Question 1
First, suppose we represent integrity constraint I1 by the formula
\[ \forall x \ (p(x) \rightarrow (m(x) \lor f(x))) \]
Clearly, every database \( \text{Th}(D_i) \) satisfies I1 by the weakest consistency definition of IC satisfaction, because \( \exists x \ (p(x) \land \neg m(x) \land \neg f(x)) \notin \text{Th}(D_i) \) for any of the bases \( D_i \).
With the strongest entailment/theoremhood definition, only \( \text{Th}(D_2) \) satisfies I1, because clearly \( \exists x \ (p(x) \rightarrow (m(x) \lor f(x))) \notin \text{Th}(D_i) \) for any of the other \( D_i \).
Now suppose we read the integrity constraint I1 as a metalevel statement. Instead of reading it as a statement about what is true in the world being represented, read it as a constraint on what is in the database: for every \( p(x) \) in the database, there is either a record in the database that \( x \) is male or there is a record in the database that \( x \) is female. In other words: if \( p(x) \in \text{Th}(D_i) \) then either \( m(x) \in \text{Th}(D_i) \) or \( f(x) \in \text{Th}(D_i) \). Now we have:

- \( D_1 \) not satisfied: \( p(b) \in \text{Th}(D_1) \) but \( m(b) \notin \text{Th}(D_1) \) and \( f(b) \notin \text{Th}(D_1) \).
- \( D_2 \) not satisfied: \( p(a) \in \text{Th}(D_2) \) but \( m(a) \notin \text{Th}(D_2) \) and \( f(a) \notin \text{Th}(D_2) \). We have \( m(a) \lor f(a) \in \text{Th}(D_2) \) but that is not enough (with I1 as formulated above). Similarly for \( p(b) \).
- \( D_3 \) satisfied: we have \( p(x) \in \text{Th}(D_3) \) for \( x = a, x = b \), and in both cases \( m(x) \in \text{Th}(D_3) \).
- \( D_4 \) satisfied: trivially, because there is no \( x \) such that \( p(x) \in \text{Th}(D_4) \).

For I2, proceed similarly. First, suppose we represent I2 by the formula
\[ \forall x \ (p(x) \rightarrow \exists n \ h(x, n)) \]
Every database \( \text{Th}(D_i), \ i = 5, 6, 7 \), satisfies I2 by the consistency definition of integrity constraint satisfaction.
With the entailment/theoremhood definition, \( \text{Th}(D_6) \) does not satisfy I2, and \( \text{Th}(D_6) \) and \( \text{Th}(D_7) \) do.
Now suppose we read I2 as a metalevel constraint on what is in the database: for every \( p(x) \) in the database there is record of \( x \)'s home telephone number in the database; in other words, if \( p(x) \in \text{Th}(D_i) \) then there is a constant \( n \) such that \( h(x, n) \in \text{Th}(D_i) \).
With this reading, \( \text{Th}(D_5) \) and \( \text{Th}(D_6) \) do not satisfy I2 but \( \text{Th}(D_7) \) does.

Now consider \( \text{cwa}_p(D_i) \). For convenience, I will write \( \text{neg}_p(D_i) \) for \( \{ \neg \alpha \mid \alpha \in P, \ \alpha \notin \text{Th}(D_i) \} \), so \( \text{cwa}_p(D_i) = \text{Th}(D_i \cup \text{neg}_p(D_i)) \).

- \( \text{neg}_p(D_1) = \{ \neg h(a, 456), \neg h(b, 123), \neg h(b, 456) \} \). \( \text{Th}(D_1 \cup \text{neg}_p(D_1)) \) is consistent but \( \text{Th}(D_1 \cup \text{neg}_p(D_1)) \cup \{ I \} \) is inconsistent. So I1 is not satisfied by the consistency definition. It is not satisfied by the other two definitions either.
- \( \text{neg}_p(D_2) = \{ \neg f(a), \neg f(b) \} \). \( \text{Th}(D_2 \cup \text{neg}_p(D_2)) \) is inconsistent. So this database fails to satisfy I1 by the consistency definition (obviously). It satisfies I1 by both metalevel and entailment/theoremhood definitions—trivially, since everything is a consequence of an inconsistent set of formulas.
- \( \text{neg}_p(D_3) = \{ \neg f(a), \neg f(b) \} \). \( \text{Th}(D_3 \cup \text{neg}_p(D_3)) \) is consistent. It satisfies I1 by the consistency and metalevel definitions, but not by the entailment/theoremhood definition.
- \( \text{neg}_p(D_4) = \{ \neg p(a), \neg p(b), \neg m(a), \neg m(b), \neg f(a), \neg f(b) \} \). \( \text{Th}(D_4 \cup \text{neg}_p(D_4)) \) is consistent. It satisfies I1 by the consistency and metalevel definitions, but not by the entailment/theoremhood definition.

For integrity constraint I2 and databases \( D_5 - D_7 \):

- \( \text{neg}_p(D_5) = \{ \neg h(a, 456), \neg h(b, 123), \neg h(b, 456) \} \). \( \text{Th}(D_5 \cup \text{neg}_p(D_5)) \) is consistent. \( \text{Th}(D_5 \cup \text{neg}_p(D_5)) \) is also consistent with I2 (there is nothing in \( D_5 \) that says 123 and 456 are the only possible telephone numbers). So it satisfies I2 by the consistency definition, but not by the other two definitions.
- \( \text{neg}_p(D_6) = \{ \neg h(a, 123), \neg h(a, 456), \neg h(b, 123), \neg h(b, 456) \} \). \( \text{Th}(D_6 \cup \text{neg}_p(D_6)) \) is consistent. It satisfies I2 by both the consistency and entailment/definitions, but not by the metalevel definition.
- \( \text{neg}_p(D_7) = \{ \neg h(a, 123), \neg h(b, 123) \} \). \( \text{Th}(D_7 \cup \text{neg}_p(D_7)) \) is consistent. It satisfies I2 by all three definitions of integrity constraint satisfaction.

(Please check the above for typos/mistakes. I typed it in a hurry.)
Question 2

(i) First half. Assume $Y \subseteq \text{Th}(X) \Rightarrow (X \subseteq \text{Cn}(A) \Rightarrow Y \subseteq \text{Cn}(A))$. Show $\text{Th}(\text{Cn}(A)) \subseteq \text{Cn}(A)$, i.e., $Y \subseteq \text{Th}(\text{Cn}(A)) \Rightarrow Y \subseteq \text{Cn}(A)$ for all $Y$.

Take the special case $X = \text{Cn}(A)$. We have:

$$Y \subseteq \text{Th}(\text{Cn}(A)) \Rightarrow (\text{Cn}(A) \subseteq \text{Cn}(A) \Rightarrow Y \subseteq \text{Cn}(A))$$

But $\text{Cn}(A) \subseteq \text{Cn}(A)$ trivially, so $Y \subseteq \text{Th}(\text{Cn}(A)) \Rightarrow Y \subseteq \text{Cn}(A)$ as required.

The other half: Suppose $\text{Th}(\text{Cn}(A)) \subseteq \text{Cn}(A)$. We need to show that if $Y \subseteq \text{Th}(X)$ and $X \subseteq \text{Cn}(A)$ then $Y \subseteq \text{Cn}(A)$. By monotony of $\text{Th}$, $X \subseteq \text{Cn}(A)$ implies $\text{Th}(X) \subseteq \text{Th}(\text{Cn}(A))$. So we have:

$$Y \subseteq \text{Th}(X) \subseteq \text{Th}(\text{Cn}(A)) \subseteq \text{Cn}(A)$$

So $Y \subseteq \text{Th}(X)$ implies $Y \subseteq \text{Cn}(A)$ as required.

(ii) First part: we need to show that if $(\alpha \rightarrow \beta) \in \text{Cn}(D)$ and $\alpha \in \text{Cn}(D)$ then $\beta \in \text{Cn}(D)$. But $\{\alpha \rightarrow \beta, \alpha\} \subseteq \text{Cn}(D)$ implies $\beta \in \text{Cn}(D)$ because $\{\alpha \rightarrow \beta, \alpha\} \models_{\text{PL}} \beta$ and part (i) above.

Second part: suppose $\text{Cn}(D)$ is consistent and suppose $\alpha \in \text{Cn}(D)$ implies $\beta \in \text{Cn}(D)$. Assume for contradiction that $\neg (\alpha \rightarrow \beta) \in \text{Cn}(D)$. $\neg (\alpha \rightarrow \beta)$ is truth-functionally equivalent to $\alpha \land \neg \beta$. So by part (i), if $\neg (\alpha \rightarrow \beta) \in \text{Cn}(D)$ then $\alpha \in \text{Cn}(D)$ and $\neg \beta \in \text{Cn}(D)$. But if $\alpha \in \text{Cn}(D)$ then $\beta \in \text{Cn}(D)$, so we have $\beta \in \text{Cn}(D)$ and $\neg \beta \in \text{Cn}(D)$, which contradicts the assumption that $\text{Cn}(D)$ is consistent.

(iii) First part: assume $\text{Cn}(D)$ is complete. We show the contrapositive, i.e., show that if $\alpha \in \text{Cn}(D)$ and $\beta \notin \text{Cn}(D)$ then $\neg (\alpha \rightarrow \beta) \in \text{Cn}(D)$. Since $\text{Cn}(D)$ is complete, $\beta \notin \text{Cn}(D)$ implies $\neg \beta \in \text{Cn}(D)$. So if $\alpha \in \text{Cn}(D)$ and $\beta \notin \text{Cn}(D)$ then, by part (i), $\alpha \land \neg \beta \in \text{Cn}(D)$, i.e., $\neg (\alpha \rightarrow \beta) \in \text{Cn}(D)$.

Second part: Assume $\text{Cn}(D)$ is complete. Assume that $\alpha \in \text{Cn}(D)$ implies $\beta \in \text{Cn}(D)$. Show $(\alpha \rightarrow \beta) \in \text{Cn}(D)$. Two cases: case (a) $\alpha \in \text{Cn}(D)$: then $\beta \in \text{Cn}(D)$ and so $\alpha \rightarrow \beta \in \text{Cn}(D)$ because $\{\beta\} \models_{\text{PL}} (\alpha \rightarrow \beta)$. Case (b) $\alpha \notin \text{Cn}(D)$: then because $\text{Cn}(D)$ is complete, $\neg \alpha \in \text{Cn}(D)$. And then $(\alpha \rightarrow \beta) \in \text{Cn}(D)$ because $\{\neg \alpha\} \models_{\text{PL}} (\alpha \rightarrow \beta)$. 

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