Tutorial 4

11·2005

Douglas de Jager
dvd Ø3
Differential Equations

1st Order

Separable Variables

These equations take the form: \( f(y) \frac{dy}{dx} = g(x) \)
where \( f \) is a function just of \( y \) and \( g \) is a function just of \( x \).

Integrating both sides with respect to \( x \) gives:
\[
\int f(y) \frac{dy}{dx} \, dx = \int g(x) \, dx
\]
\[
\Rightarrow \int f(y) \, dy = \int g(x) \, dx
\]

This last line gives us the general solution.

\[E.g. 1\]
\[
\Rightarrow \int y \, \frac{dy}{dx} = \int x^2 \, dx
\]
\[
\Rightarrow \int y \, dy = \int x^2 \, dx
\]
\[
\Rightarrow \frac{1}{2} y^2 = \frac{1}{3} x^3 + C
\]

\[E.g. 2\]
\[
2 \frac{dy}{dx} + y = 0
\]
\[
\Rightarrow 2 \frac{dy}{dx} = -y
\]
\[
\Rightarrow \frac{1}{y} \frac{dy}{dx} = -\frac{1}{2}
\]
\[
\Rightarrow \int \frac{1}{y} \, dy = \int -\frac{1}{2} \, dx
\]
\[
\Rightarrow \ln|y| = -\frac{x}{2} + C
\]
\[
\Rightarrow y = Ce^{-\frac{x}{2}}
\]
Exact 1st Order D.E.

Let \( v(x) \) and \( u(x) \) be functions of \( x \).

Let \( u(x,y) \) be a function of \( x \) and \( y \) defined as follows: \( u(x,y) = u(x) \cdot y \),
that is, \( y \) times \( u(x) \).

Now, recall the product rule:
\[
\frac{d}{dx} (uv) = v \frac{du}{dx} + u \frac{dv}{dx}
\]

We term an exact 1st order d.e. an equation of the form:
\[
f(x) \frac{dy}{dx} + g(x) y = h(x)
\]

such that there is some function product \( u(x,y) \cdot v(x) \) given which
\[
\frac{d}{dx} (uv) = f(x) \frac{dy}{dx} + g(x) y
\]

That is to say, some function product \( uv \) which has as its derivative
the LHS of the d.e.

Using the anti-derivative rule, we can integrate an exact d.e.
with respect to \( x \) to give general solution:

\[
u(x,y) \cdot v(x) = \int h(x) \, dx
\]

E.g.

\[
x \frac{dy}{dx} + y = e^x
\]

LHS is the derivative of \( uv \) where \( \begin{cases} u &= y \\ v &= x \end{cases} \)

\[\Rightarrow \quad xy = \int e^x \, dx \]
\[\Rightarrow \quad xy = e^x + k\]
Inexact (Integrating Factor)

Consider an inexact equation of the form: \( \frac{dy}{dx} + q(x)y = h(x) \)

Let \( I(x) \) be a function of \( x \) such that \( I(x)\frac{dy}{dx} + I(x)q(x)y = I(x)h(x) \)
is exact.

We term such an \( I(x) \) an \textit{integrating factor}.

Now, integrating factors and the associated general solution of inexact d.e.s happen to take particular forms. Consider:

**I.F.**

\[
\begin{align*}
I \frac{dy}{dx} + Iqy & \text{ with } v\frac{du}{dx} + u\frac{dv}{dx} \\
\begin{cases}
v = I \quad &\text{**} \\
\frac{du}{dx} = \frac{dy}{dx} \\
u = y \\
\frac{dv}{dx} = qI \quad &\text{***} \\
\end{cases}
\end{align*}
\]

**I.F.** gives us that

\[
\frac{dI}{dx} = qI \quad &\text{***}
\]

Solving ***** as a 1st order d.e. with separable variables gives:

\[
\begin{align*}
S \frac{1}{I} \frac{dI}{dx} &= S q \frac{dx}{dx} \\
\Rightarrow \ln|I| &= S q \frac{dx}{dx} \\
\Rightarrow I &= e^{S q \frac{dx}{dx}} \\
\end{align*}
\]

Form taken by integrating factor

\[
\begin{align*}
\text{Gen. Sol.} \quad \text{using the assignment of } u, v, \frac{du}{dx}, \frac{dv}{dx} \text{ above (between ** and ***)}, \\
\text{and employing the methodology used for exact equations, we get:}
\end{align*}
\]

\[
\begin{align*}
I \frac{dy}{dx} + Iqy &= Ih \\
\Rightarrow S (I \frac{dy}{dx} + Iqy) dx &= S Ih dx \\
\Rightarrow Iy &= S Ih dx \quad \text{This is the gen. sol}
\end{align*}
\]
We'll consider 3 cases:

**Case (1)**

Suppose $y = Ae^{\alpha x} + Be^{\beta x}$

where $A, \alpha, B, \beta$ are arbitrary real constants, and $\alpha$ and $\beta$ are distinct ($\alpha \neq \beta$).

\[ \frac{dy}{dx} = A\alpha e^{\alpha x} + B\beta e^{\beta x} \]

\[ \frac{d^2y}{dx^2} = A\alpha^2 e^{\alpha x} + B\beta^2 e^{\beta x} \]

Substituting * and ** into *** to remove $A$ and $B$ yields:

\[ \frac{d^2y}{dx^2} - (\alpha + \beta) \frac{dy}{dx} + \alpha \beta y = 0 \]

We term this equation with $\alpha + \beta$ on RHS homogeneous. We can solve inhomogeneous equations, but this is beyond scope of course.

Compare **** with the quadratic equation $\alpha^2 + (\alpha + \beta)\alpha + \beta \beta = 0$

We term this the **auxiliary** quadratic equation and use it to recognize the general solution of equations of the form:

\[ \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \]

If the auxiliary equation $\alpha^2 + (\alpha + \beta)\alpha + \beta \beta = 0$ has real, distinct roots (i.e., $\Delta = \beta^2 - 4\alpha \beta > 0$), then we can quote (by way of the anti-derivative rule) that the general solution is:

\[ y = Ae^{\alpha x} + Be^{\beta x} \]
It is beyond the scope of the course to show that this solution, with \( a \neq 0 \) and \( B \), is the general solution of \( **** \).

The reader will need either just to accept the fact, or, if interested, to consider the principle of superposition:

Let \( y_1 \) and \( y_2 \) be two solutions of \( **** \).

(i) For any constants \( c_1 \) and \( c_2 \), \( c_1 y_1 + c_2 y_2 \) is also a solution of \( **** \).

(ii) Let \( y \) be any other solution of \( **** \). If \( y_2 \) is not a constant multiple of \( y_1 \) and \( y_1 \neq 0 \), then there exists some constant \( k_1 \) and some constant \( k_2 \) such that \( y = k_1 y_1 + k_2 y_2 \).
Case (2)

Suppose \( y := e^{\alpha x} (A + Bx) \)

where \( A, B, \alpha \) are arbitrary, real constants.

\[ \Rightarrow \frac{dy}{dx} = \alpha y + B e^{\alpha x} \]

\[ \Rightarrow \frac{d^2y}{dx^2} = \alpha \frac{dy}{dx} + B \alpha e^{\alpha x} \]

\[ = \alpha \frac{dy}{dx} + \alpha \left( \frac{dy}{dx} - \alpha y \right) \]

\[ = 2 \alpha \frac{dy}{dx} - \alpha^2 y \]

\[ \Rightarrow \frac{d^2y}{dx^2} - 2 \alpha \frac{dy}{dx} + \alpha^2 y = 0 \]

So, by reasoning similarly to case (1),

\[ y = e^{\alpha x} (A + Bx) \]

gives us the general solution of the differential equation

\[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0 \]

provided the auxiliary equation \( \alpha^2 + \beta^2 + c = 0 \) has equal roots, namely \( \alpha \) \((\beta^2 - 4 \alpha c = 0)\)
Case (3): Not covered in lectures.

For the interested reader, consider \( \frac{dy}{dx} \) and \( \frac{d^2 y}{dx^2} \) of

\[
 y = A e^{px} \cos(qx + r) \\
 = e^{px} (B \cos qx + C \sin qx)
\]

By compound angle formula

By proceeding analogously to earlier cases, it can be shown that

\[
 \frac{a d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0
\]

has general solution

\[
 y = A e^{px} \cos(qx + r)
\]

if \( b^2 - 4ac < 0 \) (auxiliary equation has complex roots)