Parametric Completeness for Separation Theories

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Joint work with James Brotherston (UCL)
Mathematical logics expressivity trade-off

- Weaker languages cannot capture interesting properties, but
- Richer languages have higher complexity, may lack sensible proof theories and may be unavoidably \textit{incomplete} (cf. Gödel).
Logics: Expressivity vs Complexity

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This talk

- Study this gap in the context of separation logic
Separation Theories

Separation Logic (SL)

- Compositional program logic for heap-manipulating programs (C, C++, Java, ...)

Models of Separation Logic and BBI

- Models of BBI: partial commutative relational monoids
- Concrete model: Heaps: Location $\mapsto$ Values
- In-between: separation theories satisfying some of functionality
  - cancellativity
  - single-unit

Parametric Completeness for Separation Theories

Introduction
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- Compositional program logic for heap-manipulating programs (C, C++, Java, ...)
- Hoare triples \( \{A\} \text{ program } \{B\} \)
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Definability of Classes of Models

Given a logical language $\mathcal{L}$, and an intended class $\mathcal{C}$ of models for that language,

1. Is $\mathcal{C}$ \textit{finitely axiomatisable}, a.k.a. \textit{definable} in $\mathcal{L}$?
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(Note that these questions are not connected, in general.)
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Pure separation logic

- $\mathcal{L}$ is **Boolean BI (BBI)**;
- the intended models are given by **separation theories**
The rest of the talk goes as follows:

1. First, we recall the standard presentation of BBI.

2. We introduce separation theories, which describe practically interesting classes of models, and show that many such theories are not definable in BBI.

3. We then propose an extension of BBI based on hybrid logic, which adds a theory of naming to BBI, and show that these properties become definable in this extension.

4. We show how to axiomatise validity in our hybrid system(s). Moreover, we do this such that completeness is parametric in the axioms defining separation theories.
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Boolean BI
**BBI formula**

\[ A ::= P \mid \top \mid \bot \mid \neg A \mid A_1 \land A_2 \mid A_1 \lor A_2 \mid A_1 \rightarrow A_2 \]
\[ \mid I \mid A_1 \ast A_2 \mid A_1 \leftarrow A_2 \]
## (Propositional) Boolean BI

### BBI formula

\[ A ::= P \mid \top \mid \bot \mid \neg A \mid A_1 \land A_2 \mid A_1 \lor A_2 \mid A_1 \to A_2 \mid \mathbf{I} \mid A_1 \cdot A_2 \mid A_1 \not\cdot A_2 \]

### Separating Conjunction

\[ A_1 \cdot A_2 \iff \]

\[ A_1 \mid \mid A_2 \]
(Propositional) Boolean BI

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| I \mid A_1 \ast A_2 \mid A_1 \rightarrow A_2
\]

**Magic Wand**
Proof theory of BBI

**Provability** for the multiplicatives is given by

\[
\begin{align*}
A \ast B & \vdash B \ast A \\
A \ast (B \ast C) & \vdash (A \ast B) \ast C \\
A & \vdash A \ast I \\
A \ast I & \vdash A \\
A_1 \vdash B_1 & \quad A_2 \vdash B_2 \\
\hline
A_1 \ast A_2 & \vdash B_1 \ast B_2 \\
A \ast B & \vdash C \\
A & \vdash B \multimap C \\
A \ast B & \vdash C
\end{align*}
\]
### BBI-models

**BBI model** $\langle W, \circ, E \rangle$

A **relational commutative monoid**, i.e. a tuple $\langle W, \circ, E \rangle$ where

- $\circ : W \times W \to \mathcal{P}(W)$
  
  $\left( \text{lifted to } W_1 \circ W_2 \overset{\text{def}}{=} \bigcup_{w_1 \in W_1, w_2 \in W_2} w_1 \circ w_2 \right)$

- $\circ$ commutative and associative

- $E \subseteq W$ and $\forall w \in W. \ w \circ E = \{ w \}$ (multi-units)
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- \( e \) is the empty heap that is undefined everywhere.
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| $M, w \models \rho P$ | $\iff$ | $w \in \rho(P)$ |

Theorem Galmiche and Larchey-Wendling 2006

Provability in BBI coincides with validity in BBI-models.

Parametric Completeness for Separation Theories • Boolean BI
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<td>$M, w \models_{\rho} A_1 \rightarrow A_2$</td>
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$A$ is **valid in** $M$ iff $M, w \models_\rho A$ for all $\rho$ and $w \in W$. 

Theorem Galmiche and Larchey-Wendling 2006

Provability in BBI coincides with validity in BBI-models.
# Semantics of BBI

Forcing relation $M, w \models_{\rho} A$

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M = \langle W, \circ, E \rangle \\
M, w \models_{\rho} P \iff w \in \rho(P) \\
M, w \models_{\rho} A_1 \land A_2 \iff M, w \models_{\rho} A_1 \text{ and } M, w \models_{\rho} A_2 \\
\vdots \\
M, w \models_{\rho} I \iff w \in E \\
M, w \models_{\rho} A_1 \ast A_2 \iff w \in w_1 \circ w_2 \text{ and } M, w_1 \models_{\rho} A_1 \text{ and } M, w_2 \models_{\rho} A_2 \\
M, w \models_{\rho} A_1 \rightarrow A_2 \iff \forall w', w'' \in W. \text{ if } w'' \in w \circ w' \text{ and } M, w' \models_{\rho} A_1 \text{ then } M, w'' \models_{\rho} A_2
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**Theorem** Galmiche and Larchey-Wendling 2006

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(Un)definable properties in BBI
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Divisibility: for every \(w \not\in E\) there are \(w_1, w_2 \not\in E\) such that \(w \in w_1 \circ w_2\).
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Divisibility: for every \(w \notin E\) there are \(w_1, w_2 \notin E\) such that \(w \in w_1 \circ w_2\);

Cross-split property: whenever \((a \circ b) \cap (c \circ d) \neq \emptyset\), there exist \(ac, ad, bc, bd\) such that \(a \in ac \circ ad, b \in bc \circ bd, c \in ac \circ bc\) and \(d \in ad \circ bd\).
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Cross-split property:
### Separation Algebras throughout the Ages

**Definition Separation algebra (Calcagno et al. 07)**

A *separation algebra* is a BBI-model that is **partial functional**, **cancellative**, and with a **single unit**.

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### Separation Algebras throughout the Ages

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### Separation Algebras throughout the Ages

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Separation algebra (Calcagno et al. 07)  

A *separation algebra* is a BBI-model that is **partial functional**, **cancellative**, and with a **single unit**.

**Definition**  
Separation algebra (Dockins et al. 09)  

A *separation algebra* is a BBI-model that is **partial functional** and **cancellative**.

**Definition**  
Separation algebra (Dinsdale-Young et al. 13)  

A *separation algebra* is a BBI-model that is **partial functional**.
Definable properties

A class $\mathcal{C}$ of BBI-models is said to be $\mathcal{L}$-definable if there exists an $\mathcal{L}$-formula $A$ such that for all BBI-models $M$,

$$A \text{ is valid in } M \iff M \in \mathcal{C}.$$
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Proposition

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- **Divisibility**: \( \neg I \vdash \neg I \ast \neg I \)

**Proof.**

Just directly verify the needed biimplication.
Undefinability via disjoint union

To show a property is not BBI-definable, we show it is not preserved by some validity-preserving model construction.
# Undefinability via disjoint union

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**Proposition**

If $A$ is valid in $M_1$ and in $M_2$, and $M_1 \uplus M_2$ is defined, then it is also valid in $M_1 \uplus M_2$. 
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**Proof.**

Structural induction on $A$. 

---

Parametric Completeness for Separation Theories • (Un)definable properties in BBI
**Lemma**

Let $\mathcal{C}$ be a class of BBI-models, and suppose that there exist BBI-models $M_1$ and $M_2$ such that $M_1, M_2 \in \mathcal{C}$ but $M_1 \uplus M_2 \not\in \mathcal{C}$. Then $\mathcal{C}$ is not BBI-definable.
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**Proof.**

If $\mathcal{C}$ were definable via $A$ say, then $A$ would be true in $M_1$ and $M_2$ but not in $M_1 \uplus M_2$, contradicting previous Proposition.
Undefinability of single-unit property

**Lemma**

Let $\mathcal{C}$ be a class of BBI-models, and suppose that there exist BBI-models $M_1$ and $M_2$ such that $M_1, M_2 \in \mathcal{C}$ but $M_1 \uplus M_2 \notin \mathcal{C}$. Then $\mathcal{C}$ is not BBI-definable.

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If $\mathcal{C}$ were definable via $A$ say, then $A$ would be true in $M_1$ and $M_2$ but not in $M_1 \uplus M_2$, contradicting previous Proposition.

**Theorem**

The single unit property is not BBI-definable.
**Lemma**

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If $C$ were definable via $A$ say, then $A$ would be true in $M_1$ and $M_2$ but not in $M_1 \uplus M_2$, contradicting previous Proposition.

**Theorem**

The single unit property is not BBI-definable.

**Proof.**

The disjoint union of any two single-unit BBI-models (e.g. two copies of $\mathbb{N}$ under addition) is not a single-unit model, so we are done by the above Lemma.
We adapt the notion of **bounded morphism** from modal logic to BBI-models, and can show it is also validity-preserving.
Undefinability via bounded morphisms

We adapt the notion of **bounded morphism** from modal logic to BBI-models, and can show it is also validity-preserving.

**Theorem**

*None of the following separation theory properties (or any combination thereof) is BBI-definable:*

- functionality;
- cancellativity;
- disjointness.

**Proof.** E.g., for functionality, we build models $M$ and $M'$ such that there is a bounded morphism from $M$ to $M'$, but $M$ is functional while $M'$ is not. See paper for details.
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Hybrid BBI
HyBBI: a hybrid extension of BBI

- We saw that BBI is not expressive enough to accurately capture many separation theories.
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### HyBBI formula (extends BBI)

\[ A ::= \ldots | \ell | @\ell A \]
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- Valuations interpret nominals as *individual worlds* in a BBI-model.

### Forcing relation (extended)

\[
M, w \models \rho \ell \iff w = \rho(\ell)
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<td>$M, w \models _\rho \ell \iff w = \rho(\ell)$</td>
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### Forcing relation (extended)

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M, w \models_\rho \ell & \iff w = \rho(\ell) \\
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\end{align*}
\]

- **Fact**: HyBBI is a *conservative extension* of BBI.
Definable properties in HyBBI

A formula is **pure** if it contains no propositional variables. Pure formulas have particularly nice properties wrt. completeness.
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**Functionality:**\[ \Diamond_{\ell} (j \ast k) \land \Diamond_{\ell'} (j \ast k) \vdash \Diamond_{\ell \ell'} \]
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- **Cancellativity:** \( \ell * j \land \ell * k \vdash \Diamond j k \)
- **Single unit:** \( \Diamond_{\ell_1} I \land \Diamond_{\ell_2} I \vdash \Diamond_{\ell_1} \ell_2 \)
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Overlapping conjunction

\[ A_1 \uplus A_2 \iff M, w \models \rho \ A_1 \uplus A_2 \iff \exists w_1, w_2, w_3, w', w'' \in W. \]

\[ w' \in w_1 \circ w_2 \text{ and } w'' \in w_2 \circ w_3 \text{ and } w \in w' \circ w_3 \text{ and } M, w' \models \rho \ A_1 \text{ and } M, w'' \models \rho \ A_2 \]
Overlapping conjunction

\[ A_1 \uplus A_2 \iff A_1 \cap A_2 \]

\[ M, w \models \rho A_1 \uplus A_2 \iff \exists w_1, w_2, w_3, w', w'' \in \mathcal{W}. \]

\[ w' \in w_1 \circ w_2 \text{ and } w'' \in w_2 \circ w_3 \text{ and } w \in w' \circ w_3 \]

and \( M, w' \models \rho A_1 \) and \( M, w'' \models \rho A_2 \)

By naming the shared part, one can easily define the overlapping conjunction:

\[ (\ell_s \rightarrow A_1) \uplus (\ell_s \rightarrow A_2) \uplus \ell_s \]
Overlapping conjunction

\[ A_1 \uplus^* A_2 \iff \exists w_1, w_2, w_3, w', w'' \in W. \]
\[ w' \in w_1 \circ w_2 \text{ and } w'' \in w_2 \circ w_3 \text{ and } w \in w' \circ w_3 \text{ and } M, w' \models \rho A_1 \text{ and } M, w'' \models \rho A_2 \]

By naming the shared part, one can easily define the overlapping conjunction:

\[ (\ell_s \rightarrow A_1) \ast (\ell_s \rightarrow A_2) \ast \ell_s \]

(*but where does \ell_s come from?..*)
A word about cross-split

We have brushed over the **cross-split** property:

\[(a \circ b) \cap (c \circ d) \neq \emptyset, \text{ implies } \exists ac, ad, bc, bd \text{ with } a \in ac \circ ad, b \in bc \circ bd, c \in ac \circ bc, d \in ad \circ bd.\]
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$$(a \circ b) \cap (c \circ d) \neq \emptyset,$$

implies $\exists ac, ad, bc, bd$ with $a \in ac \circ ad$, $b \in bc \circ bd$, $c \in ac \circ bc$, $d \in ad \circ bd$.

$$\forall \begin{array}{c|c}\hline a & b \\
\hline\end{array} \quad \begin{array}{c|c|c|c|c}
\hline c & d & ac & bc & ad & bd \\
\hline & & & & \end{array} \quad \exists \begin{array}{c|c}\hline ac & bc \\
\hline ad & bd \\
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\[\forall \begin{array}{c|c}
 a & b \\
 \hline
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\end{array} \quad \exists \begin{array}{c|c|c|c}
 ac & bc & ad & bd \\
\hline
\end{array}\]

We conjecture this is not definable in BBI or in HyBBI.
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We conjecture this is not definable in BBI or in HyBBI. If we add the ↓ binder to HyBBI, defined by

\[M, w \models_{\rho} \downarrow \ell. A \iff M, w \models_{\rho[\ell := w]} A\]
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\[
\forall \begin{array}{c}
\circlearrowleft a \\
\circlearrowleft b \\
\circlearrowleft c \\
\circlearrowleft d
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We conjecture this is not definable in BBI or in HyBBI. If we add the ↓ binder to HyBBI, defined by

\[
M, w \models_\rho \downarrow \ell. A \iff M, w \models_\rho[\ell:=w] A
\]

then cross-split is definable as the pure formula

\[
(a * b) \land (c * d) \vdash @_a(\top * \downarrow ac. @_a(\top * \downarrow ad. @_a(ac * ad) \\
\land @_b(\top * \downarrow bc. @_b(\top * \downarrow bd. @_b(bc * bd) \\
\land @_c(ac * bc) \land @_d(ad * bd))))
\]
Overlapping conjunction (bis)

Proposition \( A_1 \cup^* A_2 \) is definable via the following Hybrid BBI formula, where \( \ell \) and \( \ell_s \) do not occur in \( A_1 \) or \( A_2 \):

\[
\downarrow \ell \top^* \downarrow \ell_s
\]

\[
@\ell (\ell_s \backslash A_1^*) \backslash (\ell_s \backslash A_2^*) \backslash \ell_s
\]

where \( A^\backslash B \) def = \( \neg (A^* \neg B) \)
Overlapping conjunction (bis)

A_1 \uplus A_2 \iff A_1 \cap A_2

Proposition

A_1 \uplus A_2 \text{ is definable via the following HyBBI(\downarrow) formula, where } \ell \text{ and } \ell_s \text{ do not occur in } A_1 \text{ or } A_2:

\downarrow \ell. \top \ast \downarrow \ell_s. @\ell (\ell_s \ominus A_1) \ast (\ell_s \ominus A_2) \ast \ell_s

(where \( A \ominus B \overset{\text{def}}{=} \neg (A \rightarrow \neg B) \))
Parametric completeness for HyBBI(\downarrow)
Axiomatic proof systems for HyBBI(\(\downarrow\))

Our axiom system \(K_{\text{HyBBI}(\downarrow)}\) is chosen to make the completeness proof as clean as possible.
Axiomatic proof systems for HyBBI(\(\downarrow\))

Our axiom system \(K_{\text{HyBBI}(\downarrow)}\) is chosen to make the completeness proof as clean as possible. Some example axioms and rules:

\[(K_\@) \quad \@_\ell (A \rightarrow B) \vdash \@_\ell A \rightarrow \@_\ell B\]
Our axiom system $K_{HyBBI(\downarrow)}$ is chosen to make the completeness proof as clean as possible. Some example axioms and rules:

- **($K@$)**
  \[ \bowtie_\ell (A \rightarrow B) \vdash \bowtie_\ell A \rightarrow \bowtie_\ell B \]

- **($@$-intro)**
  \[ \ell \land A \vdash \bowtie_\ell A \]
Axiomatic proof systems for HyBBI(↓)

Our axiom system \( K_{\text{HyBBI}(\downarrow)} \) is chosen to make the completeness proof as clean as possible. Some example axioms and rules:

\[
\begin{align*}
(K_\@) & \quad \@_\ell (A \to B) \vdash \@_\ell A \to \@_\ell B \\
(\@\text{-intro}) & \quad \ell \land A \vdash \@_\ell A \\
(Bridge \ast) & \quad \@_\ell (k \ast k') \land \@_k A \land \@_k' B \vdash \@_\ell (A \ast B)
\end{align*}
\]
Axiomatic proof systems for HyBBI($\downarrow$)

Our axiom system $K_{\text{HyBBI}(\downarrow)}$ is chosen to make the completeness proof as clean as possible. Some example axioms and rules:

\( (K_{\neg}) \quad \neg_{\ell}(A \rightarrow B) \vdash \neg_{\ell}A \rightarrow \neg_{\ell}B \)

\( (@\text{-intro}) \quad \ell \wedge A \vdash \neg_{\ell}A \)

\( (\text{Bridge } \ast) \quad \neg_{\ell}(k \ast k') \wedge \neg_{k}A \wedge \neg_{k'}B \vdash \neg_{\ell}(A \ast B) \)

\( (\text{Bind } \downarrow) \quad \vdash \neg_{j}(\downarrow. B \leftrightarrow B[j/\ell]) \)
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Our axiom system $K_{\text{HyBBI}(↓)}$ is chosen to make the completeness proof as clean as possible. Some example axioms and rules:

- **($K_@$)** $\vdash \_\ell (A \rightarrow B) \rightarrow \_\ell A \rightarrow \_\ell B$
- **(@-intro)** $\vdash \ell \land A \rightarrow \_\ell A$
- **(Bridge *)** $\vdash \_\ell (k \ast k') \land \_k A \land \_k' B \rightarrow \_\ell (A \ast B)$
- **(Bind ↓ )** $\vdash \_j (\_\ell \downarrow \ell. B \leftrightarrow B[j/\ell])$

\[
\vdash \_\ell (k \ast k') \land \_k A \land \_k' B \rightarrow C \quad k, k' \text{ not in } A, B, C \text{ or } \{\ell\}
\]

\[
\_\ell (A \ast B) \vdash C \quad \text{(Paste *)}
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(\ell\text{-intro}) & \quad \ell \land A \vdash \ell A \\
(Bridge \, \star) & \quad \ell(k * k') \land \ell kA \land \ell k'B \vdash \ell(A * B) \\
(Bind \, \ell) & \quad \vdash \ell j(\downarrow \ell. B \leftrightarrow B[j/\ell])
\end{align*}
\]

\[
\frac{\ell (k * k') \land \ell kA \land \ell k'B \vdash C}{\ell (A * B) \vdash C}
\]

\( k, k' \) not in \( A, B, C \) or \( \{\ell\} \) (Paste \, \star)

<table>
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<tr>
<th>Proposition</th>
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<td>Any ( K_{HyBBI(↓)} )-provable sequent is valid in all BBI-models.</td>
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Completeness

Standard modal logic approach to completeness via **maximal consistent sets (MCSs)**:

1. Show that any consistent set of formulas can be extended to an MCS (known as the **Lindenbaum construction**);
2. Define a **canonical model** whose worlds are MCSs, and a valuation such that a proposition $P$ is true at $w$ iff $P \in w$.
3. **Truth Lemma**: $A$ is true at $w$ iff $A \in w$ for any formula $A$.
4. If $A$ is unprovable, $\{\neg A\}$ is consistent so there is an MCS $w \supset \{\neg A\}$. Then $A$ is false at $w$ in the canonical model, hence invalid.

(In our case, we also have to show that the canonical model is really a BBI-model.)
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Parametric Completeness for Separation Theories

- Parametric completeness for HyBBI($\downarrow$)
Completeness

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\[ \square \]
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Following the above approach (non-trivial; details in paper) we obtain the following, for any set of pure axioms Ax:

**Theorem Parametric completeness**

If A is valid in the class of BBI-models satisfying Ax, then it is provable in \( K_{\text{HyBBI}}(↓) + Ax \).

**Corollary**

By a suitable choice of axioms, we have a sound and complete axiomatic proof system for any given separation theory from our collection. In particular, we obtain sound and complete proof systems for separation algebras.
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- search for nice structural proof theories;
- investigate possible applications to program analysis.
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Thanks for listening!

Draft paper available from authors’ webpages:

J. Brotherston and J. Villard.
Parametric completeness for separation theories.
To be presented at POPL’14.
Parametric Completeness for Separation Theories

Jules Villard

University College London
Programming Principles, Logic and Verification Group

Joint work with James Brotherston (UCL)