C240 Computability and Complexity : Tutorial 1 Solution notes

1. The 'paradox' is that (a) the list contains all programs, so $P = P_n$ for some n, but (b) P is written so that its output differs from P_n at the nth bit, so $P \neq P_n$. This is impossible, so something must be wrong with the argument. In fact the 'bug' is in line 3 – we don't know whether this line, when executed, will terminate. P may hang here forever. If $P = P_n$, then P may (will) hang here on the nth repeat, so P never actually prints the nth bit.
2. We know = P_n for some n. Say n=19. To know what to output as its 19^{th} bit, P must know the 19^{th} bit of P_{19}'s output – ie. The 19^{th} bit of its own output. Of course P may never get to the 19^{th} run round the loop because maybe an earlier P (say P_5) also happens to run forever without ever outputting 5 bits. But if it does, then on this 19^{th} run round the loop, P will generate P19 – ie. A copy, P' say, of itself. It must then run this copy from the start, until it outputs 19 bits, (that's what line 3 does). But this means that P_{19} is run again up to the 19^{th} run round the loop. Then the code of P' tells it to generate a copy of P_{19}, ie. *another* copy of P, and so on…So if P gets as far as trying to output bit 19 it goes into an infinite loop, without ever outputting again..

3. If we could write a program H such that for any P, H(P) = 1 if P halts, and H(P) = 0 if P doesn't halt, we could add the lines:

   2.1 if H("run P_n as far as the nth bit of the output") = 0 then output 1; next repeat; end if.

   (we pass the entire text of the interpreter, plus the text of P_n, to H.We assume this halts if P_n prompts for input before outputting bit n. Now the problem in Q1 would not arise, and a real paradox results. So there is no such H.

4. Observe that if paradoxical sentence (iii) is false then you can convince yourself that it is true, so your logic is not too hot! But if it is true, you have no way of establishing that it's true – that's what it says. This shows that if your thinking never makes mistakes, (like a mathematical proof-inducer) then there are true statements whose truth you can't establish!. Cf. Gödel's incompleteness theorem. The other paradoxes are for you to lose sleep over!