Question 1

I might not have chosen the best notation. \( A \vdash_{PL} \alpha \) is shorthand for \( \alpha \in \text{Th}(A) \). But by definition, \( \alpha \in \text{Th}(A) \) iff \( A \models \alpha \). So we have \( A \vdash_{PL} \alpha \) iff \( A \models \alpha \). I hope this isn’t confusing.

Here are the properties of \( \text{Th} \) listed in the lecture notes (except for ‘compactness’ which is omitted here):

- \( A \subseteq \text{Th}(A) \) (inclusion)
  \( A \vdash_{PL} A \) (reflexivity)
  Trivial: we just need to show \( A \models \alpha \) for every \( \alpha \in A \), which is immediate from the definition of \( \models \).

- \( \text{Th}(\text{Th}(A)) \subseteq \text{Th}(A) \) (closure). (Idempotence is closure together with inclusion.)
  As noted on the question sheet, it is easier to prove the more general case of ‘cut’ (below). If we have inclusion, then closure is just a special case of ‘cut’.

- If \( A \cup B \) then \( \text{Th}(A) \subseteq \text{Th}(B) \) (monotony)
  If \( A \vdash_{PL} \alpha \) then \( A \cup X \vdash_{PL} \alpha \)
  First formulation above: we need to show that if \( A \subseteq B \) and \( A \models \alpha \) then \( B \models \alpha \).
  Suppose \( \mathcal{M} \) is a models such that \( \mathcal{M} \models B \). Since \( A \subseteq B \), \( \mathcal{M} \models A \) also. But we have \( \mathcal{M} \models \alpha \), so \( \mathcal{M} \models \alpha \).
  Or simply, from the second formulation above, we just need to show that if \( A \models \alpha \) then \( A \cup X \models \alpha \). Since any model of \( A \cup X \) is also a model of \( A \), every \( \alpha \) which is true in all models of \( A \) must also be true in all models of \( A \cup X \).

- If \( B \subseteq \text{Th}(A) \) then \( \text{Th}(B) \subseteq \text{Th}(A) \) (transitivity/syllogism)
  We need to show that if \( A \models B \) then, for any \( \alpha, B \models \alpha \) implies \( A \models \alpha \). So suppose \( A \models B \) and \( B \models \alpha \). We need to show \( A \models \alpha \).
  Suppose \( \mathcal{M} \models A \). We need to show \( \mathcal{M} \models \alpha \). If \( A \models B \) and \( \mathcal{M} \models A \) then (by definition) \( \mathcal{M} \models B \) also. And \( B \models \alpha \) and \( \mathcal{M} \models B \) implies \( \mathcal{M} \models \alpha \).

- If \( A \cup B \subseteq \text{Th}(A) \) then \( \text{Th}(B) \subseteq \text{Th}(A) \) (cut’)
  This is a weakened form of transitivity/syllogism, sometimes called ‘cumulative transitivity’. Because it’s just a special case, it follows from the above. If you want to see it directly:
  Suppose \( A \cup B \subseteq \text{Th}(A) \). We need to show \( B \models \alpha \) implies \( A \models \alpha \). So, suppose \( B \models \alpha \) and \( \mathcal{M} \models A \). We need to show \( \mathcal{M} \models \alpha \). This is exactly as for transitivity/syllogism above.

### SOLUTIONS

- \( \beta \in \text{Th}(A \cup \{\alpha\}) \) iff \( (\alpha \rightarrow \beta) \in \text{Th}(A) \) (deduction)
  We need to show \( A \cup \{\alpha\} \models \beta \) iff \( A \models (\alpha \rightarrow \beta) \).
  Left-to-right. Suppose \( A \cup \{\alpha\} \models \beta \) and \( \mathcal{M} \models A \). We need to show \( \mathcal{M} \models (\alpha \rightarrow \beta) \), i.e., that either \( \mathcal{M} \not\models \alpha \) or \( \mathcal{M} \models \beta \).
  We have assumed \( \mathcal{M} \models A \). There are two cases: either \( \mathcal{M} \models \alpha \) or \( \mathcal{M} \not\models \alpha \). If \( \mathcal{M} \not\models \alpha \) then we have \( \mathcal{M} \models (\alpha \rightarrow \beta) \) as required.
  Right-to-left. Suppose \( \mathcal{M} \models (\alpha \rightarrow \beta) \).
  Suppose \( \mathcal{M} \models A \).
  Then \( \mathcal{M} \models A \).
  And hence \( \mathcal{M} \models (\alpha \rightarrow \beta) \), i.e., either \( \mathcal{M} \not\models \alpha \) or \( \mathcal{M} \models \beta \).
  But \( \mathcal{M} \models A \).

- \( \mathcal{L} = \{p, \neg p \} \) (because e.g. \( p \lor \neg p \in \text{Th}(\emptyset) \))
  \( \text{Th}(\emptyset) \) is the set of all tautologies.

- \( \{\alpha, \beta\} \subseteq \text{Th}(A) \) iff \( (\alpha \land \beta) \in \text{Th}(A) \)
  \( \{\alpha, \beta\} \subseteq \text{Th}(A) \) iff \( A \models \alpha \) and \( A \models \beta \)
  if for all \( \mathcal{M} \) such that \( \mathcal{M} \models A \) we have \( \mathcal{M} \models \alpha \),
  and for all \( \mathcal{M} \) such that \( \mathcal{M} \models A \) we have \( \mathcal{M} \models \beta \)
  if for all \( \mathcal{M} \) such that \( \mathcal{M} \models A \) we have \( \mathcal{M} \models \alpha \land \beta \)
  if \( (\alpha \land \beta) \in \text{Th}(A) \)

- \( \{\alpha, \alpha \rightarrow \beta\} \subseteq \text{Th}(A) \) then \( \beta \in \text{Th}(A) \)
  Suppose \( A \models \alpha \) and \( A \models \alpha \rightarrow \beta \).
  We need to show \( A \models \beta \).
  So suppose \( \mathcal{M} \models A \). Clearly we have both \( \mathcal{M} \models \alpha \) and \( \mathcal{M} \models \alpha \rightarrow \beta \).
  But \( \mathcal{M} \models \alpha \rightarrow \beta \) if either \( \mathcal{M} \not\models \alpha \) or \( \mathcal{M} \models \beta \).
  And we have \( \mathcal{M} \models A \), so it must be that \( \mathcal{M} \models \beta \), as required.

- \( \emptyset \vdash_{PL} \alpha \leftrightarrow \beta \) then \( A \vdash_{PL} \alpha \) iff \( A \vdash_{PL} \beta \)
  We need to show that if \( \emptyset \models \alpha \leftrightarrow \beta \) then \( A \models \alpha \) iff \( A \models \beta \).
  There are lots of ways to do it. For example, by monotony of \( \text{Th} \), \( \emptyset \models \alpha \leftrightarrow \beta \) implies \( A \models \alpha \leftrightarrow \beta \).
  Now we have (by a result earlier), \( A \models (\alpha \leftrightarrow \beta) \land (\beta \leftrightarrow \alpha) \) iff \( A \models \alpha \leftrightarrow \beta \) and \( A \models \beta \leftrightarrow \alpha \).
  So we can show (with a couple more easy steps) that \( A \models (\alpha \rightarrow \beta) \) and \( A \models \beta \).
  Or: \( \emptyset \models \alpha \leftrightarrow \beta \) means that \( \alpha \) and \( \beta \) have the same truth value in all models. And from this it follows immediately that \( A \models \alpha \) iff \( A \models \beta \) for any set of formulas \( A \).

- \( \emptyset \subseteq \text{Th}(A) \) (every \( \text{Th}(A) \) contains all tautologies)
  Trivial. (For example, follows from monotony of \( \text{Th} \).)
More or less trivial. We just need to show $\{\alpha\} \models \alpha \lor \beta$. Suppose $M$ is any model such that $M \models \alpha$. Then $M \models \alpha \lor \beta$ also.

\textbf{‘Disjunction in the premises’ (‘OR’)}

$\text{Th}(A \cup \{\alpha\}) \cap \text{Th}(A \cup \{\beta\}) \subseteq \text{Th}(A \cup \{\alpha \lor \beta\})$

We need to show that if $A \cup \{\alpha\} \models \gamma$ and $A \cup \{\beta\} \models \gamma$ then $A \cup \{\alpha \lor \beta\} \models \gamma$.

There are several ways to do it. For example: assume LHS and suppose $M \models A \lor \{\alpha \lor \beta\}$. We need to show $M \models \gamma$. $M \models A \lor \{\alpha \lor \beta\}$ implies $M \models A$ and $M \models \alpha$ or $M \models \beta$ (or both). Suppose $M \models A$ and $M \models \alpha$; then $M \models A \lor \{\alpha\}$, and $M \models \gamma$. Suppose $M \models A$ and $M \models \beta$: then $M \models A \lor \{\beta\}$, and again $M \models \gamma$.

Or (another method) use the property ‘deduction’ above:

$A \cup \{\alpha\} \models \gamma$ and $A \cup \{\beta\} \models \gamma$ if $A \models \alpha \rightarrow \gamma$ and $A \models \beta \rightarrow \gamma$

if $A \models (\alpha \lor \beta) \rightarrow \gamma$

if $A \models (\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)$

if $A \cup \{\alpha \lor \beta\} \models \gamma$

The penultimate step uses the property that $(\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)$ is equivalent to $(\alpha \lor \beta) \rightarrow \gamma$, i.e., that

$\models ((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)) \iff ((\alpha \lor \beta) \rightarrow \gamma)$

which is very easily checked.

\textbf{Question 2}

It is just elementary set theory, and the definitions and properties given.

(i) \textit{Inclusion} $A \subseteq \text{Th}(A)$. This is just $\text{models}(A) \subseteq \text{models}(A)$, which is obviously true for all $A$.

(ii) ‘Closure.’ As suggested in the question, this will follow from ‘transitivity’:

$B \subseteq \text{Th}(A) \Rightarrow \text{Th}(B) \subseteq \text{Th}(A)$

Take the special case where $B = \text{Th}(A)$. Then transitivity gives

$\text{Th}(A) \subseteq \text{Th}(A) \Rightarrow \text{Th}(\text{Th}(A)) \subseteq \text{Th}(A)$

and the left hand side is obviously true, for all $A$.

‘Transitivity’

Suppose $B \subseteq \text{Th}(A)$, i.e., $\text{models}(A) \subseteq \text{models}(B)$.

Suppose $\alpha \in \text{Th}(B)$, i.e., $\text{models}(B) \subseteq \text{models}(\{\alpha\})$.

We need to show $\alpha \in \text{Th}(A)$. Putting the two assumptions together we have:

$\text{models}(A) \subseteq \text{models}(B) \subseteq \text{models}(\{\alpha\})$

and so $\text{models}(A) \subseteq \text{models}(\{\alpha\})$, which is $\alpha \in \text{Th}(A)$.

Or, if you prefer, you can show $\text{Th}(B) \subseteq \text{Th}(A)$ by showing that, for all $C$:

$C \subseteq \text{Th}(B) \Rightarrow C \subseteq \text{Th}(A)$

We want

$\text{models}(A) \subseteq \text{models}(B) \Rightarrow (\text{models}(B) \subseteq \text{models}(C) \Rightarrow \text{models}(A) \subseteq \text{models}(C))$

which is equivalent to saying

$\text{models}(A) \subseteq \text{models}(B) \subseteq \text{models}(C) \Rightarrow \text{models}(A) \subseteq \text{models}(C)$

(iii) Monotony

Suppose $A \subseteq B$. We want to show $\text{Th}(A) \subseteq \text{Th}(B)$.

$A \subseteq B$ implies $\text{models}(B) \subseteq \text{models}(A)$ (stated in the question).

Suppose $\alpha \in \text{Th}(A)$, i.e., $\text{models}(A) \subseteq \text{models}(\{\alpha\})$.

Then we have

$\text{models}(B) \subseteq \text{models}(A) \subseteq \text{models}(\{\alpha\})$

from which follows $\text{models}(B) \subseteq \text{models}(\{\alpha\})$, i.e., $\alpha \in \text{Th}(B)$.

\textbf{Question 3}

(i) ‘Supraclassical’ $\text{Th}(A) \subseteq C_{\text{prel}}(A)$.

Suppose $\alpha \in \text{Th}(A)$, i.e., $\text{models}(A) \subseteq \text{models}(\{\alpha\})$.

From the question, $\text{models}_{\text{prel}}(A) \subseteq \text{models}(A)$.

So we have

$\text{models}_{\text{prel}}(A) \subseteq \text{models}(A) \subseteq \text{models}(\{\alpha\})$

from which follows $\text{models}_{\text{prel}}(A) \subseteq \text{models}(\{\alpha\})$, i.e., $\alpha \in C_{\text{prel}}(A)$ as required.

As in the previous question, you can also do it by showing that $C \subseteq \text{Th}(A)$ implies $C \subseteq C_{\text{prel}}(A)$ for any $C$. The rest is more or less identical.

(ii) Well obviously: if $A \subseteq B$ were to imply $\text{models}_{\text{prel}}(B) \subseteq \text{models}_{\text{prel}}(A)$ for all $A$ and $B$, $C_{\text{prel}}$ would be monotonic. We just proved that in the previous question.
Question 4

We need to check that

- if $A \vdash B$ and $A \cup B \vdash \alpha$ then $A \vdash \alpha$ (‘cut’)

is expressed equivalently in terms of $Cn$ as:

1. if $B \subseteq Cn(A)$ then $Cn(A \cup B) \subseteq Cn(A)$

which in turn is equivalent (assuming $A \subseteq Cn(A)$) to

2. if $A \subseteq B \subseteq Cn(A)$ then $Cn(B) \subseteq Cn(A)$

First, let us check that (1) and (2) are equivalent. It just relies on some elementary set theory. In case it is not obvious:

Suppose (1). Show (2).

Suppose $A \subseteq B \subseteq Cn(A)$. We have $A \subseteq Cn(A)$ (given in the question). So $A \cup B \subseteq Cn(A)$.

Obviously $A \subseteq A \cup B$. So we have $A \subseteq A \cup B \subseteq Cn(A)$, which implies $Cn(A \cup B) \subseteq Cn(A)$ by (2).

It remains to show ‘cut’ is equivalently expressed as (1). Expressed in terms of $Cn$ instead of the notation $\vdash$, ‘cut’ is:

- if $B \subseteq Cn(A)$ and $\alpha \in Cn(A \cup B)$ then $\alpha \in Cn(A)$

Now, maybe you can see it straightaway. In case not: I am usually a bit casual when stating these properties. I mean of course that, for all $A$ and $B$, and for all $\alpha$:

- if $B \subseteq Cn(A)$ and $\alpha \in Cn(A \cup B)$ then $\alpha \in Cn(A)$

That is equivalent to saying, for all $A$ and $B$

- if $B \subseteq Cn(A)$ then, for all $\alpha$ (if $\alpha \in Cn(A \cup B)$ then $\alpha \in Cn(A)$)

which is just, for all $A$ and $B$

- if $B \subseteq Cn(A)$ then $Cn(A \cup B) \subseteq Cn(A)$

which is (1).

Question 5

Most of the parts of this question make use of $B \subseteq Cn(A)$ implies $Cn(B) \subseteq Cn(A)$. This follows since $Cn(B) \subseteq Cn(Cn(A))$ by monotony, and $Cn(Cn(A)) = Cn(A)$ by idempotence.

(i) $Cn(D) \subseteq Cn(B)$ follows immediately from $D \subseteq B$ by monotony. To show $Cn(B) \subseteq Cn(D)$:

$B \subseteq Cn(A)$ so $Cn(B) \subseteq Cn(A)$ (monotony + idempotence). But $A \subseteq D$ implies $Cn(A) \subseteq Cn(D)$ by monotony, so we have $Cn(B) \subseteq Cn(A) \subseteq Cn(D)$.

(ii) $A \subseteq Cn(B)$ and $B \subseteq Cn(B)$ (inclusion) so $A \cup B \subseteq Cn(B)$ (elementary set theory). So then $Cn(A \cup B) \subseteq Cn(B)$ (monotony + idempotence).

(iii) Rewrite using the $Cn$ notation: we need to show that if $\{\alpha\} \subseteq Cn(X)$ then $Cn(X \cup \{\alpha\}) \subseteq Cn(X)$. This is just a special case of (ii).

(iv) $B \subseteq Cn(B)$ (inclusion) so if $Cn(B) = Cn(D)$ then also $B \subseteq Cn(D)$. Similarly $D \subseteq Cn(D)$ implies $D \subseteq Cn(B)$. For the other half: $B \subseteq Cn(D)$ implies $Cn(B) \subseteq Cn(D)$ and $D \subseteq Cn(B)$ implies $Cn(D) \subseteq Cn(B)$ (monotony + idempotence).

(v) Rewrite using $Cn$ notation: we want to show that if $B \subseteq Cn(A)$ and $C \subseteq Cn(B)$ then $C \subseteq Cn(A)$. From $B \subseteq Cn(A)$ we have $Cn(B) \subseteq Cn(A)$ (monotony + idempotence). So we have $C \subseteq Cn(B) \subseteq Cn(A)$ and so $C \subseteq Cn(A)$ as required.

Question 6

(i) Rewrite using $Cn$ notation: we need to show that if $Y \subseteq \text{Th}(X)$ then $Cn(A \cup Y) \subseteq Cn(A \cup X)$. As noted in the question statement, it suffices to show $Cn(A \cup \text{Th}(X)) \subseteq Cn(A \cup X)$. This is because if $Y \subseteq \text{Th}(X)$ then monotony implies $Cn(A \cup Y) \subseteq Cn(A \cup \text{Th}(X))$, and so $Cn(A \cup Y) \subseteq Cn(A \cup X)$ as required. For the other half of the equivalence, take $Y = \text{Th}(X)$.

Now $A \subseteq A \cup X \subseteq Cn(A \cup X)$, and for supraclassical $Cn$, $\text{Th}(X) \subseteq Cn(X) \subseteq Cn(X \cup A)$. So $A \cup \text{Th}(X) \subseteq Cn(X \cup A)$ (set theory). $Cn(A \cup \text{Th}(X)) \subseteq Cn(X \cup A)$ follows from monotony + idempotence of $Cn$.

(ii) $\{\alpha, \beta\} \subseteq \text{Th}\{\alpha \land \beta\}$ so by part (i) $Cn(A \cup \{\alpha, \beta\}) \subseteq Cn(A \cup \{\alpha \land \beta\})$. Similarly, $\{\alpha \land \beta\} \subseteq \text{Th}\{\alpha, \beta\}$ and so $Cn(A \cup \{\alpha \land \beta\}) \subseteq Cn(A \cup \{\alpha, \beta\})$.

(iii) Since $Cn$ is supraclassical, this is just a special case of transitivity of $Cn$ (part 3(v)).

(iv) $\{A\} \subseteq \text{Th}(A)$ so $A \subseteq Cn(D)$ implies $\{A\} \subseteq Cn(D)$ by part (iii) (or transitivity of $Cn$). And $A \subseteq \text{Th}\{\{A\}\}$ so $\{A\} \subseteq Cn(D)$ implies $A \subseteq Cn(D)$.

(A has to be finite, otherwise $\{A\}$ is not a well-formed formula.)
Question 7

All three required properties follow straightforwardly:

**inclusion:** $A \subseteq T \cup A \subseteq \text{Th}(T \cup A) = \text{Cn}_T(A)$.

**idempotence:** $\text{Cn}_T(\text{Cn}_T(A)) = \text{Th}(\text{Th}(T \cup A))$. Since $T \subseteq \text{Th}(T \cup A)$, $T \cup \text{Th}(T \cup A) = \text{Th}(T \cup A)$. And $\text{Cn}_T(\text{Cn}_T(A)) = \text{Th}(\text{Th}(T \cup A)) = \text{Th}(T \cup A)$. $\text{Cn}_T(A)$ is $\text{Th}(T \cup A)$.

**monotony:** if $A \subseteq B$ then $T \cup A \subseteq T \cup B$, and $\text{Cn}_T(A) = \text{Th}(T \cup A) \subseteq \text{Th}(T \cup B) = \text{Cn}_T(B)$.

And further;

**supraclassicality:** $A \subseteq T \cup A$, so $\text{Th}(A) \subseteq \text{Th}(T \cup A) = \text{Cn}_T(A)$.

**deduction:** $A \cup \{\alpha\} \vdash_T \beta$ iff $T \cup A \cup \{\alpha\} \vdash_{PL} \beta$ iff $T \cup A \vdash_{PL} \alpha \rightarrow \beta$ (deduction Th) iff $A \vdash_T \alpha \rightarrow \beta$.

**compactness:** if $A \vdash_T \alpha$ then $T \cup A \vdash_{PL} \alpha$ and (compactness of Th) $T' \cup A' \vdash_{PL} \alpha$ for some finite $T' \subseteq T$, $A' \subseteq A$. By monotony, $T \cup A \vdash_{PL} \alpha$, and so $A' \vdash_T \alpha$ for $A'$ a finite subset of $A$.

The last part of the question: this is a standard property of all supraclassical consequence relations with deduction.

$X \cup \{\alpha\} \vdash_T \beta$ and $X \cup \{\gamma\} \vdash_T \beta$ iff $X \vdash_T \alpha \rightarrow \beta$ and $X \vdash_T \gamma \rightarrow \beta$, i.e. iff $X \vdash_T (\alpha \rightarrow \beta) \land (\gamma \rightarrow \beta)$ by part 4(iv). It is easy to check that $(\alpha \rightarrow \beta) \land (\gamma \rightarrow \beta)$ is truth-functionally equivalent to $(\alpha \lor \gamma) \rightarrow \beta$. And so (by part 4(iii)) $X \vdash_T \alpha \rightarrow \beta$ and $X \vdash_T \gamma \rightarrow \beta$ iff $X \vdash_T (\alpha \lor \gamma) \rightarrow \beta$ iff (deduction) $X \cup (\alpha \lor \gamma) \vdash_T \beta$ as required.