491 Knowledge Representation

Tutorial Exercise

Consequence relations

Discussion:

When A is a set of sentences (formulas of the language \(L\)), \(X \vdash A\) is shorthand for \(X \vdash \alpha\) for all \(\alpha \in A\), i.e., for \(A \subseteq \text{Cn}(X)\). \(\text{Th}(X)\) is the set of classical truth-functional consequences of \(X\). \(X \vdash_{PL} A\) is shorthand for \(A \subseteq \text{Th}(X)\).

1. Check that \(\text{Th}(X) \overset{\text{def}}{=} \{ \alpha \in L \mid X \models \alpha \}\) does indeed satisfy the properties of \(\text{Th}\) listed in the lecture notes.

You may ignore compactness. To show closure/idempotence, it is easier to show the more general property ‘cut’. (You might want to look at Question 2 first.)

2. From 2008 Exam:

Let \(\text{models}(A)\) denote the set of models of \(A\), i.e., the set of interpretations in which every formula of \(A\) is true. One can see that \(\alpha \in \text{Th}(A)\) iff \(\text{models}(A) \subseteq \text{models}(\{\alpha\})\), and more generally, that \(B \subseteq \text{Th}(A)\) iff \(\text{models}(A) \subseteq \text{models}(B)\). Moreover, if \(A \subseteq B\) then \(\text{models}(B) \subseteq \text{models}(A)\) (or if you prefer, \(\text{models}(A \cup X) \subseteq \text{models}(A)\)).

Use these observations to show that \(\text{Th}\) is a classical consequence operator.

You may find it helpful to prove first the transitivity of \(\text{Th}: B \subseteq \text{Th}(A)\) implies \(\text{Th}(B) \subseteq \text{Th}(A)\), or equivalently, if \(B \subseteq \text{Th}(A)\) and \(\alpha \in \text{Th}(B)\) then \(\alpha \in \text{Th}(A)\).

3. In preferential entailment, the consequences of a set \(A\) of formulas are defined in terms of some preferred subset of models of \(A\). \(B \subseteq \text{Cn}_{\text{pref}}(A)\) iff all formulas in \(B\) are true in all preferred models of \(A\), i.e., iff \(\text{models}_{\text{pref}}(A) \subseteq \text{models}(B)\). The detailed definition of \(\text{models}_{\text{pref}}\) varies, though it is always the case that \(\text{models}_{\text{pref}}(A) \subseteq \text{models}(A)\).

Some properties of \(\text{Cn}_{\text{pref}}\) follow immediately. For example:

- \(\text{models}_{\text{pref}}(A) \subseteq \text{models}(A)\) means that \(A \subseteq \text{Cn}_{\text{pref}}(A)\) (‘inclusion’).

Also more or less immediate:

(i) Show that preferential entailment is always ‘supraclassical’:

\[
\text{Th}(A) \subseteq \text{Cn}_{\text{pref}}(A)
\]

(ii) In general, \(A \subseteq B\) does not imply \(\text{models}_{\text{pref}}(B) \subseteq \text{models}_{\text{pref}}(A)\) (or, if you prefer, in general \(\text{models}_{\text{pref}}(A \cup X) \not\subseteq \text{models}_{\text{pref}}(A)\)).

Why does it follow that, in general, preferential entailment is non-monotonic? (Rather obvious.)

There are some other properties of preferential entailment that can easily be discovered. I will leave them until we discuss properties of non-monotonic consequence relations later in the course.

4. In the lecture notes it is claimed that

- if \(A \vdash B\) and \(A \cup B \vdash \alpha\) then \(A \vdash \alpha\) (‘cut’)

is expressed in terms of \(\text{Cn}\) as:

- if \(B \subseteq \text{Cn}(A)\) then \(\text{Cn}(A \cup B) \subseteq \text{Cn}(A)\)

which in turn is equivalent (assuming \(A \subseteq \text{Cn}(A)\)) to

- if \(A \cup B \subseteq \text{Cn}(A)\) then \(\text{Cn}(B) \subseteq \text{Cn}(A)\)

(which is called ‘cumulative transitivity’).

Check this.

5. Suppose \(\text{Cn}\) is a classical consequence operator. Prove:

(i) If \(A \subseteq D \subseteq B \subseteq \text{Cn}(A)\) then \(\text{Cn}(D) = \text{Cn}(B)\)

(ii) If \(A \subseteq \text{Cn}(B)\) then \(\text{Cn}(A \cup B) = \text{Cn}(B)\)

(iii) If \(X \vdash \alpha\) and \(X \cup \{\alpha\} \vdash \beta\) then \(X \vdash \beta\)

(iv) \(\text{Cn}(B) = \text{Cn}(D)\) iff \(B \subseteq \text{Cn}(D)\) and \(D \subseteq \text{Cn}(B)\) (i.e., \(D \vdash B\) and \(B \vdash D\))

(v) \(\vdash\) is transitive: if \(A \vdash B\) and \(B \vdash C\) then \(A \vdash C\)

(Read what these properties are saying. You could draw a picture for (i), (ii) and (iii) you have seen already on this sheet.)

6. Suppose \(\text{Cn}\) is also supraclassical, i.e. \(\text{Th}(A) \subseteq \text{Cn}(A)\) for all \(A\). Prove:

(i) If \(X \vdash_{PL} Y\) then \(A \cup Y \vdash \alpha\) implies \(A \cup X \vdash \alpha\)

(or equivalently, \(\text{Cn}(A \cup \text{Th}(X)) \subseteq \text{Cn}(A \cup X)\))

(ii) Hence, \(\text{Cn}(A \cup \{\alpha, \beta\}) = \text{Cn}(A \cup \{\alpha \land \beta\})\)

(and so also more generally: \(\text{Cn}(\{\alpha_1, \alpha_2, \ldots, \alpha_n\}) = \text{Cn}(\{\alpha_1 \land \alpha_2 \land \cdots \land \alpha_n\})\))

(iii) If \(X \vdash_{PL} Y\) then \(A \vdash X\) implies \(A \vdash Y\)

(iv) Hence, for finite \(A\), \(A \subseteq \text{Cn}(D)\) iff \(D \vdash A\)

7. Suppose \(T\) is a set of formulas of the language \(L\). Consider the operator \(\text{Cn}_T\) defined as follows:

\[
\text{Cn}_T(A) \overset{\text{def}}{=} \text{Th}(T \cup A)
\]

(You can think of formulas \(T\) as expressing some fixed ‘background’ knowledge.)

Show that \(\text{Cn}_T\) is a classical consequence operator and that it satisfies the further properties of supraclassicality, deduction, and compactness.

Hence show that

\[
X \cup \{\alpha\} \vdash_T \beta \text{ and } X \cup \{\gamma\} \vdash_T \beta \text{ iff } X \cup \{\alpha \lor \gamma\} \vdash_T \beta
\]