491 Knowledge Representation

# Non-monotonic Consequence Relations

Marek Sergot
Department of Computing
Imperial College, London

March 2000 v1.1d

Many consequence relations used in practical reasoning (AI, databases) fail monotony: they are non-monotonic. Let's look at some general properties that (non-)monotonic consequence relations can have.

Further reading:
David Makinson, General Patterns in Nonmonotonic Reasoning.
*Handbook of Logic in Artificial Intelligence and Logic Programming* (Vol 3), Gabbay, Hogger, Robinson (eds). Oxford University Press, 1994.
David Makinson, *Bridges from Classical to Nonmonotonic Logic*.
King's College London Publications, 2005.

$\mathrm{Cn}(A)$ is the set of all consequences of $A$. $A \vdash \alpha$ means the same as $\alpha \in \mathrm{Cn}(A)$.
$\mathrm{Th}(A)$ stands for the set of all classical truth-functional consequences of $A$.

When I want to emphasise that a consequence relation is non-monotonic, I write $\mathrm{Cn}^*$, and $A \mid\!\sim \alpha$ for $\alpha \in \mathrm{Cn}^*(A)$.

## Reminder: Classical consequence relations

A consequence operator Cn is 'classical' if it satisfies the following three properties:

- if $\alpha \in A$ then $A \vdash \alpha$ (reflexivity)
- if $A \vdash B$ and $A \cup B \vdash \alpha$ then $A \vdash \alpha$ ('cut')
- if $A \subseteq B$ then $A \vdash \alpha \Rightarrow B \vdash \alpha$ (monotony)
  (Or: if $A \vdash \alpha$ then $A \cup X \vdash \alpha$, any $X$)

Written in terms of the corresponding consequence operator Cn, the conditions are:

- $A \subseteq \mathrm{Cn}(A)$ (inclusion)
- if $A \subseteq B \subseteq \mathrm{Cn}(A)$ then $\mathrm{Cn}(B) \subseteq \mathrm{Cn}(A)$ ('cumulative transitivity')
- $A \subseteq B \Rightarrow \mathrm{Cn}(A) \subseteq \mathrm{Cn}(B)$ (monotony)

('Cumulative transitivity/cut' generalises 'closure'. And monotony implies they are equivalent.)



## Classical truth-functional consequence Th

Classical truth-functional consequence Th is a classical consequence operator.
In addition, Th has the following properties:

- $\beta \in \mathrm{Th}(A \cup \{\alpha\})$ iff $(\alpha \to \beta) \in \mathrm{Th}(A)$ (deduction)
- if $\alpha \in \mathrm{Th}(A)$ then $\alpha \in \mathrm{Th}(A')$ for some finite $A' \subseteq A$ (compactness)

Some consequence operators have the additional properties of deduction and compactness, some do not.

## Cautious monotony

Obviously monotony no good for default reasoning.

*coffee* $\mid\!\sim$ *tastes-nice*
*coffee* $\cup$ {*diesel-oil*} $\not\mid\!\sim$ *tastes-nice*

But suppose $A \mid\!\sim \alpha$ and $A \mid\!\sim X$. Then $X$ is already in $\mathrm{Cn}^*(A)$ and so (perhaps) $A \cup X \mid\!\sim \alpha$ should hold.
Suggests the following property of *cautious monotony* (Gabbay):

- $A \mid\!\sim \alpha$ and $A \mid\!\sim X \Rightarrow A \cup X \mid\!\sim \alpha$
- $A \subseteq B \subseteq \mathrm{Cn}^*(A) \Rightarrow \mathrm{Cn}^*(A) \subseteq \mathrm{Cn}^*(B)$

Are there any examples of well-known default reasoning systems that don't satisfy such a reasonable looking property? Yes, the following do *not*:

- normal and extended logic programs
- Reiter defaults
- autoepistemic logic (which we did not cover)
- preferential entailment (in general) (which I mentioned but did not cover)

Any examples that do?

- maxiconsistent default assumptions ('Poole systems'), which we did not cover.
- preferential entailment with some additional restrictions (details omitted)

Here is another property ('rational monotony'):

- if $A \mid\!\sim \alpha$ and $A \not\mid\!\sim \neg\beta$ then $A \cup \{\beta\} \mid\!\sim \alpha$

And another ('negation rationality')

- if $A \mid\!\sim \alpha$ then either $A \cup \{\beta\} \mid\!\sim \alpha$ or $A \cup \{\neg\beta\} \mid\!\sim \alpha$

Most of the standard default reasoning systems don't have either of these properties either!

## 'Cumulative transitivity' (aka 'Cut')

- if $A \mathrel{|\!\sim} B$ and $A \cup B \mathrel{|\!\sim} \alpha$ then $A \mathrel{|\!\sim} \alpha$
- $A \subseteq B \subseteq \mathrm{Cn}^*(A) \Rightarrow \mathrm{Cn}^*(B) \subseteq \mathrm{Cn}^*(A)$

Easy to check 'closure/idempotence' is a special case of 'cut', and equivalent when $\mathrel{|\!\sim}$ is monotonic (in an earlier set of notes).

'Cut' means that the length, intricacy, manner of derivation does not affect the conclusions.

Not surprising that 'cut' does not hold for e.g. most forms of probabilistic reasoning — in general, for any form of reasoning in which the conclusions get weaker as the chain of inference gets longer.

## Cumulative consequence relations

Satisfy both cautious monotony and cut:

- $A \subseteq B \subseteq \mathrm{Cn}^*(A) \Rightarrow \mathrm{Cn}^*(B) = \mathrm{Cn}^*(A)$

## Interaction with truth-functional connectives

Which of these properties would one expect to hold?
(depends on intended reading of $\mathrel{|\!\sim}$ of course).

- $A \mathrel{|\!\sim} \alpha$ and $A \mathrel{|\!\sim} \beta \Leftrightarrow A \mathrel{|\!\sim} (\alpha \wedge \beta)$
- $(A \mathrel{|\!\sim} \alpha$ or $A \mathrel{|\!\sim} \beta) \Rightarrow A \mathrel{|\!\sim} (\alpha \vee \beta)$
- $A \mathrel{|\!\sim} \alpha$ and $\alpha \vdash \beta \Rightarrow A \mathrel{|\!\sim} \beta$
  *canary* $\mathrel{|\!\sim}$ *yellow* and *yellow* $\vdash$ *not-blue* $\Rightarrow$ *canary* $\mathrel{|\!\sim}$ *not-blue*
- $A \mathrel{|\!\sim} \alpha$ and $\alpha \mathrel{|\!\sim} \beta \Rightarrow A \mathrel{|\!\sim} \beta$
  *alcoholic* $\mathrel{|\!\sim}$ *adult* and *adult* $\mathrel{|\!\sim}$ *healthy* $\Rightarrow$ *alcoholic* $\mathrel{|\!\sim}$ *healthy*

What about these?

- $\alpha \mathrel{|\!\sim} \beta \Rightarrow \neg\beta \mathrel{|\!\sim} \neg\alpha$ (contrapositives)
- $A \cup \{\beta\} \mathrel{|\!\sim} \alpha$ and $A \cup \{\neg\beta\} \mathrel{|\!\sim} \alpha \Rightarrow A \mathrel{|\!\sim} \alpha$ ('reasoning by cases')
- $A \cup \{\beta\} \mathrel{|\!\sim} \alpha \Rightarrow A \mathrel{|\!\sim} (\beta \rightarrow \alpha)$ ('deduction thm')
- $A \mathrel{|\!\sim} \alpha$ and $A \cup \{\beta\} \mathrel{|\!\sim} \neg\alpha \Rightarrow A \mathrel{|\!\sim} \neg\beta$
  *bird* $\mathrel{|\!\sim}$ *flies* and *bird* $\cup$ {*penguin*} $\mathrel{|\!\sim}$ $\neg$*flies* $\Rightarrow$ *bird* $\mathrel{|\!\sim}$ $\neg$*penguin*



It turns out that the key properties are these:

- $\mathrm{Th}(A) \subseteq \mathrm{Cn}^*(A)$ ('supraclassical')
- $\mathrm{Th}(\mathrm{Cn}^*(A)) \subseteq \mathrm{Cn}^*(A)$ ('left absorption')
- $\mathrm{Cn}^*(\mathrm{Th}(A)) \subseteq \mathrm{Cn}^*(A)$ ('right absorption')
- $\mathrm{Cn}^*(A) \cap \mathrm{Cn}^*(B) \subseteq \mathrm{Cn}^*(\,\mathrm{Th}(A) \cap \mathrm{Th}(B)\,)$ ('distribution')
  $\mathrm{Cn}^*(A) \cap \mathrm{Cn}^*(B) \subseteq \mathrm{Cn}^*(A \cap B)$

It can be argued (e.g. by Makinson) that "an approach to default reasoning deserves the name 'logical' only if it leads to an inference operation $\mathrm{Cn}^*$ satisfying the full absorption principle (both left and right absorption): in other words, only if the propositions that we are allowed to infer from a set $A$ form a classical theory ($\mathrm{Cn}^*(A) = \mathrm{Th}(\mathrm{Cn}^*(A))$), which, moreover, depends only upon the logical content of $A$ rather than upon its manner of presentation ($\mathrm{Cn}^*(A) = \mathrm{Cn}^*(\mathrm{Th}(A))$)".

## Extension families

Here is a common construction: given $A$ there is a set $ext(A)$ of *extensions* $E$ of $A$ such that $\mathrm{Th}(A) \subseteq E$. Details of the definition vary. Reiter default logic is an example.
Usually (not always)

- $A \subseteq E$
- $E = \mathrm{Th}(E)$

So then $\mathrm{Th}(A) \subseteq \mathrm{Th}(E) \subseteq E$.
Now a *sceptical* or *cautious* consequence relation is produced by intersecting all these extensions:

$$\mathrm{Cn}^*(A) \stackrel{\text{def}}{=} \bigcap ext(A)$$

Obviously, by construction, any such $\mathrm{Cn}^*$ is supraclassical:

- $\mathrm{Th}(A) \subseteq \mathrm{Cn}^*(A)$

Notice that by elementary set theory, $ext(X) \subseteq ext(Y)$ implies $\bigcap ext(Y) \subseteq \bigcap ext(X)$. The following now follow (among other things):

- if $A \subseteq B \subseteq \mathrm{Cn}^*(A)$ implies $ext(A) \subseteq ext(B)$, then $\mathrm{Cn}^*$ satisfies cut.
- if $A \subseteq B \subseteq \mathrm{Cn}^*(A)$ implies $ext(B) \subseteq ext(A)$, then $\mathrm{Cn}^*$ satisfies cautious monotony.
- if $A \subseteq B \subseteq \mathrm{Cn}^*(A)$ implies $ext(B) = ext(A)$, then $\mathrm{Cn}^*$ is cumulative.
- (and a result about 'distribution' which I omit).

**Example** Reiter default logic has the first but not the second.

## Three ways of constructing a nonmonotonic logic

1. 'Preferential entailment'

   $$KB \models_\Delta \alpha \text{ --- } \alpha \text{ is true in all } \textit{preferred} \text{ models of } KB$$

   The 'preferred' models are usually *minimal* in some appropriate sense.

2. Define $\models_\Delta$ in terms of classical consequence $\models$. Usually:

   $$KB \models_\Delta \alpha \quad \text{iff} \quad KB \cup \mathit{ext}(KB) \models \alpha$$
   $$\mathrm{Cn}_\Delta(KB) \stackrel{\text{def}}{=} \mathrm{Th}(KB \cup \mathit{ext}(KB))$$

   $\mathit{ext}(KB)$ is some kind of additional background knowledge.

   Non-monotonic because, in general:

   $$KB \cup \mathit{ext}(KB) \not\subseteq KB \cup X \cup \mathit{ext}(KB \cup X)$$

3. Extend the language with a new form of
   - defeasible rule (like Default Logic)
   - defeasible conditional (e.g. $\leadsto$).