491 Knowledge Representation

Tutorial Exercise

Logic databases
SOLUTIONS

Question 1

First, suppose we represent integrity constraint I1 by the formula

$$\forall x\,(\,p(x) \rightarrow (m(x) \vee f(x))\,)$$

Clearly, every database $\mathrm{Th}(D_i)$ satisfies I1 by the weakest *consistency* definition of IC satisfaction, because $\exists x\,(\,p(x) \wedge \neg m(x) \wedge \neg f(x)\,) \notin \mathrm{Th}(D_i)$ for any of the bases D_i.

With the strongest *entailment/theoremhood* definition, only $\mathrm{Th}(D_2)$ satisfies I1, because clearly $\forall x\,(\,p(x) \rightarrow (m(x) \vee f(x))\,) \notin \mathrm{Th}(D_i)$ for any of the other D_i.

Now suppose we read the integrity constraint I1 as a *metalevel* statement. Instead of reading it as a statement about what is true in the world being represented, read it as a constraint on what is in the database: for every $p(x)$ in the database, there is either a record in the database that x is male or there is a record in the database that x is female. In other words: if $p(x) \in \mathrm{Th}(D_i)$ then either $m(x) \in \mathrm{Th}(D_i)$ or $f(x) \in \mathrm{Th}(D_i)$. Now we have:

- D_1 not satisfied: $p(b) \in \mathrm{Th}(D_1)$ but $m(b) \notin \mathrm{Th}(D_1)$ and $f(b) \notin \mathrm{Th}(D_1)$.
- D_2 not satisfied: $p(a) \in \mathrm{Th}(D_2)$ but $m(a) \notin \mathrm{Th}(D_2)$ and $f(a) \notin \mathrm{Th}(D_2)$. We have $m(a) \vee f(a) \in \mathrm{Th}(D_2)$ but that is not enough (with I1 as formulated above). Similarly for $p(b)$.
- D_3 satisfied: we have $p(x) \in \mathrm{Th}(D_3)$ for $x = a, x = b$, and in both cases $m(x) \in \mathrm{Th}(D_3)$.
- D_4 satisfied: trivially, because there is no x such that $p(x) \in \mathrm{Th}(D_4)$.

For I2, proceed similarly. First, suppose we represent I2 by the formula

$$\forall x\,(\,p(x) \rightarrow \exists n\, h(x, n)\,)$$

Every database $\mathrm{Th}(D_i)$, $i = 5, 6, 7$, satisfies I2 by the consistency definition of integrity constraint satisfaction.

With the entailment/theoremhood definition, $\mathrm{Th}(D_5)$ does not satisfy I2, and $\mathrm{Th}(D_6)$ and $\mathrm{Th}(D_7)$ do.

Now suppose we read I2 as a metalevel constraint on what is in the database: for every $p(x)$ in the database there is record of x's home telephone number in the database; in other words, if $p(x) \in \mathrm{Th}(D_i)$ then there is a constant n such that $h(x, n) \in \mathrm{Th}(D_i)$.

With this reading, $\mathrm{Th}(D_5)$ and $\mathrm{Th}(D_6)$ do not satisfy I2 but $\mathrm{Th}(D_7)$ does.

Now consider $cwa_{\mathcal{P}}(D_i)$. For convenience, I will write $neg_{\mathcal{P}}(D_i)$ for $\{\neg\alpha \mid \alpha \in \mathcal{P},\ \alpha \notin \mathrm{Th}(D_i)\}$), so $cwa_{\mathcal{P}}(D_i) = \mathrm{Th}(D_i \cup neg_{\mathcal{P}}(D_i))$.

- $neg_{\mathcal{P}}(D_1) = \{\neg m(b), \neg f(a), \neg f(b)\}$. $\mathrm{Th}(D_1 \cup neg_{\mathcal{P}}(D_1))$ is consistent but $\mathrm{Th}(D_1 \cup neg_{\mathcal{P}}(D_1)) \cup \{\mathrm{I1}\}$ is inconsistent. So I1 is not satisfied by the consistency definition. It is not satisfied by the other two definitions either.
- $neg_{\mathcal{P}}(D_2) = \{\neg m(a), \neg m(b), \neg f(a), \neg f(b)\}$. $\mathrm{Th}(D_2 \cup neg_{\mathcal{P}}(D_2))$ is inconsistent. So this database fails to satisfy I1 by the consistency definition (obviously). It satisfies I1 by both metalevel and entailment/theoremhood definitions—trivially, since everything is a consequence of an inconsistent set of formulas.
- $neg_{\mathcal{P}}(D_3) = \{\neg f(a), \neg f(b)\}$. $\mathrm{Th}(D_3 \cup neg_{\mathcal{P}}(D_3))$ is consistent. It satisfies I1 by the consistency and metalevel definitions, but not by the entailment/theoremhood definition.
- $neg_{\mathcal{P}}(D_4) = \{\neg p(a), \neg p(b), \neg m(a), \neg m(b), \neg f(a), \neg f(b)\}$. $\mathrm{Th}(D_4 \cup neg_{\mathcal{P}}(D_4))$ is consistent. It satisfies I1 by the consistency and metalevel definitions, but not by the entailment/theoremhood definition.

For integrity constraint I2 and databases D_5–D_7:

- $neg_{\mathcal{P}}(D_5) = \{\neg h(a, 456), \neg h(b, 123), \neg h(b, 456)\}$. $\mathrm{Th}(D_5 \cup neg_{\mathcal{P}}(D_5))$ is consistent. $\mathrm{Th}(D_5 \cup neg_{\mathcal{P}}(D_5))$ is also consistent with I2 (there is nothing in D_5 that says 123 and 456 are the only possible telephone numbers). So it satisfies I2 by the consistency definition, but not by the other two definitions.
- $neg_{\mathcal{P}}(D_6) = \{\neg h(a, 123), \neg h(a, 456), \neg h(b, 123), \neg h(b, 456)\}$. $\mathrm{Th}(D_6 \cup neg_{\mathcal{P}}(D_6))$ is consistent. It satisfies I2 by both the consistency and entailment/definitions, but not by the metalevel definition.

  Why not by the metalevel definition? Because we have $p(a) \in \mathrm{Th}(D_6 \cup neg_{\mathcal{P}}(D_6))$ but there is no constant n such that $h(a, n) \in \mathrm{Th}(D_6 \cup neg_{\mathcal{P}}(D_6))$. And similarly for $p(b)$. We do have the weaker $\exists x\, h(a, x) \in \mathrm{Th}(D_6 \cup neg_{\mathcal{P}}(D_6))$ but that is not enough for the metalevel I2 (as we formulated it above).
- $neg_{\mathcal{P}}(D_7) = \{\neg h(a, 123), \neg h(b, 123)\}$. $\mathrm{Th}(D_7 \cup neg_{\mathcal{P}}(D_7))$ is consistent. It satisfies I2 by all three definitions of integrity constraint satisfaction.

(Please check the above for typos/mistakes. I typed it in a hurry.)

Question 2

(i) First half. Assume $Y \subseteq \text{Th}(X) \;\Rightarrow\; (X \subseteq \text{Cn}(A) \;\Rightarrow\; Y \subseteq \text{Cn}(A))$. Show $\text{Th}(\text{Cn}(A)) \subseteq \text{Cn}(A)$, i.e., $Y \subseteq \text{Th}(\text{Cn}(A)) \;\Rightarrow Y \subseteq \text{Cn}(A)$ for all Y.

Take the special case $X = \text{Cn}(A)$. We have:

$$Y \subseteq \text{Th}(\text{Cn}(A)) \;\Rightarrow\; (\text{Cn}(A) \subseteq \text{Cn}(A) \;\Rightarrow\; Y \subseteq \text{Cn}(A))$$

But $\text{Cn}(A) \subseteq \text{Cn}(A)$ trivially, so $Y \subseteq \text{Th}(\text{Cn}(A)) \;\Rightarrow Y \subseteq \text{Cn}(A)$ as required.

The other half: Suppose $\text{Th}(\text{Cn}(A)) \subseteq \text{Cn}(A)$. We need to show that if $Y \subseteq \text{Th}(X)$ and $X \subseteq \text{Cn}(A)$ then $Y \subseteq \text{Cn}(A)$. By monotony of Th, $X \subseteq \text{Cn}(A)$ implies $\text{Th}(X) \subseteq \text{Th}(\text{Cn}(A))$. So we have:

$$Y \subseteq \text{Th}(X) \subseteq \text{Th}(\text{Cn}(A)) \subseteq \text{Cn}(A)$$

So $Y \subseteq \text{Th}(X)$ implies $Y \subseteq \text{Cn}(A)$ as required.

(ii) First part: we need to show that if $(\alpha \to \beta) \in \text{Cn}(D)$ and $\alpha \in \text{Cn}(D)$ then $\beta \in \text{Cn}(D)$. But $\{\alpha \to \beta, \alpha\} \subseteq \text{Cn}(D)$ implies $\beta \in \text{Cn}(D)$ because $\{\alpha \to \beta, \alpha\} \vdash_{PL} \beta$ and part (i) above.

Second part: suppose $\text{Cn}(D)$ is consistent and suppose $\alpha \in \text{Cn}(D)$ implies $\beta \in \text{Cn}(D)$. Assume for contradiction that $\neg(\alpha \to \beta) \in \text{Cn}(D)$. $\neg(\alpha \to \beta)$ is truth-functionally equivalent to $\alpha \wedge \neg\beta$. So by part (i), if $\neg(\alpha \to \beta) \in \text{Cn}(D)$ then $\alpha \in \text{Cn}(D)$ and $\neg\beta \in \text{Cn}(D)$. But if $\alpha \in \text{Cn}(D)$ then $\beta \in \text{Cn}(D)$, so we have $\beta \in \text{Cn}(D)$ and $\neg\beta \in \text{Cn}(D)$, which contradicts the assumption that $\text{Cn}(D)$ is consistent.

(iii) First part: assume $\text{Cn}(D)$ is complete. We show the contrapositive, i.e., show that if $\alpha \in \text{Cn}(D)$ and $\beta \notin \text{Cn}(D)$ then $\neg(\alpha \to \beta) \in \text{Cn}(D)$. Since $\text{Cn}(D)$ is complete, $\beta \notin \text{Cn}(D)$ implies $\neg\beta \in \text{Cn}(D)$. So if $\alpha \in \text{Cn}(D)$ and $\beta \notin \text{Cn}(D)$ then, by part (i), $\alpha \wedge \neg\beta \in \text{Cn}(D)$, i.e., $\neg(\alpha \to \beta) \in \text{Cn}(D)$.

Second part: Assume $\text{Cn}(D)$ is complete. Assume that $\alpha \in \text{Cn}(D)$ implies $\beta \in \text{Cn}(D)$. Show $(\alpha \to \beta) \in \text{Cn}(D)$. Two cases: case (a) $\alpha \in \text{Cn}(D)$: then $\beta \in \text{Cn}(D)$ and so $\alpha \to \beta \in \text{Cn}(D)$ because $\{\beta\} \vdash_{PL} (\alpha \to \beta)$. Case (b) $\alpha \notin \text{Cn}(D)$: then because $\text{Cn}(D)$ is complete, $\neg\alpha \in \text{Cn}(D)$. And then $(\alpha \to \beta) \in \text{Cn}(D)$ because $\{\neg\alpha\} \vdash_{PL} (\alpha \to \beta)$.