs:
  
  $\vec{y} = B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10x} + B_2 \begin{pmatrix} 8 \\ -5 \end{pmatrix} e^{-3x}$

**Partial Derivatives**

- Used in (amongst others):
  - Computational Techniques (2nd Year)
  - Optimisation (3rd Year)
  - Computational Finance (4th Year)

**Differentiation Contents**

- What is a (partial) differentiation used for?

- Useful (partial) differentiation tools:
  - Differentiation from first principles
  - Partial derivative chain rule
  - Derivatives of a parametric function
  - Multiple partial derivatives

**Optimisation**

- Example: look to find best predicted gain in portfolio given different possible shareholdings in portfolio
Differentiation

- Gradient on a curve $f(x)$ is approximately:

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Definition of derivative

- The derivative at a point $x$ is defined by:

$$\frac{df}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

- Take $f(x) = x^n$

We want to show that:

$$\frac{df}{dx} = nx^{n-1}$$

Partial Differentiation

- Ordinary differentiation $\frac{df}{dx}$ applies to functions of one variable i.e. $f \equiv f(x)$

- What if function $f$ depends on one or more variables e.g. $f \equiv f(x_1, x_2)$

- Finding the derivative involves finding the gradient of the function by varying one variable and keeping the others constant

- For example for $f(x, y) = x^2y + xy^3$; partial derivatives are written:

$$\frac{\partial f}{\partial x} = 2xy + y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + 3xy^2$$
Partial Derivative: example

- \( f(x, y) = x^2 + y^2 \)

Further Examples

- \( f(x, y) = (x + 2y)^2 \)
  \[ \frac{\partial f}{\partial x} = 2(x + 2y) \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} \right) = 2(x + 2y) \]
- If \( x \) and \( y \) are themselves functions of \( t \) then:
  \[ \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \]
- So if \( f(x, y) = x^2 + 2y \) where \( x = \sin t \) and \( y = \cos t \) then:
  \[ \frac{df}{dt} = 2x \cos t - 2 \sin t = 2 \sin t \cos t - 1 \]

Extended Chain Rule

- If \( f \) is a function of \( x \) and \( y \) where \( x \) and \( y \) are themselves functions of \( s \) and \( t \) then:
  \[ \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \]
- which can be expressed as a matrix equation:
  \[ \begin{pmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial f}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \]
- Useful for changes of variable e.g. to polar coordinates
The modulus of this matrix is called the **Jacobian**:

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\
\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t}
\end{vmatrix}
\]

Just as when performing a substitution on the integral:

\[
\int f(x) \, dx
\]

we would use: \( du \equiv \frac{df(x)}{dx} \, dx \)

So if converting between multiple variables in an integration, we would use \( du \equiv J \, dx \).

**Formal Definition**

Similar to ordinary derivative. For a two variable function \( f(x, y) \):

\[
\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}
\]

and in the \( y \)-direction:

\[
\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}
\]

**Further Notation**

Multiple partial derivatives (as for ordinary derivatives) are expressed:

- \( \frac{\partial^2 f}{\partial x^2} \) is the second partial derivative of \( f \)
- \( \frac{\partial^2 f}{\partial x \partial y} \) is the \( n \)th partial derivative of \( f \)
- \( \frac{\partial f}{\partial x \partial y} \) is the partial derivative obtained by first partial differentiating by \( y \) and then \( x \)
- \( \frac{\partial f}{\partial y \partial x} \) is the partial derivative obtained by first partial differentiating by \( x \) and then \( y \)

If \( f(x, y) \) is a *nice* function then:

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}
\]