Performance Analysis

Peter Harrison, Maria Vigliotti and Jeremy Bradley

Room 372. Email: jb@doc.ic.ac.uk

Department of Computing, Imperial College London

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Recall Jackson's theorem

- For a steady-state probability $\pi(r_1, \ldots, r_N)$ of there being r_1 jobs in node 1, r_2 nodes at node 2, etc.:

$$\pi(r_1, r_2, \ldots, r_N) = \prod_{i=1}^{N} (1-\rho_i)\rho_i^{r_i} = \prod_{i=1}^{N} \pi_i(r_i)$$

where $\pi_i(r_i)$ is the steady-state probability there being n_i jobs at node i independently

PEPA and Product Form

- A product form result links the overall steady-state of a system to the product of the steady state for the components of that system
  - e.g. Jackson's theorem
- In PEPA, a simple product form can be got from:

$$P_1 \underset{\emptyset}{\bowtie} P_2 \underset{\emptyset}{\bowtie} \cdots \underset{\emptyset}{\bowtie} P_n$$

- $\pi(P_1^{r_1}, P_2^{r_2}, \ldots, P_n^{r_n}) = \frac{1}{G} \prod_{i=1}^{n} \pi(P_1^{r_1}) \cdots \pi(P_n^{r_n})$
- where $\pi(P_i^{r_i})$ is steady state prob. that component P_i is in state r_i

PEPA and RCAT

- RCAT: *Reversed Compound Agent Theorem*
- RCAT can take the more general cooperation:

$$P \underset{L}{\bowtie} Q$$

- ...and find a product form, given structural conditions, in terms of the individual components P and Q

What does RCAT do?

- RCAT expresses the reversed component $\overline{P \underset{L}{\bowtie} Q}$ in terms of $\overline{P}$ and $\overline{Q}$ (almost)
- This is powerful since it avoids the need to expand the state space of $P \underset{L}{\bowtie} Q$
- This is useful since from the forward and reversed processes, $P \underset{L}{\bowtie} Q$ and $\overline{P \underset{L}{\bowtie} Q}$, we can find the steady state distribution $\pi(P_i, Q_i)$
- $\pi(P_i, Q_i)$ is the steady state distribution of both the forward and reversed processes (by definition)

Recall: Reversed processes

- The reversed process of a stationary Markov process $\{X_t : t \geq 0\}$ with state space S, generator matrix Q and stationary probabilities $\vec{\pi}$ is a stationary Markov process with generator matrix Q' defined by:

$$q'_{ij} = \frac{\pi_j q_{ji}}{\pi_i} \qquad : i, j \in S$$

and with the same stationary probabilities $\vec{\pi}$.

Kolmogorov's Generalised Criteria

A stationary Markov process with state space S and generator matrix Q has reversed process with generator matrix Q' if and only if:

1. $q'_i = q_i$ for every state $i \in S$
2. For every finite sequence of states $i_1, i_2, ..., i_n \in S$,

$$q_{i_1 i_2} q_{i_2 i_3} \dots q_{i_{n-1} i_n} q_{i_n i_1} = q'_{i_1 i_n} q'_{i_n i_{n-1}} \dots q'_{i_3 i_2} q'_{i_2 i_1}$$

where $q_i = -q_{ii} = \sum_{j\,:\,j \neq i} q_{ij}$

Finding π from the reversed process

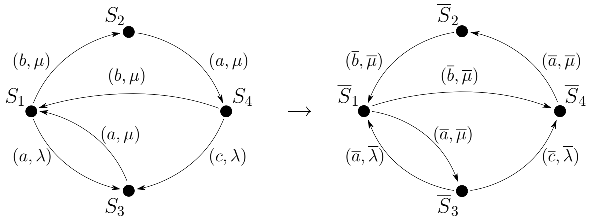
- Once reversed process rates Q' have been found, can be used to extract $\vec{\pi}$
- In an irreducible Markov process, choose a reference state 0 arbitrarily
- Find a sequence of connected states, in either the forward or reversed process, $0, \dots, j$ (i.e. with either $q_{i,i+1} > 0$ or $q'_{i,i+1} > 0$ for $0 \leq i \leq j - 1$) for any state j and calculate:

$$\pi_j = \pi_0 \prod_{i=0}^{j-1} \frac{q_{i,i+1}}{q'_{i+1,i}} = \pi_0 \prod_{i=0}^{j-1} \frac{q'_{i,i+1}}{q_{i+1,i}}$$

Reversing a sequential component

- Reversing a sequential component, S, is straightforward:

$$\overline{S} \stackrel{\text{def}}{=} \sum_{i\,:\,R_i \xrightarrow{(a_i,\lambda_i)} S} (\overline{a}_i, \overline{\lambda}_i).\overline{R}_i$$



Activity substitution

- We need to be able to substitute a PEPA activity $\alpha = (a, r)$ for another $\alpha' = (a', r')$:

$$(\beta.P)\{\alpha \leftarrow \alpha'\} = \begin{cases} \alpha'.(P\{\alpha \leftarrow \alpha'\}) & : \text{if } \alpha = \beta \\ \beta.(P\{\alpha \leftarrow \alpha'\}) & : \text{otherwise} \end{cases}$$

$$(P + Q)\{\alpha \leftarrow \alpha'\} = P\{\alpha \leftarrow \alpha'\} + Q\{\alpha \leftarrow \alpha'\}$$

$$(P \underset{L}{\bowtie} Q)\{\alpha \leftarrow \alpha'\} = P\{\alpha \leftarrow \alpha'\} \underset{L\{\alpha \leftarrow \alpha'\}}{\bowtie} Q\{\alpha \leftarrow \alpha'\}$$

where $L\{(a, \lambda) \leftarrow (a', \lambda')\} = (L \setminus \{a\}) \cup \{a'\}$
if $a \in L$ and L otherwise

- A set of substitutions can be applied with:

$$P\{\alpha \leftarrow \alpha', \beta \leftarrow \beta'\}$$

RCAT Conditions (Informal)

For a cooperation $P \underset{L}{\bowtie} Q$, the reversed process $\overline{P \underset{L}{\bowtie} Q}$ can be created if:

1. Every passive action in P or Q that is involved in the cooperation $\underset{L}{\bowtie}$ must always be enabled in P or Q respectively.
2. Every reversed action $\overline{a}$ in $\overline{P}$ or $\overline{Q}$, where a is active in the original cooperation $\underset{L}{\bowtie}$, must:
   (a) always be enabled in $\overline{P}$ or $\overline{Q}$ respectively
   (b) have the same rate throughout $\overline{P}$ or $\overline{Q}$ respectively

RCAT Notation

In the cooperation, $P \underset{L}{\bowtie} Q$:

- $\mathcal{A}_P(L)$ is the set of actions in L that are also active in the component P
- $\mathcal{A}_Q(L)$ is the set of actions in L that are also active in the component Q
- $\mathcal{P}_P(L)$ is the set of actions in L that are also passive in the component P
- $\mathcal{P}_Q(L)$ is the set of actions in L that are also passive in the component Q
- $\overline{L}$ is the reversed set of actions in L, that is $\overline{L} = \{\overline{a} \mid a \in L\}$

RCAT Conditions (Formal)

For a cooperation $P \underset{L}{\bowtie} Q$, the reversed process $\overline{P \underset{L}{\bowtie} Q}$ can be created if:

1. Every passive action type in $\mathcal{P}_P(L)$ or $\mathcal{P}_Q(L)$ is always enabled in P or Q respectively (i.e. enabled in all states of the transition graph)
2. Every reversed action of an active action type in $\mathcal{A}_P(L)$ or $\mathcal{A}_Q(L)$ is always enabled in $\overline{P}$ or $\overline{Q}$ respectively
3. Every occurrence of a reversed action of an active action type in $\mathcal{A}_P(L)$ or $\mathcal{A}_Q(L)$ has the same rate in $\overline{P}$ or $\overline{Q}$ respectively

RCAT (I)

For $P \underset{L}{\bowtie} Q$, the reversed process is:

$$\overline{P \underset{L}{\bowtie} Q} = R^* \underset{\overline{L}}{\bowtie} S^*$$

where:

$$\begin{aligned}
R^* &= \overline{R}\{(\overline{a}, \overline{p}_a) \leftarrow (\overline{a}, \top) \mid a \in \mathcal{A}_P(L)\} \\
S^* &= \overline{S}\{(\overline{a}, \overline{q}_a) \leftarrow (\overline{a}, \top) \mid a \in \mathcal{A}_Q(L)\} \\
R &= P\{(a, \top) \leftarrow (a, x_a) \mid a \in \mathcal{P}_P(L)\} \\
S &= Q\{(a, \top) \leftarrow (a, x_a) \mid a \in \mathcal{P}_Q(L)\}
\end{aligned}$$

where the reversed rates, $\overline{p}_a$ and $\overline{q}_a$, of reversed actions are solutions of Kolmogorov equations.

RCAT (II)

x_a are solutions to the linear equations:

$$x_a = \begin{cases} \overline{q}_a & : \text{if } a \in \mathcal{P}_P(L) \\ \overline{p}_a & : \text{if } a \in \mathcal{P}_Q(L) \end{cases}$$

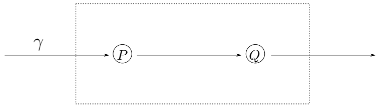
and $\overline{p}_a$, $\overline{q}_a$ are the symbolic rates of action types $\overline{a}$ in $\overline{P}$ and $\overline{Q}$ respectively.

RCAT in words

To obtain $\overline{P \underset{L}{\bowtie} Q} = R^* \underset{\overline{L}}{\bowtie} S^*$:

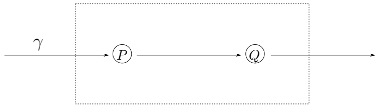
1. substitute all the cooperating passive rates in P, Q with symbolic rates, x_{action}, to get R, S
2. reverse R and S, to get $\overline{R}$ and $\overline{S}$
3. solve non-linear equations to get reversed rates, $\{\overline{r}\}$ in terms of forward rates $\{r\}$
4. solve non-linear equations to get symbolic rates $\{x_{action}\}$ in terms of forward rates
5. substitute all the cooperating active rates in $\overline{R}$, $\overline{S}$ with $\top$ to get R^*, S^*

Example: Tandem queues (I)



- Jobs arrive to node P with activity (e, γ)
- Jobs are serviced at node P with rate μ_1
- Jobs move between node P and Q with action a
- Jobs are serviced at node Q with rate μ_2
- Jobs depart Q with action d

Example: Tandem queues (II)



- PEPA description, $P_0 \underset{\{a\}}{\bowtie} Q_0$, where:

$$
\begin{aligned}
P_0 &\stackrel{\text{def}}{=} (e,\gamma).P_1 \\
P_n &\stackrel{\text{def}}{=} (e,\gamma).P_{n+1} + (a,\mu_1).P_{n-1} && : n > 0 \\
Q_0 &\stackrel{\text{def}}{=} (a,\top).Q_1 \\
Q_n &\stackrel{\text{def}}{=} (a,\top).Q_{n+1} + (d,\mu_2).Q_{n-1} && : n > 0
\end{aligned}
$$

Example: Tandem queues (III)

- Replace passive rates in cooperation with variables:

$$
\begin{aligned}
R &= P\{(a,\top) \leftarrow (a, x_a) \mid a \in \mathcal{P}_P(L)\} \\
S &= Q\{(a,\top) \leftarrow (a, x_a) \mid a \in \mathcal{P}_Q(L)\}
\end{aligned}
$$

- Transformed PEPA model:

$$
\begin{aligned}
R_0 &\stackrel{\text{def}}{=} (e,\gamma).R_1 \\
R_n &\stackrel{\text{def}}{=} (e,\gamma).R_{n+1} + (a,\mu_1).R_{n-1} && : n > 0 \\
S_0 &\stackrel{\text{def}}{=} (a,x_a).S_1 \\
S_n &\stackrel{\text{def}}{=} (a,x_a).S_{n+1} + (d,\mu_2).S_{n-1} && : n > 0
\end{aligned}
$$

Example: Tandem queues (IV)

- Reverse components R and S to get:

$$
\begin{aligned}
\overline{R}_0 &\stackrel{\text{def}}{=} (\overline{a},\overline{\mu}_1).\overline{R}_1 \\
\overline{R}_n &\stackrel{\text{def}}{=} (\overline{a},\overline{\mu}_1).\overline{R}_{n+1} + (\overline{e},\overline{\gamma}).\overline{R}_{n-1} && : n > 0 \\
\overline{S}_0 &\stackrel{\text{def}}{=} (\overline{d},\overline{\mu}_2).\overline{S}_1 \\
\overline{S}_n &\stackrel{\text{def}}{=} (\overline{d},\overline{\mu}_2).\overline{S}_{n+1} + (\overline{a},\overline{x}_a).\overline{S}_{n-1} && : n > 0
\end{aligned}
$$

- Now need to find in this order:
  1. reverse rates in terms of forward rates
  2. variable x_a in terms of forward rates

Example: Tandem queues (V)

- Finding reverse rates using Kolmogorov
  - Compare forward/reverse leaving rate from states R_0, S_0:

$$\begin{aligned} exit\_rate(R_0) = exit\_rate(\overline{R}_0) : &\quad \overline{\mu}_1 = \gamma \\ exit\_rate(S_0) = exit\_rate(\overline{S}_0) : &\quad \overline{\mu}_2 = x_a \end{aligned}$$

  - Compare rate cycles in R, $\overline{R}$ and S, $\overline{S}$:

$$\begin{aligned} R_0 \to R_1 \to R_0 : &\quad \gamma\mu_1 = \overline{\mu}_1\overline{\gamma} \\ S_0 \to S_1 \to S_0 : &\quad x_a\mu_2 = \overline{\mu}_2\overline{x}_a \end{aligned}$$

  - Giving: $\overline{\gamma} = \mu_1$ and $\overline{x}_a = \mu_2$

Example: Tandem queues (VI)

- Finding symbolic rates – recall:

$$x_a = \begin{cases} \overline{q}_a & : \text{if } a \in \mathcal{P}_P(L) \\ \overline{p}_a & : \text{if } a \in \mathcal{P}_Q(L) \end{cases}$$

- In this case, $a \in \mathcal{P}_Q(L)$, so $x_a = \overline{p}_a =$ reversed rate of a-action in $\overline{R}$
- Thus $x_a = \overline{\mu}_1 = \gamma$
- This agrees with rate of customers leaving forward network – why?

Example: Tandem queues (VII)

- Constructing $\overline{P \bowtie_L Q}$
  - $\overline{P_0 \bowtie_{\{a\}} Q_0} = R_0^* \bowtie_{\{\overline{a}\}} S_0^*$ where:

$$\begin{aligned} R_0^* &\stackrel{\text{def}}{=} (\overline{a}, \top).R_1^* \\ R_n^* &\stackrel{\text{def}}{=} (\overline{a}, \top).R_{n+1}^* + (\overline{e}, \mu_1).R_{n-1}^* &&: n > 0 \\ S_0^* &\stackrel{\text{def}}{=} (\overline{d}, \gamma).S_1^* \\ S_n^* &\stackrel{\text{def}}{=} (\overline{d}, \gamma).S_{n+1}^* + (\overline{a}, \mu_2).S_{n-1}^* &&: n > 0 \end{aligned}$$

Example: Tandem queues (VIII)

- Finding the steady state distribution:
  - Need to use the following formula:

$$\pi_j = \pi_0 \prod_{i=0}^{j-1} \frac{q_{i,i+1}}{q'_{i+1,i}}$$

  ...to find the steady state distribution
  - First need to construct a sequence of events to a generic state (n, m) in network
    - where (n, m) represents n jobs in node P and m in node Q

Example: Tandem queues (IX)

- Generic state can be reached by:
  1. $n + m$ arrivals or e-actions to node P (forward rate $= \gamma$, reverse rate $= \mu_1$)
  2. followed by m departures or a-actions from node P and arrivals to node Q (forward rate $= \mu_1$, reverse rate $= \mu_2$)

Thus:
$$\pi(n,m) = \pi_0 \prod_{i=0}^{n+m-1} \frac{\gamma}{\mu_1} \times \prod_{i=0}^{m-1} \frac{\mu_1}{\mu_2} = \pi_0 \left(\frac{\gamma}{\mu_1}\right)^n \left(\frac{\gamma}{\mu_2}\right)^m$$

References

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