Completeness of Resolution

We will show by construction:

If clauses $S$ have no models then there is a resolution proof of $\emptyset$ from $S$.

Most methods to show completeness rely on some very useful properties:

(a) A set of clauses $S$ has no models iff $S$ has no $H$-models.
    (Useful Theorem (*))
    so it is sufficient to look at Herbrand Interpretations

(b) If a set of clauses $S$ is $H$-unsatisfiable (has no $H$-models) then there is a
    finite subset of ground instances of $S$ also $H$-unsatisfiable  
    (compactness).
    find the appropriate ground instances:
    construct a finite closed semantic tree $G$ for ground instances of $S$

(c) A resolution refutation for a set of clauses $S$ has a similar structure to a
    ground resolution refutation using ground instances of $S$
    (see slide 5aiii for an example).
    find a ground refutation:
    construct a ground resolution refutation from $G$
    and lift it to give a resolution refutation from $S$

Example of the relationship between a refutation of ground instances
of clauses $S$ and a resolution refutation of $S$ (used for Step (c))

1. $Dca \lor Dcb$
2. $\neg Dxy \lor Cxy$
3. $\neg Tu \lor \neg Cub$
4. $Tc$
5. $\neg Dcz$

Ground instances: $(u == c, x == c, y == b, z == a)$

$Dca \lor Dcb \quad \neg Dcb \lor Ccb \quad \neg Tu \lor \neg Cub \quad Tc \quad \neg Dca$

Example:

- $Dca \lor Dcb$
- $\neg Dxy \lor Cxy$
- $\neg Cub \lor \neg Tu$
- $\neg Dxb \lor \neg Tx$
- $Tc$
- $Dcb \lor Dca$
- $\neg Dza$
- $\neg Dcb$
- $\neg Dca$
- $\neg Dza$
- $Dcb$
- $\neg Dcb$

Ground proof:

Resolution proof:

Each clause in the ground proof is an instance of a corresponding clause in the resolution proof.
A **Semantic Tree** is an enumeration of all H-interpretations (HI) over \(\text{Sig}(L)\), where \(S\) uses language \(L\). Each branch represents an HI over \(\text{Sig}(L)\).

**Example:** Let \(\text{Sig}(L) = \langle \{P, Q, R, S\}, \{f\}, \{a, b\} \rangle\) and

Given \(Px \lor Ry \lor \neg Qxy\), \(\neg Sz \lor \neg Rz\), \(Pu \lor Q(v)v\), \(Sa\), \(Sb\), \(\neg Pf(a) \lor \neg Pf(b)\)

Each finite portion of a branch of a semantic tree gives a partial Herbrand Interpretation of \(S\). A branch is **terminated** (marked \(x\)) if it cannot be a model for \(S\).

eg. the leftmost branch falsifies \(\neg Sa \lor \neg Ra\), instance of \(\neg Sz \lor \neg Rz\).

Which other branches falsify a given clause?

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**Observations about a Semantic Tree**

1. If atom \(A\) is tested, then the clause falsified by the \(A=F\) branch will contain \(A\) and the clause falsified by the \(A=T\) branch will contain \(\neg A\). (Why?)

2. Any interpretation that uses the assignments in a terminated branch is impossible as a model of \(S\).

3. If every branch in a semantic tree for clauses in \(S\) is closed then \(S\) is unsatisfiable.

   Why?
   
   Every HI will be an extension of some branch of the tree and hence makes some clause in \(S\) false

4. We would like to know that if \(S\) is unsatisfiable we can find a finite closed tree.

   Could the tree have an infinite set of branches when \(S\) is unsatisfiable?
   
   Let’s see .......

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**Is a Semantic tree (for unsatisfiable \(S\)) always finite?**

If \(S\) has no H-Models then each H-Interpretation must falsify a clause in \(S\).

For Step b of the completeness proof want to collect from a completed tree the set of ground clauses made false by the given general clauses

**How do we know this set is finite?**

Could the tree have an infinite set of branches?

Let’s see what we know:

- To make a clause \(C\) false it is sufficient to make 1 ground instance of \(C\) false.

- Since clauses in \(S\) are finite, the falsifying part of the interpretation is found after consideration of a finite number of atoms.

**BUT:** Can we be sure there are a finite number of ground instances of \(S\) sufficient to be falsified by all the H-Interpretations over the signature of \(S\)?

The “Umbrella” Property says we can!

In other words, the closed tree is finite.

(We prove this next relying on König’s Lemma)
We can show: If $S$ is unsatisfiable then there is a finite closed semantic tree for $S$ (Called compactness.) We’ll use König’s Lemma: A finitely branching tree can have an infinite number of nodes only if some branch is infinite (has an infinite length)

Assume $S$ has no models:
- If the Semantic tree for $S$ were infinite, then there would be an infinite branch. (Directly from König’s Lemma)
- We claim such an infinite branch would yield a model because:
  - Assume for contradiction the branch did not give a model.
  - Then the branch could have been finite (by earlier observations on 5biii)

How to find a resolvent from a semantic tree

The two children of a failure node will resolve (see Notes 1/2 below the tree)
The resolvent will be false at the failure node. Why?
Therefore, the resolvent cannot be a tautology. Why?
The resolvent can be added to the falsifying ground instances and will allow a smaller tree to be obtained, since the failure node will now become a closure node.

Note 1: the false clause must include $P$
Note 2: the false clause must include $\neg P$

How to obtain a ground refutation from a complete semantic tree (see ppt)

1. $Dca \lor Dcb$  2. $\neg Dxy \lor Cxy$  3. $\neg Tx \lor \neg Cxb$  4. $Tc$  5. $\neg Dcz$

Ground instances found from tree:
1g. $Dca \lor Dcb$  2g. $\neg Dcb \lor Ccb$  3g. $\neg Tc \lor \neg Ccb$  4g. $Tc$  5g. $\neg Dca$

A semantic tree:

Check that at each failure node the two false clauses below it can be resolved and that if atom $A$ is tested, then left clause contains $\neg A$ and right clause contains $A$. (Assumes left branch of test makes $A=T$, right branch makes $A=F$.) Also check that the resolvent is false at the processed node.
Properties of the Semantic Tree method (1)

- Each tree gives rise to a ground resolution proof (refutation) using instances of the given clauses.
- Can derive a full resolution refutation from a semantic tree proof by LIFTING because of the following invariant property:
  - Each failed clause instance is either an instance of a given clause, or an instance of the resolvent of the involved clauses at the leaves (ie the closure nodes).

See example below (and optional slides 5e for the general case)

LIFTING LEMMA
(Example)
Resolve instances of C1 and C2 to give ¬Sb v P(b) v ¬Qf(b)b which is an instance of ¬Sz v Px v ¬Qxz, the resolvent of C1 and C2

\[ \neg R_b = \text{false} \]

\[ P_f(b) \lor R_b \lor \neg Q_f(b)b \]

\[ \neg R_b \lor \neg S_b \]

\[ \neg S_z \lor P_x \lor \neg Q_x z \]

Properties of the Semantic Tree method (2)

Sometimes a factoring step is required to complete a refutation:

- It's indicated if the ground instance has fewer literals than the general clause.
- Recall that at ground level, factoring is just merging of identical literals, whereas in general it requires a substitution to make 2 or more literals identical.

Example:
- In slide 5ciii at (Z) the new ground clause ¬ Sb v P(b) is obtained:
  - by resolution of ¬Sb v P(b) v ¬Qf(b)b and P(b) v Qf(b)b to give ¬Sb v P(b) v P(b)
  - then by merging to give ¬Sb v P(b)

- The general resolvent clause is formed from ¬Sz v P(x) v ¬Qxz and P(v) v Qf(v)v, giving ¬Sz v P(f(z)) v Pu which factors to ¬Sz v P(z) with binding (u==f(z)).

Exercise:
Fill in the missing ground clauses and resolvents at places marked ?.

S = P_x v R_y v ¬Q_x y, ¬S_z v ¬R_z, P_u v Q_f(v)v, S_a, S_b, ¬P_f(a) v ¬P_f(b)

Q. What is the problem with using the semantic tree method for showing unsatisfiability? What feature of resolution makes resolution better?
Properties of the Semantic Tree method (3)

Sometimes a resolvent falsifies a node higher than the failure node at which it was formed - enables the tree to contract more quickly.

When/how can this happen?
• Since the order of atoms chosen is arbitrary, any particular choice may not give the shortest or best tree. Sometimes (as above) this is detected.
• (Because there are usually several different refutations, different orders of atoms can give completely different trees)

Completing the Semantic Tree method

Suppose a fully closed semantic tree has been generated using clauses in T. Let T' be the used ground instances (i.e. the ground instances falsified at the leaves).

If the resolvent at a failure node, e.g. \( Q \lor R \lor S \) of Slide 5bvi, is added by resolution to T' then T can be contracted since the resolvent is falsified at the failure node.

The failure node is higher than the two parent clauses and gives rise to a new closure node. This in turn can be used to derive a resolvent and hence a smaller semantic tree.

Eventually, since there were only a finite number of closure nodes at the start and each step removes at least one, a tree of the form on the right will be derived, from which [] is deduced.

Summary of Slides 5

1. Completeness of resolution can be shown in several ways. Most proofs demonstrate completeness using two steps. First a refutation is found using ground instances of the given clauses. This ground refutation is then lifted, to use the original clauses.

2. There is a close relationship between the ground refutation and the lifted refutation.

3. A Semantic Tree formed from a set of unsatisfiable clauses will be finite. Each branch in the tree will falsify some ground instance of one of the given clauses.

4. A failure node A in a semantic tree is a node such that both descendants of A, formed by considering some atom D=T or D=F, are leaves and the falsifying ground instance of the leaf which considered D=T contains the literal \( \neg D \) and the falsifying instance for the other leaf contains the atom D. The resolvent of the two falsifying instances falsifies the branch ending at A.

5. A semantic tree can be used to obtain a ground refutation of ground instances of given clauses and also to find the corresponding refutation.

6. The refutation obtained from a semantic tree indicates where factoring is needed. The refutation never derives a tautology.

7. Semantic Trees could be used to show unsatisfiability of a set of clauses S. But it is not a very practical method in general, since if a “bad” order of atoms is selected the tree could be very large. For this purpose, there is no need to form resolvents, of course, it is enough to know that every branch in the tree falsifies some ground instance of S.

8. Inductive proofs can also be used to show the completeness of ground resolution. (See optional slides!)
A different completeness proof for ground resolution (ie not using semantic trees) uses induction on the number of different atoms occurring in the given set of clauses. As identical literals in a clause are merged, each literal in a clause occurs only once.) You can assume also that there are no tautologies as such clauses can be removed without affecting satisfiability (or, as noted on Slide 5biv, no tautologies are needed), so all literals in a clause involve different atoms. The base case (for one atom A) is easy - all clauses must be unit clauses of the form A or ¬A. If S is unsatisfiable it must contain both kinds and a refutation can easily be found by induction on the number of different literals in a clause are merged, each literal in a clause occurs only once. You can assume also

For S': First construct S1 = {C | C in S and C does not contain B} and then delete any occurrences of ¬B from clauses in S1 to give S'.

For S": First construct S2 = {C | C in S and C does not contain ¬B} and then delete any occurrences of B from clauses in S2 to give S".

(Exercise: show S' and S" must be unsatisfiable - actually, this is the crucial step.

Hint: e.g. for S' assume S' is satisfiable and hence has a model, and then derive a contradiction. )

Hence by the Ind. Hyp. there is a resolution refutation of S' and of S", as all occurrences of B have been removed from S' and S", so they have <k atoms occurring. Now replace the removed literals ¬B/B into the refutations. That of S' will derive ¬B (or still be a refutation) and that of S" will derive B or still be a refutation. In case both ¬B and B are derived a refutation can be found by resolving them. In all cases a refutation is found for S.

There is another proof that uses induction on the number of different clauses. We may use the same idea as before.

The induction step (for k+1 atoms) assumes as Ind. Hyp. that a refutation can be obtained for an unsatisfiable set of ground clauses S with <k atoms. Now, there must be at least one atom (say B) that occurs both positively and negatively in different clauses (if not S can't be unsatisfiable - why?). Form two sets of unsatisfiable clauses, S' and S", as follows:

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