Algorithmic Deduction based on Labelled Tableaux
Project GR/J14745 - Final Report for EPSRC

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This report describes the research results obtained during the EPSRC project GR/J14745, giving firstly a summary of the results, with respect to the objectives given in the Research Grant Report and a brief introduction of the project's research field (in Section 1.2).

1 Introduction

1.1 Summary of the achievements

A uniform theorem prover incorporating free variables and unification techniques has been developed for a family of propositional multiplicative substructural logics, fulfilling objectives 1 and 2. Extensions to the additive operators are ongoing work. The same methodology has been exploited to give a theorem prover based on Natural Deduction, which incorporate new adaptations of various unification techniques. The latter again contributes to objective 2, whereas the application to Natural Deduction provides an example of some of the advantages of the approach, as required by objective 4. Furthermore, a new family of resource logics has been investigated which arise from combining substructural implication systems with the classical theory of modalities, so achieving objective 3. Finally, other advantages of the approach towards application area (objective 4) have been shown by (i) developing a modal labelled deductive system which could be used for formal specifications of Distributed DB and defining a translation approach to LDS which should prove a useful classically-based theorem prover for various non-classical logics. These results are also summarized in Table 1 given below.

Achievements	Project's objectives
Theorem Prover for prop. multiplicative linear logic	1
Unification algorithms for various logics	2
Extension to include free variables	2
Extension to additive connectives	1
Modal Logic within LDS	3
Natural deduction and LDS	4
Combination of modal and substructural logic	4
Translation approach to LDS	1/2/3/4

Table 1: Achievements of the project's objectives

1.2 Labelled Analytic Deduction

This project arises from the combination of two lines of research: the methodology of Labelled Deductive Systems [19] and the recent developments in the field of theorem-proving based on *analytic methods*, such as the method of analytic tableaux or the system **KE** [14], or procedural approaches to Natural Deduction (ND) [3].

The methodology of Labelled Deductive Systems (LDS for short) is a unifying framework for the study of logics and of their interactions. It was proposed by Gabbay a few years ago in response to conceptual pressure from various application areas and their needs, and is now proving a very powerful approach both for theoretical and applied research.

In the LDS approach the basic units of logical derivation are not just formulae but *labelled formulae*, where the labels belong to a given "labelling algebra". The derivation rules act on the labels as well as on the formulae, according to certain fixed rules of propagation. By virtue of the extra power of the labelling algebras, standard (classical or intuitionistic) proof systems can be extended to cover a much wider territory without modifying their structure, the non-standard features being expressed by the algebra of the labels. The role of the labels is that of incorporating *metalevel* information explicitly into the object language. This information can encode the structure of the data, record dependencies within the proof, allow for a stricter control on the derivation process (possibly motivated by some computational application), or incorporate semantic or pragmatic notions into the proof-theory. The labelling algebra represents this metalevel information as a *separate* component of a standard derivation system and can, therefore, be treated as an independent parameter. From the theoretical point of view LDS provides a taxonomy of logics and brings out hidden connections between a variety of different logical systems proposed for a variety of different purposes (whether philosophical, or mathematical or practical). On the other hand, it also provides a flexible tool for developing logical systems which suits particular applications, so extending the potential of a logic-based approach in applied research. Gabbay's theory of Labelled Deductive Systems was initially formulated in terms of a (labelled) generalization of the traditional Natural Deduction rules. Our initial problem was the following: can labels be used in a similar way to generalize classical *analytic* deduction?

The analytic tradition in classical logic is associated with the idea that all deductive arguments can be modelled by a system of *elimination rules* which "analyse" complex formulae by specifying the *consequences* of their truth and falsity in terms of the truth and falsity of their subformulae. See [29] and discussion in [14]. A theory of analytic deduction can be built "around" the cut rule. The subformula principle — which is what counts both for the theory and the applications — does not require the *elimination* of cuts, but only their *restriction* to "analytic" applications, i.e. applications involving exclusively subformulae of the theorem. The extra inferential power generated by analytic cuts can then be exploited to simplify the elimination rules. In [14] these ideas led to the formulation of a new system of analytic deduction for classical logic, the system **KE**which was proved to be essentially more efficient than the classical tableau method (in the sense that the analytic **KE** system can linearly simulate the tableau method, but the classical tableau method cannot p-simulate the analytic **KE** system).

The main aim of our project was to develop a systematic theory of *Labelled Analytic Deduction* as part of the LDS program. In the next sections we shall consider the main results of our research. Specifically, Section 2 addresses research objectives 1 and 2 as given in the reserach grant report. Section 3 addresses objective 3 and Sections 4 and 5 address objective 4. In Section 6 we describe our continuing work.

2 Uniform Procedures for Substructural Logics

We started our investigation with the family of substructural logics. The term "substructural logics" is now of widespread use to indicate a family of logics weaker than classical logic (including intuitionistic, relevant[1] and linear logic), which result from dropping some or all of the traditional structural rules of the classical sequent calculus (see [15] for an introduction). The study of these logics yields a subtle analysis of the logical operators and of the structure of proofs which is arguably more in tune with the needs of applications and is, therefore, proving useful in several applications of logic to Computer Science and AI (see, for instance, [22] for an application of relevance logic to modular reasoning systems, and [1] for an overview of computational applications of Linear Logic). Our research showed how a *uniform* and *transparent* system of analytic deduction for all these logics can be obtained via a generalization of the classical **KE** system via the LDS methodology[2]. We believe that our methods will prove convenient for the fast-growing community of researchers who are involved in the development and implementation of algorithmic proof systems for substructural logics. Moreover, the our basic results are not specific to substructural logics and we regard them as part of a general theory of Labelled Analytic Deduction which will be further developed in future research.

The main features of the labelled KE method for the family of substructural logics are the following:

- each logic in the family is associated with a "labelling algebra";
- the tree-expansion rules (for labelled formulae) are the same for all the logics in the family;
- the difference between one logic and the other is captured by the conditions under which a branch is declared closed; such conditions depend only on the labelling algebra associated with each logic;
- classical and intuitionistic negations are characterized uniformly, by means of the same tree-expansion rules, and their difference is reduced to a difference in the labelling algebra used in closing a branch.

2.1 Development of a Theorem Prover

In accordance with the LDS approach, the rules of the labelled **KE** method operate both on formulae and on their associated labels. We developed a theorem prover based on two components (i) an *inferential engine* based on the classical **KE** system, and (ii) a *labelling engine* depending on the algebra of the labels. As for the first component, we based our theorem prover on a fairly standard implementation of the classical **KE** system (see for example [26]). As for the second component, the methods we developed within our project required the solution of systems of inequations in a given class of algebras, a problem we addressed by means of unification-like techniques (especially AC-unification). So far, this objective has been fully achieved for the special, if important, case of (multiplicative) propositional Linear Logic. The algorithms developed are described in [17, 4, 5]. and [7]. The resulting theorem

[1] Related work on tableau-like methods for relevance logics is contained in [16, 23, 31].

[2] For a related LDS-based approach to categorial theorem proving, in terms of labelled proof nets, the reader is referred to [24]

prover is described in [9]. The use of unification enabled the development of resource abduction in which the instantiation of labels indicates the abductive data needed to prove a given theorem.

The work of Russo [27] on modal logics introduces the possibility of a completely different approach, in which a given problem is partially compiled into a first order theory. The sentences of the theory are derived from compiling the application of the appropriate inference rules to a representation of each sub-formula of the formula to be proven. The effect is to enable an extension of the Davis Putnam procedure [10] to be used to refute the translated sentences in which only ground atoms occur. On this point see [6].

A standard tableaux approach to modal logics using prefixed tableaux [18] and in which prefixes are seen as labels was adapted to the **KE**system in [25].

3 Modal Logics

3.1 A uniform approach

Modal logic introduces the monadic modalities $\Box$, meaning "necessarily", or "always", and $\Diamond$, meaning "possibly", or "sometimes". A semantics can be given in terms of *possible worlds*, which are sets of formulas that locally obey the rules of classical logic and which are also subject to special rules for the modal operators.

The notion of modality is central in pure and applied logic. Many systems presented to formalise some application area require the addition of modality to the language for a variety of reasons: to cater for changes of the system in time, or perhaps for the dependency of the system on the context, or even to bring metalevel notions into the object level. However, it is often the case the logic involved is *not* classical logic, but some other logical system which provides a more direct modelling of the processes involved in a particular application area. The introduction of modalities into such non-classical logical systems adds a new dimension which can explicitly and naturally account for "accessibility" relations involved in the processes that are being modelled.

In our research we addressed the problem of "grafting" modalities onto substructural implication logics. In particular, we investigated the semantics of the logical systems obtained by introducing the modalities $\Box$ and $\Diamond$ into the family of *substructural* implication logics. Then, in the spirit of the LDS (Labelled Deductive Systems) methodology, we "imported" this semantics into the classical proof system **KE**. This led to the formulation of a uniform labelled refutation system for the new logics which is a natural extension of a system for substructural implication discussed above.

Formally we have a system with the conditional operator $\rightarrow$ and we want to add unary operators $\Box$ and $\Diamond$ which behave like modalities. We would like to do this in a way which is compatible with the spirit of the system, i.e. to define modalities from within the system in some natural way. Substructural logics can be characterized by structures of "objects" which can be understood as "pieces of information" or, sometimes, as "resources". A natural way of defining modalities within such semantics consists therefore in adding an accessibility relation between pieces of information. This is the approach adopted in [13]. By analogy with classical modal logics the intuitive idea is that the verification of a proposition of the form "$\Box A$" or "$\Diamond A$" by means of a given piece of information (or resource) x, depends on what is verified by other pieces of information (or resources) "accessible" from x. Unlike classical modal logics, however, the verification of non-modal formulae, such as a conditional formula, by a given piece of information x also depends on what is or is not verified elsewhere in the structure (as with the intuitionistic conditional).

Previous work on this topic had been concentrating on intuitionistic modal logics (see [30] for an overview). In [13] we started developing a more general approach which fits naturally into the framework for substructural logics presented in [12].

3.2 A First Order Approach

A different approach is taken in [28], in which the labels represent the possible worlds of the traditional semantics. Although this is a more standard approach, it also brings benefits in the form of a uniform treatment of non-axiomatisable modal logics as well as those that are axiomatisable. It uses the translation approach described in Section 2.1 and is based on natural deduction instead of tableau methods.

The approach for propositional logic is easily extended to predicate logic with quantifiers, and again gives a uniformity over all types of extension, with constant or varying domains. Although it hasn't been proven, the simplicity of the resulting translation means the extended Davis Putnam procedure developed for the substructural logics should prove a useful theorem prover.

4 Natural Deduction

Although the KE method is relatively more natural than standard tableau methods, in that much of the reasoning is in a linear fashion, e.g. 'from A, conclude B, then C, then D and so on', until the final goal is reached, humans like to reason procedurally. That is, they usually argue thus:

> to prove A, try to prove B, and to prove B try to prove C, and so on.

This is a more structured method and leads to well-understood proofs, which, although derived backwards from the goal, can be read forwards from the data.

It seemed interesting, therefore, to compare the KE approach with that of natural deduction, and in particular with a format for natural deduction developed by Gabbay in [19] and Broda in [3]

There is, indeed, a close correspondence between procedural natural deduction and KE tableaux, if the latter are formed in a particular way. This was initially described and applied to substructural logics in [8]. It is described fully in [2] leading to the completeness theorem for ND. A sound theorem prover based on the same idea is presented in [4].

In addition the extra structure led us to develop a very simple algorithm for solving abductive style problems [].

5 Multi-Implication Logics

The last decade has seen a proliferation of different logical systems proposed for a variety of different purposes, both theoretical and practical. These logics appear to satisfy the needs of different application areas or to capture different interpretations of the logical operators. This is particularly apparent in the case of the conditional operator. The different implication logics which have been proposed in the literature seem to succeed in modelling some aspect of the ordinary use of this operator, or in suggesting useful non-standard interpretations. What emerges from these developments is a class of operators bearing a family resemblance to each other, each of which may fit a different application. So, a crucial problem, both in pure and applied logic, is that of developing a general methodology for combining logical systems— each with its own operators and its own semantics— into larger systems of "modular reasoning" where the user can easily switch from one context to the other. The proof-theoretical side of this problem can be formulated as follows: how can we combine different proof systems (e.g. tableau systems) into more general ones, possibly preserving the desirable properties of the originals? In [21, 20] Gabbay investigated the general notion of fibred semantics which provides the theoretical foundations for the development of such hybrids. In a recent position paper [11] we merged this line of research with our results on substructural implication (described in Section 2 above), and investigated the notion of "fibred tableau" that arises from their interaction.

6 Ongoing Research

6.1 Extensions to include additives

The theorem prover developments described in Section 2.1 were for the multiplicative fragments of substructural logics. It is relatively easy to give sequent rules for the additive fragments, but has proved rather harder to give the labelling discipline needed for LDS. However this problem is presently close to solution and the results will be presented in a forthcoming paper.

6.2 Many-Valued Logics

Recent work pursued in collaboration with Dr. Nicola Olivetti (University of Torino) shows that our approach can be naturally extended to characterize *many-valued logics*— a family of logics which is of crucial importance in today's applied research— as specializations of substructural logics.

The basic idea is to transform the algebraic semantics of many-valued logics— based on MV-algebras— into a Kripke-style semantics, by means of the general transformation methods outlined in [2], and then to turn this semantics into a suitable labelling discipline. We expect this labelling discipline to be essentially the same as the one used in our work on substructural logics.

6.3 Application to Distributed Databases

In Russo's work [27, 28], labels may be interpreted as nodes representing individual collection of data, which may then model distributed database systems. Modalities and accessibility relations between nodes may be used to model the association between the databases. We intend to apply this approach to provide a logical model of reasoning among distributed databases, both sequential and concurrent.

7 Project-Related Publications and Reports

References

[1] V. Alexiev. Applications of linear logic to computation: An overview. *Bulletin of the IGPL*, 2(1):77–107, 1994.

[2] K. Broda, M. D'Agostino, and A. Russo. *Logic, Language and Reasoning, Essays in Honor of Dov Gabbay*, chapter Transformation methods in LDS. Kluwer Academic Publisher, 1996. To appear.

[3] K. Broda, S. Eisenbach, H. Khoshnevisan, and S. Vickers. *Reasoned Programming*. Prentice Hall, 1994.

[4] K. Broda, M. Finger, and A. Russo. Lds – natural deduction for substructural logics. In *WOLLIC, Proceedings of the IIIrd Workshop on Logic, Language, Information and Computation*, volume (ISSN 0945-9103) of *Journal of the Interest Group in Pure and Applied Logics*, 1996. (Submitted).

[5] K. Broda, M. Finger, and A. Russo. LDS – natural deduction for substructural logics. *Journal of the Interest Group in Pure and Applied Logics (ISSN 0945-9103)*, 1996. (Abstract).

[6] K. Broda and J. Fleuriot. A translation approach to intuitionistic logic. Technical report, Department of Computing, Imperial College, London, 1996.

[7] K. Broda, D'Agostino M., and M. Finger. A computational kripke semantics for multiplicative propositional linear logic. Working paper.

[8] Krysia Broda and Marcello D'Agostino. Finding proofs in substructural logics. Presented at the Workshop in Proof Theory, LInear Logic and Categorical Gramars, Rome, 1994.

[9] Krysia Broda and Marcelo Finger. Ke-tableaux for a fragment of linear logic. Workshop on Theorem Proving and Analytic Tableaux and Related Methods, Stboar, Germany, 1995. Extended Abstract.

[10] Chin-Liang Chang and Richard Lee. *Symbolic Logic and Mechanical Theorem Proving*. Academic Press, 1973.

[11] M. D'Agostino and Dov M. Gabbay. Fibred tableaux for multi-implication logics. In P. Miglioli, U. Moscato, D. Mundici, and M. Ornaghi, editors, *TABLEAUX '96. Proceedings of the 5th International Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, volume 1071 of *Lecture Notes in Artificial Intelligence*, pages 1–35. Springer-Verlag, 1996.

[12] Marcello D'Agostino and Dov M. Gabbay. A generalization of analytic deduction via labelled deductive systems.Part I: Basic substructural logics. *Journal of Automated Reasoning*, 13:243–281, 1994.

[13] Marcello D'Agostino, Dov M. Gabbay, and Alessandra Russo. Grafting modalities onto substructural implication logics. *Studia Logica*, 1996. To appear.

[14] Marcello D'Agostino and Marco Mondadori. The taming of the cut. *Journal of Logic and Computation*, 4:285–319, 1994.

[15] Kosta Dôsen. A historical introduction to substructural logics. In P. Schroeder Heister and Kosta Dôsen, editors, *Substructural Logics*, pages 1–31. Oxford University Press, 1993.

[16] J. M. Dunn. Intuitive semantics for first-degree entailment and coupled trees. *Philosophical Studies*, 29:149–168, 1976.

[17] M. Finger. Linear diophantine equations. Working paper.

[18] M. Fitting. *Proof methods for modal and intuitionistic logics.* Dordrecht, Reidel, 1983.

[19] D.M. Gabbay. Labelled Deductive Systems, Volume 1 - Foundations. Technical Report MPI-I-94-223, Max-Planck-Institut Für Informatik, 1994. Preliminary partial draft of a book intended for Oxford University Press.

[20] Dov Gabbay. Fibred semantics and the weaving of logics, part 2: Fibring non-monotonic logics. In *Proceedings of Logic Colloquium*, SILLI Book Series, pages 75–94. Cambridge University Press, 1992.

[21] Dov Gabbay. Fibred semantics and the weaving of logics, part 1: Modal and intuitionistic logic. *Journal of Symbolic Logic*, 1996. To appear.

[22] James Garson. Modularity and relevant logic. *Notre Dame Journal of Formal Logic*, 30:207–223, 1989.

[23] Michael A. McRobbie and Nuel D. Belnap. Relevant analytic tableaux. *Studia Logica*, XXXVIII:187–200, 1979.

[24] Michael Moortgat. Labelled deductive systems for categorial theorem proving. In Dekker and Stokhof, editors, *Proceedings of the 8th Amsterdam Colloquium*, 1992.

[25] Jeremy Pitt and Jim Cunningham. Distributed modal theorem proving with the system ke. In P. Miglioli, U. Moscato, D. Mundici, and M. Ornaghi, editors, *TABLEAUX '96. Proceedings of the 5th International Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, volume 1071 of *Lecture Notes in Artificial Intelligence*, pages 1–35. Springer-Verlag, 1996.

[26] Jeremy Pitt and Jim Cunningham. Theorem proving and model building with the calculus ke. *Journal of the IGPL*, 4(1), 1996.

[27] Alessandra Russo. Generalising propositional modal logic using labelled deductive systems. *Applied Logic*, 1996. To appear.

[28] Alessandra Russo. *Modal Logics as Labelled Deductive Systems*. PhD thesis, Imperial College of Science, Technology and Medicine, Department of Computing, London, 180, Queen's Gate, SW7 2BZ, London , U.K., 1996.

[29] R. Smullyan. *First-Order Logic*. Springer, Berlin, 1968.

[30] Alex K. Sympson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1993.

[31] P. B. Thistlewaite, M. A. McRobbie, and B. K. Meyer. *Automated Theorem Proving in Non Classical Logics*. Pitman, 1988.