A robust preconditioner for the stationary incompressible Navier–Stokes equations

Lawrence Mitchell[1,*]

P. E. Farrell (Oxford) L. R. Scott (Chicago) F. Wechsung (Oxford)

10 June 2019

[1]Department of Computer Science, Durham University
*`lawrence.mitchell@durham.ac.uk`

Setting

Firedrake `www.firedrakeproject.org` [...] *is an automated system for the solution of partial differential equations using the finite element method.*

- Finite element problems specified with *embedded* domain specific language, UFL (Alnæs, Logg, Ølgaard, Rognes, and Wells 2014) from the FEniCS project.
- *Runtime* compilation to optimised, low-level (C) code.
- PETSc for meshes and (algebraic) solvers.

Rathgeber et al. (2016) `arXiv:1501.01809 [cs.MS]`

Advert

3rd Firedrake user meeting is in Durham 26 & 27 September 2019.

`www.firedrakeproject.org/firedrake_19.html`

The problem

Stationary, Newtonian, incompressible Navier–Stokes

Find $(u, p) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ such that

$$
\begin{aligned}
-\nu \nabla^2 u + (u \cdot \nabla) u + \nabla p = f \quad & \text{in } \Omega, \\
\nabla \cdot u = 0 \quad & \text{in } \Omega,
\end{aligned}
$$

with suitable boundary conditions, and ν the kinematic viscosity.

The problem

Stationary, Newtonian, incompressible Navier–Stokes

Find $(u, p) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ such that

$$\begin{aligned} -\nu \nabla^2 u + (u \cdot \nabla) u + \nabla p = f \quad & \text{in } \Omega, \\ \nabla \cdot u = 0 \quad & \text{in } \Omega, \end{aligned}$$

with suitable boundary conditions, and ν the kinematic viscosity.

Multiple solutions

For $\nu \to 0$, these equations admit multiple solutions

The problem

Stationary, Newtonian, incompressible Navier–Stokes

Find $(u, p) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ such that

$$
\begin{aligned}
-\nu\nabla^2 u + (u \cdot \nabla)u + \nabla p = f \quad & \text{in } \Omega, \\
\nabla \cdot u = 0 \quad & \text{in } \Omega,
\end{aligned}
$$

with suitable boundary conditions, and ν the kinematic viscosity.

Multiple solutions

For $\nu \to 0$, these equations admit multiple solutions

Motivating question

What are the solution(s) as ν varies?

Solving for the Newton step

Newton linearisation

$$
\begin{aligned}
-\nu \nabla^2 u + (u \cdot \nabla) w + (w \cdot \nabla) u + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega.
\end{aligned}
$$

LU factorisation

- ✓ Scales well with $\nu \to 0$
- ✗ Scales poorly with dof count

Existing preconditioners

- ✗ Convergence degrades with $\nu \to 0$
- ✓ Scales well with dof count

This talk

First preconditioner to scale well with ν and dof count in 3D.

Block preconditioners

Stokes

Stokes' equations

Find $(u, p) \in H^1(\Omega)^d \times L^2(\Omega)$ such that

$$\begin{aligned} -\nu \nabla^2 u + \nabla p &= f \quad \text{in } \Omega, \\ \nabla \cdot u &= 0 \quad \text{in } \Omega, \end{aligned}$$

Discretisation

Choosing an inf-sup stable element pair yields

$$\mathcal{J} x := \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Preconditioners for block matrices

Block factorisation (Murphy, Golub, and Wathen 2000)

Build preconditioners based on

$$\mathcal{J}^{-1} = \begin{pmatrix} I & -A^{-1}B^T \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -BA^{-1} & I \end{pmatrix}$$

where S is the (usually dense!) Schur complement

$$S = -BA^{-1}B^T.$$

Block factorisation (Murphy, Golub, and Wathen 2000)

Build preconditioners based on

$$\mathcal{J}^{-1} = \begin{pmatrix} I & -A^{-1}B^T \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -BA^{-1} & I \end{pmatrix}$$

where S is the (usually dense!) Schur complement

$$S = -BA^{-1}B^T.$$

PDE-specific challenges

Find cheap, fast, approximations $\tilde{A}^{-1}$ and $\tilde{S}^{-1}$ to A^{-1} and S^{-1}.

Choosing $\tilde{A}^{-1}$ and $\tilde{S}^{-1}$

Stokes (Silvester and Wathen 1994)

Multigrid for $\tilde{A}^{-1}$, and choose $\tilde{S}^{-1} \sim M_p^{-1}$ (M_p the pressure mass matrix).

Choosing $\tilde{A}^{-1}$ and $\tilde{S}^{-1}$

Stokes (Silvester and Wathen 1994)

Multigrid for $\tilde{A}^{-1}$, and choose $\tilde{S}^{-1} \sim M_p^{-1}$ (M_p the pressure mass matrix).

Bad news

Mesh independent, but for *Navier–Stokes*, choosing $\tilde{S}^{-1} \sim M_p^{-1}$ degrades like $\mathcal{O}(\nu^{-2})$.

Choosing $\tilde{A}^{-1}$ and $\tilde{S}^{-1}$

Stokes (Silvester and Wathen 1994)

Multigrid for $\tilde{A}^{-1}$, and choose $\tilde{S}^{-1} \sim M_p^{-1}$ (M_p the pressure mass matrix).

Bad news

Mesh independent, but for *Navier–Stokes*, choosing $\tilde{S}^{-1} \sim M_p^{-1}$ degrades like $\mathcal{O}(\nu^{-2})$.

PCD for Navier–Stokes (Kay, Loghin, and Wathen 2002)

Approximate S^{-1} with convection-diffusion solves on pressure space.
Mesh independent, but degrades like $\mathcal{O}(\nu^{-1/2})$.

Performance of pressure convection-diffusion (PCD)

$1/h$	# degrees of freedom	Reynolds number		
		10	100	1000
2^4	8.34×10^2	22.0	40.4	103.3
2^5	3.20×10^3	23.0	41.3	137.7
2^6	1.25×10^4	24.5	42.0	157.0
2^7	4.97×10^4	25.5	42.7	149.0
2^8	1.98×10^5	26.0	44.0	137.0

Table 1: Average number of outer Krylov iterations per Newton step for the 2D regularized lid-driven cavity problem with PCD preconditioner. Obtained with IFISS v3.5, $Q_1 - P_0$ element pair.

Augmented Lagrangian preconditioners

SIAM J. SCI. COMPUT.
Vol. 28, No. 6, pp. 2095–2113

AN AUGMENTED LAGRANGIAN-BASED APPROACH TO THE OSEEN PROBLEM*

MICHELE BENZI† AND MAXIM A. OLSHANSKII‡

Abstract. We describe an effective solver for the discrete Oseen problem based on an augmented Lagrangian formulation of the corresponding saddle point system. The proposed method is a block triangular preconditioner used with a Krylov subspace iteration like BiCGStab. The crucial ingredient is a novel multigrid approach for the (1,1) block, which extends a technique introduced by Schöberl for elasticity problems to nonsymmetric problems. Our analysis indicates that this approach results in fast convergence, independent of the mesh size and largely insensitive to the viscosity. We present experimental evidence for both isoP2-P0 and isoP2-P1 finite elements in support of our conclusions. We also show results of a comparison with two state-of-the-art preconditioners, showing the competitiveness of our approach.

SIAM J. Sci. Comput.
Vol. 28, No. 6, pp. 2095–2113

AN AUGMENTED LAGRANGIAN-BASED APPROACH TO THE OSEEN PROBLEM*

MICHELE BENZI† AND MAXIM A. OLSHANSKII‡

Abstract. We describe an effective solver for the discrete Oseen problem based on an augmented Lagrangian formulation of the corresponding saddle point system. The proposed method is a block triangular preconditioner used with a Krylov subspace iteration like BiCGStab. The crucial ingredient is a novel multigrid approach for the (1,1) block, which extends a technique introduced by Schöberl for elasticity problems to nonsymmetric problems. Our analysis indicates that this approach results in fast convergence, independent of the mesh size and largely insensitive to the viscosity. We present experimental evidence for both isoP2-P0 and isoP2-P1 finite elements in support of our conclusions. We also show results of a comparison with two state-of-the-art preconditioners, showing the competitiveness of our approach.

Observation

- ✓ Reynolds-robust preconditioning
- ✗ No-one else appears to have implemented the full scheme (2006–2018)

Objectives

- Can we make the first general implementation of the method? ✓
- Can we extend the solver and discretisation to three dimensions? ✓

Objectives

- Can we make the first general implementation of the method? ✓
- Can we extend the solver and discretisation to three dimensions? ✓

Three ideas

1. Control Schur complement with an augmented Lagrangian term
2. Kernel-capturing multigrid relaxation
3. Robust multigrid prolongation

Farrell, Mitchell, and Wechsung (2018) `arXiv:1810.03315 [math.NA]`

Augmented Lagrangian approach

Stokes again

$$\begin{aligned} \underset{u \in V}{\text{minimise}} \quad & \int \nabla : \nabla \, \mathrm{d}x - \int f \cdot v \, \mathrm{d}x \\ \text{subject to} \quad & \nabla \cdot v = 0 \end{aligned}$$

Augmented Lagrangian approach

Stokes again, with augmented Lagrangian term

$$\begin{aligned} \underset{u \in V}{\text{minimise}} \quad & \int \nabla : \nabla \, \mathrm{d}x - \int f \cdot v \, \mathrm{d}x + \gamma \int (\nabla \cdot v)^2 \, \mathrm{d}x \\ \text{subject to} \quad & \nabla \cdot v = 0 \end{aligned}$$

Augmented Lagrangian approach

Stokes again, with augmented Lagrangian term

$$\begin{aligned} \underset{u \in V}{\text{minimise}} \quad & \int \nabla : \nabla \,\mathrm{d}x - \int f \cdot v \,\mathrm{d}x + \gamma \int (\nabla \cdot v)^2 \,\mathrm{d}x \\ \text{subject to} \quad & \nabla \cdot v = 0 \end{aligned}$$

Doesn't change solution, since $\nabla \cdot v = 0$.

Theorem (Hestenes, Fortin, Glowinski, Olshanksii, ...)

As $\gamma \to \infty$, the Schur complement is well approximated by $S^{-1} \sim -(1+\gamma)M_p^{-1}$.

...or discretely

Discrete augmented Lagrangian

$$\begin{pmatrix} A + \gamma B^T M_p^{-1} B & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

...or discretely

Discrete augmented Lagrangian

$$\begin{pmatrix} A + \gamma B^T M_p^{-1} B & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Doesn't change solution, since $Bu = 0$

New Schur complement

$$\hat{S}^{-1} = S^{-1} - \gamma M_p^{-1}$$

Again, as $\gamma \to \infty$, the Schur complement is well approximated by $\hat{S}^{-1} \sim -(1 + \gamma) M_p^{-1}$.

Discrete augmented Lagrangian

$$\begin{pmatrix} A + \gamma B^T M_p^{-1} B & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Doesn't change solution, since $Bu = 0$

New Schur complement

$$\hat{S}^{-1} = S^{-1} - \gamma M_p^{-1}$$

Again, as $\gamma \to \infty$, the Schur complement is well approximated by $\hat{S}^{-1} \sim -(1+\gamma) M_p^{-1}$.

These results still hold for Navier–Stokes!

Conservation of misery

We now must solve either: find $u \in V \subseteq H^1(\Omega; \mathbb{R}^d)$ such that

Continuous stabilisation

$$\nu(\nabla u, \nabla v) + ((w \cdot \nabla)u, v) + ((u \cdot \nabla)w, v) + \gamma(\nabla \cdot u, \nabla \cdot v) = (f, v)$$

or

Discrete stabilisation

$$\nu(\nabla u, \nabla v) + ((w \cdot \nabla)u, v) + ((u \cdot \nabla)w, v) + \gamma(\overline{\nabla \cdot u}, \overline{\nabla \cdot v}) = (f, v)$$

for all $v \in V$, where $\overline{x}$ is the L^2 projection onto the discrete pressure space.

Conservation of misery II

Good news

The Schur complement becomes easy to approximate as $\gamma \to \infty$.

Conservation of misery II

Good news

The Schur complement becomes easy to approximate as $\gamma \to \infty$.

Bad news

Our usual multigrid approaches for $\tilde{A}^{-1}$ no longer work for $A_\gamma := A + \gamma B^T M_p^{-1} B$, and get *worse* as $\gamma \to \infty$

Conservation of misery II

Good news

The Schur complement becomes easy to approximate as $\gamma \to \infty$.

Bad news

Our usual multigrid approaches for $\tilde{A}^{-1}$ no longer work for $A_\gamma := A + \gamma B^T M_p^{-1} B$, and get *worse* as $\gamma \to \infty$

	LU for A_γ^{-1}	AMG for A_γ^{-1}
$\gamma = 10^{-1}$	15	18
$\gamma = 1$	6	40
$\gamma = 10^{1}$	3	107

Conservation of misery II

Good news

The Schur complement becomes easy to approximate as $\gamma \to \infty$.

Bad news

Our usual multigrid approaches for $\tilde{A}^{-1}$ no longer work for $A_\gamma := A + \gamma B^T M_p^{-1} B$, and get *worse* as $\gamma \to \infty$

	LU for A_γ^{-1}	AMG for A_γ^{-1}
$\gamma = 10^{-1}$	15	18
$\gamma = 1$	6	40
$\gamma = 10^1$	3	107

Goal

Find a γ-robust multigrid method for A_γ^{-1}.

Robust multigrid

Stokes case

- Ignorning advection, then the top-left block corresponds to discretisation of

$$a_\gamma(u, v) = \underbrace{\int_\Omega \nu \nabla u : \nabla v \, \mathrm{d}x}_{\text{sym. pos. def.}} + \underbrace{\int_\Omega \gamma \, \mathrm{div}(u) \, \mathrm{div}(v) \, \mathrm{d}x}_{\text{sym. pos. semi-def.}}$$

Stokes case

- Ignorning advection, then the top-left block corresponds to discretisation of

$$a_\gamma(u, v) = \underbrace{\int_\Omega \nu \nabla u : \nabla v \,\mathrm{d}x}_{\text{sym. pos. def.}} \quad + \quad \underbrace{\int_\Omega \gamma\, \mathrm{div}(u)\, \mathrm{div}(v)\, \mathrm{d}x}_{\text{sym. pos. semi-def.}}$$

- The semi-definite term is singular on all solenoidal fields $\Rightarrow$ the system becomes *nearly singular* as $\gamma \to \infty$

Stokes case

- Ignorning advection, then the top-left block corresponds to discretisation of

$$a_\gamma(u, v) = \underbrace{\int_\Omega \nu \nabla u : \nabla v \, \mathrm{d}x}_{\text{sym. pos. def.}} \quad + \quad \underbrace{\int_\Omega \gamma \, \mathrm{div}(u) \, \mathrm{div}(v) \, \mathrm{d}x}_{\text{sym. pos. semi-def.}}$$

- The semi-definite term is singular on all solenoidal fields $\Rightarrow$ the system becomes *nearly singular* as $\gamma \to \infty$
- To build a γ-robust scheme we need (Schöberl 1999)
  - a γ-robust smoother;
  - a prolongation operator with γ-independent continuity constant.

Idea: kernel-capturing relaxation

Consider the problem: for $\alpha, \beta \in \mathbb{R}$, find $u \in V$ such that

$$\alpha a(u, v) + \beta b(u, v) = (f, v) \quad \forall v \in V,$$

where a is SPD, and b is symmetric positive semidefinite.

Relaxation method

Choose a subspace decomposition

$$V = \sum_i V_i,$$

solve the problem on each subspace and combine the updates.

Example

If each V_i is the span of a single basis function, this defines a Jacobi (Gauss-Seidel) iteration.

Idea: kernel-capturing relaxation

Define the kernel as

$$\mathcal{N} := \{u \in V : b(u, v) = 0 \ \forall v \in V\}.$$

Theorem (Schöberl (1999); Lee, Wu, Xu, Zikatanov (2007))

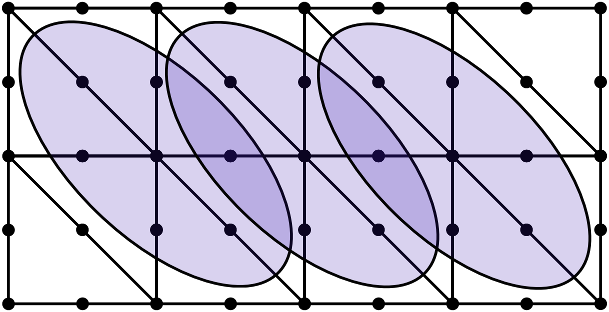
An additive subspace correction method using subspaces $\{V_i\}$ is parameter robust if every $u \in \mathcal{N}$ can be written as a sum $u = \sum_i u_i$ with $u_i \in V_i \cap \mathcal{N}$, i.e.

$$\mathcal{N} = \sum_i \mathcal{N} \cap V_i.$$

"*The subspace decomposition* captures the kernel."

Relaxation

In 2D we consider $P_2 - P_0$ elements. For these one can show that *star* patches



satisfy the kernel decomposition property.

Idea: robust prolongation

Theorem (Schöberl (1999))

Let the prolongation $P : V_H \to V_h$. A robust multigrid cycle requires

$$\nu \|\nabla P u_H\|_{L^2}^2 + \gamma \|\overline{\nabla \cdot P u_H}\|_{L^2}^2 \leq C(\nu \|\nabla u_H\|_{L^2}^2 + \gamma \|\overline{\nabla \cdot u_H}\|_{L^2}^2)$$

with C independent of both ν and γ.

Idea: robust prolongation

Theorem (Schöberl (1999))

Let the prolongation $P : V_H \to V_h$. A robust multigrid cycle requires

$$\nu \|\nabla P u_H\|_{L^2}^2 + \gamma \|\overline{\nabla \cdot P u_H}\|_{L^2}^2 \leq C(\nu \|\nabla u_H\|_{L^2}^2 + \gamma \|\overline{\nabla \cdot u_H}\|_{L^2}^2)$$

with C independent of both ν and γ.

Observation

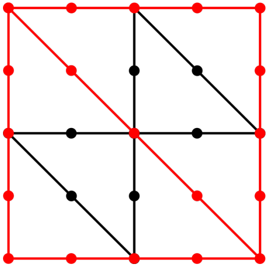
Notice that if $\nabla \cdot u_H = 0$ then we have

$$\nu \|\nabla P u_H\|_{L^2}^2 + \gamma \|\overline{\nabla \cdot P u_H}\|_{L^2}^2 \leq C \nu \|\nabla u_H\|_{L^2}^2$$

$\Rightarrow$ we need P to map coarse-grid div-free fields to (nearly) div-free fields on the fine grid.

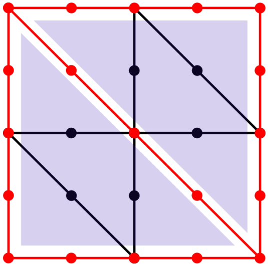
Robust prolongation: 2D

- Discrete divergence $\Leftrightarrow$ flux across facets
- $V_h \subset V_h \implies$ flux across coarse-grid faces is preserved.



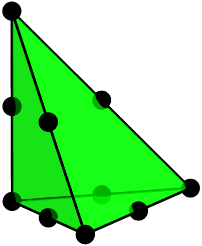
Robust prolongation: 2D

- Discrete divergence $\Leftrightarrow$ flux across facets
- $V_h \subset V_h \implies$ flux across coarse-grid faces is preserved.
- $P_2 - P_0$ not a *divergence-free* discretisation $\Rightarrow$ solve a local Stokes problem to *fix* the flux on new facets.



What about 3D

Which element should we choose?

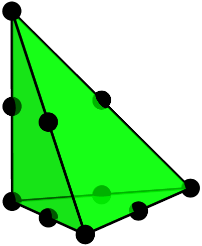


P_2

stable

What about 3D

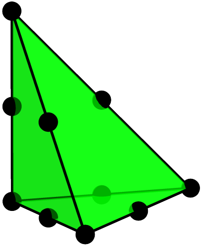
Which element should we choose?

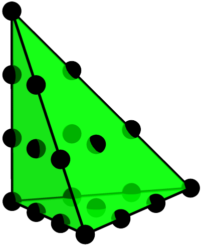


	P_2
stable	✗

What about 3D

Which element should we choose?

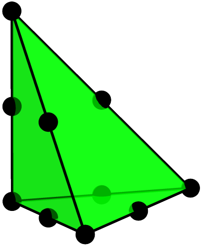
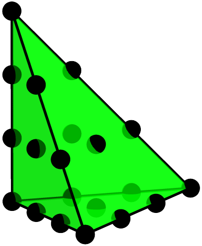
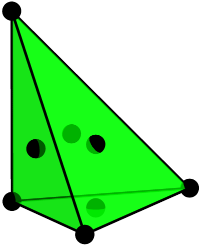




	P_2	P_3
stable	✗	✓

What about 3D

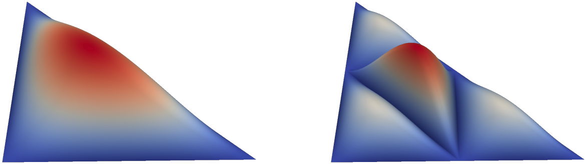
Which element should we choose?

	P_2	P_3	$P_1 \oplus \mathrm{FB}$
stable	✗	✓	✓

Prolongation — 3D

Problem: $V_H \not\subset V_h$



Solution:

1. Split into P_1 and facet bubble part, prolong the two parts separately.
2. Scale the facet bubble components to preserve flux across coarse grid facets.
3. Solve local Stokes problem to fix divergence inside coarse cell.

Numerical results

Full solver

- Continuation
  - Newton solver with line search
    - Krylov solver (FGMRES)
      - Block preconditioner
        - Approximate Schur complement inverse
        - F-cycle on augmented momentum block
          - Coarse grid solver
            - LU factorization on assembled matrix
          - Prolongation operator
            - Local solves over coarse cells
          - Relaxation
            - GMRES
              - Overlapping additive Schwarz iteration

Numerical results — 2D

# ref.	# dofs	Reynolds number 10	100	1000	5000	10000
Lid Driven Cavity						
1	1.0×10^4	3.00	4.00	5.33	8.50	11.0
2	4.1×10^4	2.50	3.67	6.00	8.00	9.50
3	1.6×10^5	2.50	3.00	5.67	7.50	9.00
4	6.6×10^5	2.50	3.00	5.00	7.00	8.00
Backwards Facing Step						
1	1.1×10^6	2.33	3.75	4.50	4.50	8.00
2	4.5×10^6	3.00	3.25	4.50	4.50	5.50
3	1.8×10^7	3.00	5.75	4.00	4.00	6.00

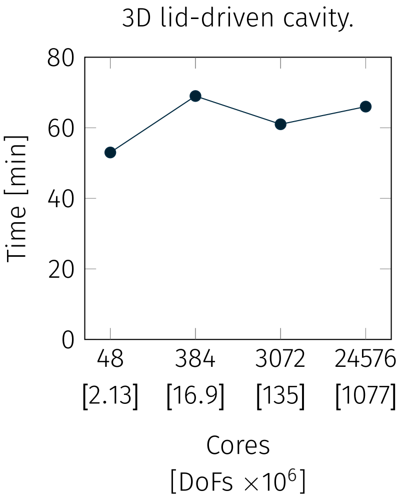
Table 2: Average number of outer Krylov iterations per Newton step for two 2D benchmark problems.

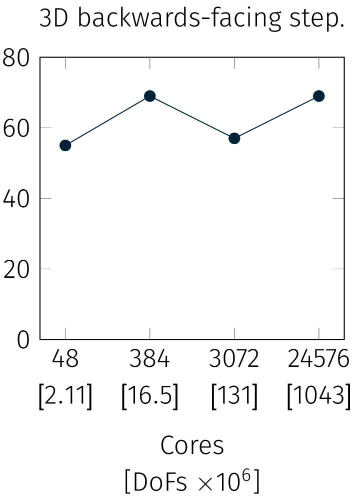
Numerical results — 3D

# ref.	# dofs	Reynolds number 10	100	1000	2500	5000
Lid Driven Cavity						
1	2.1×10^6	4.50	4.00	5.00	4.50	4.00
2	1.7×10^7	4.50	4.33	4.50	4.00	4.00
3	1.3×10^8	4.50	4.33	4.00	3.50	7.00
4	1.1×10^9	4.50	3.66	3.00	5.00	5.00
Backwards facing step						
1	2.1×10^6	4.50	4.00	4.00	4.50	7.50
2	1.7×10^7	5.00	4.00	3.33	4.00	10.00
3	1.3×10^8	6.50	4.50	3.50	3.00	8.00
4	1.0×10^9	7.50	3.50	2.50	3.00	6.00

Table 3: Average number of outer Krylov iterations per Newton step for two 3D benchmark problems.

Computational performance — 3D





50 continuation steps

Pressure robust discretisations

Why pressure robust?

Standard error estimate for Stokes (John, Linke, Merdon, Neilan, and Rebholz 2017)

$$\|\nabla(u - u_h)\|_{L^2} \leq 2 \inf_{\tilde{u}} \|\nabla(u - \tilde{u})\|_{L^2} + \nu^{-1} \inf_{\tilde{p}} \|p - \tilde{p}\|_{L^2}$$

If the element pair is *divergence-free* ($\nabla \cdot V_h \subset Q_h$), then this estimate can be improved to

$$\|\nabla(u - u_h)\|_{L^2} \leq 2 \inf_{\tilde{u}} \|\nabla(u - \tilde{u})\|_{L^2}$$

Similar optimal estimates can be shown for the time-dependent problem (Linke and Rebholz 2019).

Element choices

$H(\mathrm{div})$-conforming

Choose discrete element pair from the de Rham complex:

$$\mathbb{R} \xrightarrow{\text{id}} H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \xrightarrow{\text{null}} 0.$$

Example

Raviart and Thomas (1977)

$$V_h \times Q_h = \mathrm{RT}_k \times P_k^{\text{disc}}$$

Element choices

$H(\mathrm{div})$-conforming

Choose discrete element pair from the de Rham complex:

$$\mathbb{R} \xrightarrow{\mathrm{id}} H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \xrightarrow{\mathrm{null}} 0.$$

Example

Raviart and Thomas (1977)

$$V_h \times Q_h = \mathrm{RT}_k \times P_k^{\mathrm{disc}}$$

- ✓ Nested spaces $V_H \subset V_h$: regular prolongation suffices
- ✓ Arbitrary order
- ✓ Commuting diagram $\Rightarrow$ kernel decomposition “just works”
- ✗ Not H^1-conforming: need penalty or hybridised scheme for ∇^2 term
- ✗ Requires more code development in PETSc

Element choices

H^1-conforming

Choose discrete element pair from the Stokes complex:

$$\mathbb{R} \xrightarrow{\text{id}} H^2 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} H^1 \xrightarrow{\text{div}} L^2 \xrightarrow{\text{null}} 0.$$

Example

Scott and Vogelius (1985)

$$V_h \times Q_h = P_k \times P_{k-1}^{\text{disc}}$$

Element choices

H^1-conforming

Choose discrete element pair from the Stokes complex:

$$\mathbb{R} \xrightarrow{\text{id}} H^2 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} H^1 \xrightarrow{\text{div}} L^2 \xrightarrow{\text{null}} 0.$$

Example

Scott and Vogelius (1985)

$$V_h \times Q_h = P_k \times P_{k-1}^{\text{disc}}$$

- ✗ Low degree $k = d$ versions not inf-sup on all meshes
- ✓ Arbitrary order for k sufficiently large
- ✗ No commuting diagram (yet)
- ✓ H^1-conforming
- ✓ All code already exists

Scott–Vogelius pair

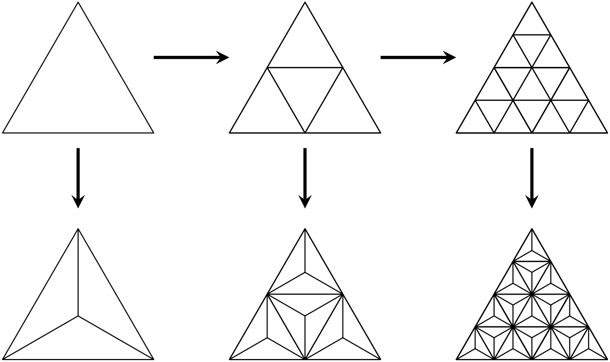
For $k \geq d$, pair is stable on *barycentrically* refined meshes.

Build a non-nested multigrid hierarchy.

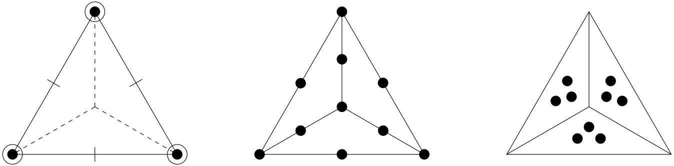
Scott–Vogelius pair

For $k \geq d$, pair is stable on *barycentrically* refined meshes.

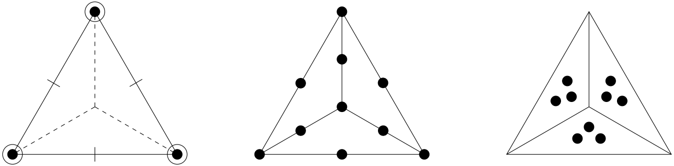
Build a non-nested multigrid hierarchy.



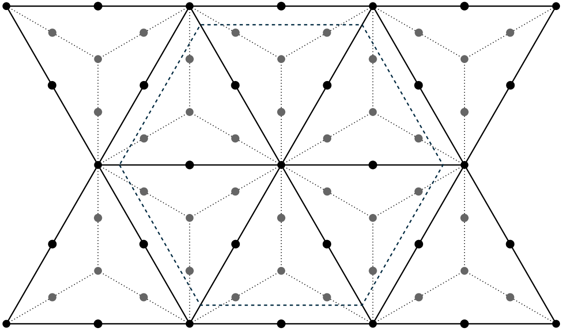
Discrete complex — 2D



Discrete complex — 2D



Requires bigger patch in smoother



Discrete complex — 3D

- Discrete exact sequence developed in Fu, Guzmán, and Neilan (2018), $k \geq 3$
- No commuting projections (yet)

$\Rightarrow$ need to work a bit harder in the theory to prove convergence of scheme

- In practice, same idea: “macro patches” work fine.

$\Rightarrow$ some new conjectures on basis for C^1 functions

Conclusions & outlook

Main result

It is possible to solve the Navier–Stokes equations in a Reynolds-robust way!

Even for exactly divergence-free discretisations

Conclusions & outlook

Main result

It is possible to solve the Navier–Stokes equations in a Reynolds-robust way!

Even for exactly divergence-free discretisations

Ongoing work

Large-scale runs for Scott-Vogelius element pair.

Tidying up proof of convergence.

Conclusions & outlook

Main result

It is possible to solve the Navier–Stokes equations in a Reynolds-robust way!

Even for exactly divergence-free discretisations

Ongoing work

Large-scale runs for Scott-Vogelius element pair.

Tidying up proof of convergence.

Other discretisations

- Scott-Vogelius rather expensive. A lot of interest in $H(\mathrm{div})$-conforming, and hybridised $H(\mathrm{div})$-conforming methods.
- Can we do this on general non-nested meshes?

$\Rightarrow$ needs divergence-preserving prolongation: Fortin operators?

References i

- Alnæs, M. S., A. Logg, K. B. Ølgaard, M. E. Rognes, and G. N. Wells (2014). "Unified Form Language: A Domain-specific Language for Weak Formulations of Partial Differential Equations". *ACM Trans. Math. Softw.* 40. doi:`10.1145/2566630`. arXiv: `1211.4047 [cs.MS]`.
- Ewing, R. E. and J. Wang (1992). "Analysis of the Schwarz algorithm for mixed finite elements methods". *ESAIM: Mathematical Modelling and Numerical Analysis* 26. doi:`10.1051/m2an/1992260607391`.
- Farrell, P. E., L. Mitchell, and F. Wechsung (2018). *An augmented Lagrangian preconditioner for the 3D stationary incompressible Navier–-Stokes equations at high Reynolds number*. Submitted to SIAM SISC. arXiv: `1810.03315 [math.NA]`.
- Fu, G., J. Guzmán, and M. Neilan (2018). *Exact smooth piecewise polynomial sequences on Alfeld splits*. arXiv: `1807.05883 [math.NA]`.
- John, V., A. Linke, C. Merdon, M. Neilan, and L. G. Rebholz (2017). "On the divergence constraint in mixed finite element methods for incompressible flows". *SIAM Review* 59. doi:`10.1137/15m1047696`.
- Kay, D., D. Loghin, and A. Wathen (2002). "A preconditioner for the steady-state Navier-Stokes equations". *SIAM Journal on Scientific Computing* 24. doi:`10.1137/S106482759935808X`.
- Lee, Y.-J., J. Wu, J. Xu, and L. Zikatanov (2007). "Robust subspace correction methods for nearly singular systems". *Mathematical Models and Methods in Applied Sciences* 17. doi:`10.1142/s0218202507002522`.
- Linke, A. and L. G. Rebholz (2019). "Pressure-induced locking in mixed methods for time-dependent (Navier–)Stokes equations". *Journal of Computational Physics* 388. doi:`10.1016/j.jcp.2019.03.010`.

References ii

- Murphy, M. F., G. H. Golub, and A. J. Wathen (2000). “A Note on Preconditioning for Indefinite Linear Systems”. *SIAM Journal on Scientific Computing* 21. doi:10.1137/S1064827599355153.
- Rathgeber, F. et al. (2016). “Firedrake: automating the finite element method by composing abstractions”. *ACM Transactions on Mathematical Software* 43. doi:10.1145/2998441. arXiv: 1501.01809 [cs.MS].
- Raviart, P. A. and J. M. Thomas (1977). “A mixed finite element method for 2nd order elliptic problems”. *Mathematical aspects of finite element methods*. Springer. doi:10.1007/BFb0064470.
- Schöberl, J. (1999). “Multigrid methods for a parameter dependent problem in primal variables”. *Numerische Mathematik* 84. doi:10.1007/s002110050465.
- Scott, L. R. and M. Vogelius (1985). “Conforming finite element methods for incompressible and nearly incompressible continua”. *Large Scale Computations in Fluid Mechanics, B. E. Engquist, ıt et al., eds.* Vol. 22 (Part 2). Providence: AMS.
- Silvester, D. and A. J. Wathen (1994). “Fast iterative solution of stabilised Stokes systems. Part II: Using general block preconditioners”. *SIAM Journal on Numerical Analysis* 31. doi:10.1137/0731070.