Mathematical Methods for Computer Science

Peter Harrison and Jeremy Bradley
Email: {pgh, jb}@doc.ic.ac.uk
Web pages: http://www.doc.ic.ac.uk/~jb/teaching/145/
Room 372. Department of Computing, Imperial College London

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## Vectors

- Used in (amongst others):
  - Computational Techniques (2nd Year)
  - Graphics (3rd Year)
  - Computational Finance (3rd Year)
  - Modelling and Simulation (3rd Year)
  - Performance Analysis (3rd Year)
  - Digital Libraries and Search Engines (3rd Year)
  - Computer Vision (4th Year)

## Vector Contents

- **What is a vector?**
- **Useful vector tools:**
  - Vector magnitude
  - Vector addition
  - Scalar multiplication
  - Dot product
  - Cross product
- **Useful results – finding the intersection of:**
  - a line with a line
  - a line with a plane
  - a plane with a plane

## What is a vector?

- A vector is used:
  - to convey *both* direction and magnitude
  - to store data (usually numbers) in an ordered form

- \( \vec{p} = (10, 5, 7) \) is a row vector

- \( \vec{p} = \begin{pmatrix} 10 \\ 5 \\ 7 \end{pmatrix} \) is a column vector

- A vector is used in computer graphics to represent the position coordinates for a point
**What is a vector?**

- The dimension of a vector is given by the number of elements it contains. E.g.
  - \((-2.4, 5.1)\) is a 2-dimensional real vector
  - \((-2.4, 5.1)\) comes from set \(\mathbb{R}^2\) (or \(\mathbb{R} \times \mathbb{R}\))
  - \(
    \begin{pmatrix}
      -2 \\
      5 \\
      7 \\
      0
    \end{pmatrix}
  \) is a 4-dimensional integer vector
  (comes from set \(\mathbb{Z}^4\) or \(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\))

**Vector Magnitude**

- The size or magnitude of a vector \(\vec{p} = (p_1, p_2, p_3)\) is defined as its length:
  \[
  |\vec{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2} = \sqrt{\sum_{i=1}^{3} p_i^2}
  \]
  - e.g. \(
    \begin{pmatrix}
      3 \\
      4 \\
      5
    \end{pmatrix}
  \) \(= \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}\)

- For an \(n\)-dimensional vector, \(\vec{p} = (p_1, p_2, \ldots, p_n)\), \(|\vec{p}| = \sqrt{\sum_{i=1}^{n} p_i^2}\)

**Vector Direction**

**Vector Angles**

- For a vector, \(\vec{p} = (p_1, p_2, p_3)\):
  - \(\cos(\theta_x) = \frac{p_1}{|\vec{p}|}\)
  - \(\cos(\theta_y) = \frac{p_2}{|\vec{p}|}\)
  - \(\cos(\theta_z) = \frac{p_3}{|\vec{p}|}\)
Vector addition

- Two vectors (of the same dimension) can be added together:
- e.g. \[
\begin{pmatrix}
1 \\
2 \\
-1
\end{pmatrix}
+ \begin{pmatrix}
1 \\
-1 \\
4
\end{pmatrix}
= \begin{pmatrix}
2 \\
1 \\
3
\end{pmatrix}
\]
- So if \( \vec{p} = (p_1, p_2, p_3) \) and \( \vec{q} = (q_1, q_2, q_3) \) then:
  \[
\vec{p} + \vec{q} = (p_1 + q_1, p_2 + q_2, p_3 + q_3)
\]

Scalar Multiplication

- A scalar is just a number, e.g. 3. Unlike a vector, it has no direction.
- Multiplication of a vector \( \vec{p} \) by a scalar \( \lambda \) means that each element of the vector is multiplied by the scalar
- So if \( \vec{p} = (p_1, p_2, p_3) \) then:
  \[
\lambda \vec{p} = (\lambda p_1, \lambda p_2, \lambda p_3)
\]
3D Unit vectors

- We use \( \vec{i}, \vec{j}, \vec{k} \) to define the 3 unit vectors in 3 dimensions.
- They convey the basic directions along \( x, y \) and \( z \) axes.
- So: \( \vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), \( \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \), \( \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \)
- All unit vectors have magnitude 1; i.e. \( |\vec{i}| = 1 \)

Vector notation

- All vectors in 3D (or \( \mathbb{R}^3 \)) can be expressed as weighted sums of \( \vec{i}, \vec{j}, \vec{k} \).
- i.e. \( \vec{p} = (10, 5, 7) \equiv \begin{pmatrix} 10 \\ 5 \\ 7 \end{pmatrix} \equiv 10\vec{i} + 5\vec{j} + 7\vec{k} \)
- \( |p_1\vec{i} + p_2\vec{j} + p_3\vec{k}| = \sqrt{p_1^2 + p_2^2 + p_3^2} \)

Dot Product

- Also known as: scalar product
- Used to determine how close 2 vectors are to being parallel/perpendicular.
- The dot product of two vectors \( \vec{p} \) and \( \vec{q} \) is:
  \[ \vec{p} \cdot \vec{q} = |\vec{p}| |\vec{q}| \cos \theta \]
- where \( \theta \) is angle between the vectors \( \vec{p} \) and \( \vec{q} \)
- For \( \vec{p} = (p_1, p_2, p_3) \) and \( \vec{q} = (q_1, q_2, q_3) \) then:
  \[ \vec{p} \cdot \vec{q} = p_1q_1 + p_2q_2 + p_3q_3 \]

Properties of the Dot Product

- \( \vec{p} \cdot \vec{p} = |\vec{p}|^2 \)
- \( \vec{p} \cdot \vec{q} = 0 \) if \( \vec{p} \) and \( \vec{q} \) are perpendicular (at right angles)
- Commutative: \( \vec{p} \cdot \vec{q} = \vec{q} \cdot \vec{p} \)
- Linearity: \( \vec{p} \cdot (\lambda \vec{q}) = \lambda (\vec{p} \cdot \vec{q}) \)
- Distributive over addition:
  \[ \vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r} \]
Vector Projection

\[ \vec{a} \cdot \hat{n} \]

- \( \hat{n} \) is a unit vector, i.e. \( |\hat{n}| = 1 \)
- \( \vec{a} \cdot \hat{n} = |\vec{a}| \cos \theta \) represents the amount of \( \vec{a} \) that points in the \( \hat{n} \) direction

What can’t you do with a vector...

The following are classic mistakes – \( \vec{u} \) and \( \vec{v} \) are vectors, and \( \lambda \) is a scalar:

- Don’t do it!
  - Vector division: \( \frac{\vec{v}}{\vec{u}} \)
  - Divide a scalar by a vector: \( \frac{\lambda}{\vec{u}} \)
  - Add a scalar to a vector: \( \lambda + \vec{u} \)
  - Subtract a scalar from a vector: \( \vec{u} - \lambda \)
  - Cancel a vector in a dot product with vector:
    \[ \frac{1}{\vec{a} \cdot \vec{n}} = \frac{1}{|\vec{a}|} \]

Example: Rays of light

- A ray of light strikes a reflective surface...
- Question: in what direction does the reflection travel?

Rays of light

[Diagram of rays of light reflecting off a surface]
Problem: find $\vec{r}$, given $\vec{s}$ and $\hat{n}$?

- angle of incidence = angle of reflection
  $\Rightarrow -\vec{s} \cdot \hat{n} = \vec{r} \cdot \hat{n}$
- Also: $\vec{r} + (-\vec{s}) = \lambda \hat{n}$ thus $\lambda \hat{n} = \vec{r} - \vec{s}$
- Taking the dot product of both sides:
  $\Rightarrow \lambda |\hat{n}|^2 = \vec{r} \cdot \hat{n} - \vec{s} \cdot \hat{n}$

But $\hat{n}$ is a unit vector, so $|\hat{n}|^2 = 1$
  $\Rightarrow \lambda = \vec{r} \cdot \hat{n} - \vec{s} \cdot \hat{n}$

...and $\vec{r} \cdot \hat{n} = -\vec{s} \cdot \hat{n}$
  $\Rightarrow \lambda = -2\vec{s} \cdot \hat{n}$

Finally, we know that: $\vec{r} + (-\vec{s}) = \lambda \hat{n}$
  $\Rightarrow \vec{r} = \lambda \hat{n} + \vec{s}$
  $\Rightarrow \vec{r} = \vec{s} - 2(\vec{s} \cdot \hat{n})\hat{n}$
**Equation of a line**

For a general point, \( \vec{r} \), on the line:

\[ \vec{r} = \vec{a} + \lambda \vec{d} \]

where: \( \vec{a} \) is a point on the line and \( \vec{d} \) is a vector parallel to the line.

**Equation of a plane**

- Equation of a plane. For a general point, \( \vec{r} \), in the plane, \( \vec{r} \) has the property that:
  
  \[ \vec{r} \cdot \hat{n} = m \]

  where:
  - \( \hat{n} \) is the unit vector perpendicular to the plane
  - \( |m| \) is the distance from the plane to the origin (at its closest point)

**How to solve Vector Problems**

1. IMPORTANT: Draw a diagram!
2. Write down the equations that you are given/apply to the situation
3. Write down what you are trying to find?
4. Try variable substitution
5. Try taking the dot product of one or more equations
   - What vector to dot with?

Answer: if eqn (1) has term \( \vec{r} \) in and eqn (2) has term \( \vec{r} \cdot \vec{s} \) in: dot eqn (1) with \( \vec{s} \).
Two intersecting lines

- Application: projectile interception
- Problem — given two lines:
  - Line 1: $\vec{r}_1 = \vec{a}_1 + t_1 \vec{d}_1$
  - Line 2: $\vec{r}_2 = \vec{a}_2 + t_2 \vec{d}_2$
- Do they intersect? If so, at what point?
- This is the same problem as: find the values $t_1$ and $t_2$ at which $\vec{r}_1 = \vec{r}_2$ or:
  $$\vec{a}_1 + t_1 \vec{d}_1 = \vec{a}_2 + t_2 \vec{d}_2$$

How to solve: 2 intersecting lines

- Separate $\vec{i}$, $\vec{j}$, $\vec{k}$ components of equation:
  $$\vec{a}_1 + t_1 \vec{d}_1 = \vec{a}_2 + t_2 \vec{d}_2$$
- ...to get 3 equations in $t_1$ and $t_2$
- If the 3 equations:
  - contradict each other then the lines do not intersect
  - produce a single solution then the lines do intersect
  - are all the same (or multiples of each other) then the lines are identical (and always intersect)

Intersection of a line and plane

- Application: ray tracing, particle tracing, projectile tracking
- Problem — given one line/one plane:
  - Line: $\vec{r} = \vec{a} + t \vec{d}$
  - Plane: $\vec{r} \cdot \vec{n} = s$
- Take dot product of line equation with $\vec{n}$ to get:
  $$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} + t(\vec{d} \cdot \vec{n})$$

Intersection of a line and plane

- With $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} + t(\vec{d} \cdot \vec{n})$ — what are we trying to find?
  - We are trying to find a specific value of $t$ that corresponds to the point of intersection
- Since $\vec{r} \cdot \vec{n} = s$ at intersection, we get:
  $$t = \frac{s - \vec{a} \cdot \vec{n}}{\vec{d} \cdot \vec{n}}$$
- So using line equation we get our point of intersection, $\vec{r}'$:
  $$\vec{r}' = \vec{a} + \frac{s - \vec{a} \cdot \vec{n}}{\vec{d} \cdot \vec{n}} \vec{d}$$
Example: intersecting planes

- Problem: find the line that represents the intersection of two planes

**Intersecting planes**

- Application: edge detection
- Equations of planes:
  - Plane 1: \( \vec{r} \cdot \hat{n}_1 = s_1 \)
  - Plane 2: \( \vec{r} \cdot \hat{n}_2 = s_2 \)
- We want to find the line of intersection, i.e. find \( \vec{a} \) and \( \vec{d} \) in: \( \vec{s} = \vec{a} + \lambda \vec{d} \)
- If \( \vec{s} = x\hat{i} + y\hat{j} + z\hat{k} \) is on the intersection line:
  - it also lies in both planes 1 and 2
  - \( \vec{s} \cdot \hat{n}_1 = s_1 \) and \( \vec{s} \cdot \hat{n}_2 = s_2 \)
- Can use these two equations to generate equation of line

Example: intersecting planes

- Equations of planes:
  - Plane 1: \( \vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 3 \)
  - Plane 2: \( \vec{r} \cdot \hat{k} = 4 \)
- Pick point \( \vec{s} = x\hat{i} + y\hat{j} + z\hat{k} \)
  - From plane 1: \( 2x - y + 2z = 3 \)
  - From plane 2: \( z = 4 \)
- We have two equations in 3 unknowns – not enough to solve the system
- But... we can express all three variables in terms of one of the other variables

Example: intersecting planes

- From plane 1: \( 2x - y + 2z = 3 \)
- From plane 2: \( z = 4 \)
- Substituting (Eqn. 2) → (Eqn. 1) gives:
  - \( 2x = y - 5 \)
- Also trivially: \( y = y \) and \( z = 4 \)
- Line: \( \vec{s} = ((y - 5)/2)\hat{i} + y\hat{j} + 4\hat{k} \)
  - \( \vec{s} = -\frac{5}{2}\hat{i} + 4\hat{k} + y(\frac{1}{2}\hat{i} + \hat{j}) \)
- ...which is the equation of a line
Cross Product

\[ \vec{p} \times \vec{q} \]

- Also known as: Vector Product
- Used to produce a 3rd vector that is perpendicular to the original two vectors
- Written as \( \vec{p} \times \vec{q} \) (or sometimes \( \vec{p} \land \vec{q} \))
- Formally:
  \[ \vec{p} \times \vec{q} = (|\vec{p}| |\vec{q}| \sin \theta) \hat{n} \]
  where \( \hat{n} \) is the unit vector perpendicular to \( \vec{p} \) and \( \vec{q} \); \( \theta \) is the angle between \( \vec{p} \) and \( \vec{q} \)

Properties of Cross Product

- \( \vec{p} \times \vec{q} \) is itself a vector that is perpendicular to both \( \vec{p} \) and \( \vec{q} \), so:
  \( \vec{p} \cdot (\vec{p} \times \vec{q}) = 0 \) and \( \vec{q} \cdot (\vec{p} \times \vec{q}) = 0 \)
- If \( \vec{p} \) is parallel to \( \vec{q} \) then \( \vec{p} \times \vec{q} = \vec{0} \)
  where \( \vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k} \)
- NOT commutative: \( \vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \)
  In fact: \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \)
- NOT associative: \( (\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c}) \)
- Left distributive: \( \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \)
- Right distributive: \( (\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a} \)

Final important vector product identity:

\[ \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \]

which says that: \( \vec{a} \times (\vec{b} \times \vec{c}) \) lies in the plane created by \( \vec{b} \) and \( \vec{c} \)
Examples of Cross Product

Area of rectangle
\[ = |\vec{a}| |\vec{b}| \]

Area of parallelogram
\[ = |\vec{a} \times \vec{b}| \]

Volume of prism
\[ = |\vec{a} \times \vec{b}| |\vec{c}| \]

Volume of parallelepiped
\[ = (\vec{a} \times \vec{b}) \cdot \vec{c} \]

View from above:
\[ h = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \cdot \vec{c} \]