Modal transition systems: a foundation for the analysis of partial systems

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Outline

1. Representing and analyzing partial systems

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2. Modal transition systems

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8. Conclusions and future work

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Modal transition systems express such aspects and generalize Dams's work (Dams 1996) to *partial* systems.

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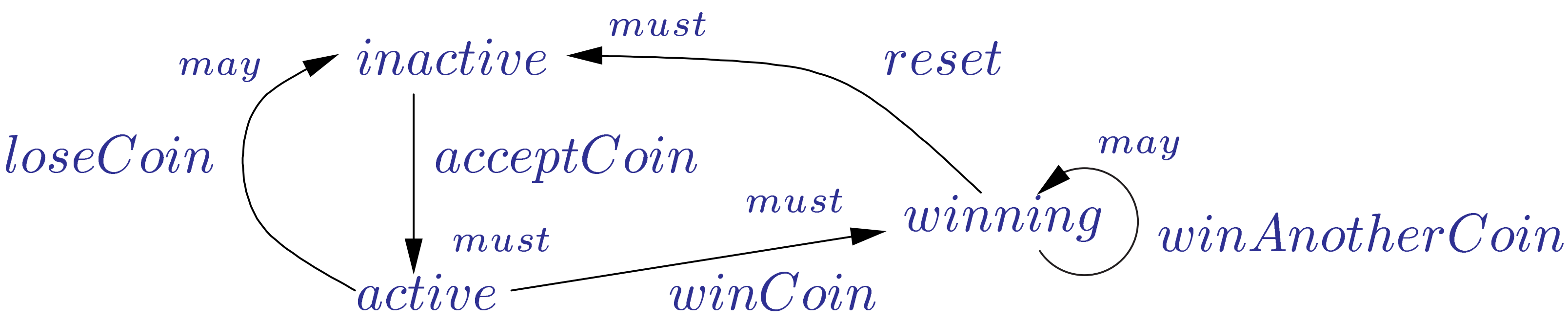
- impose a consistency condition: $\rightarrow_{\mathrm{must}} \subseteq \rightarrow_{\mathrm{may}}$ (all required behavior is also possible).

A slot machine: specification



Consistency condition: *must*-transitions comprise $\rightarrow_{\mathrm{must}}$, *may*-and *must*-transitions comprise $\rightarrow_{\mathrm{may}}$.

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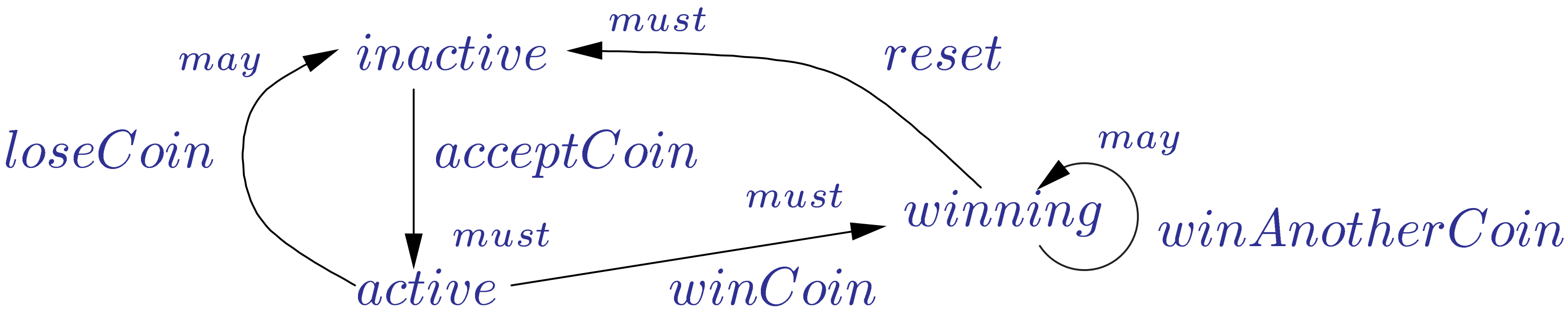

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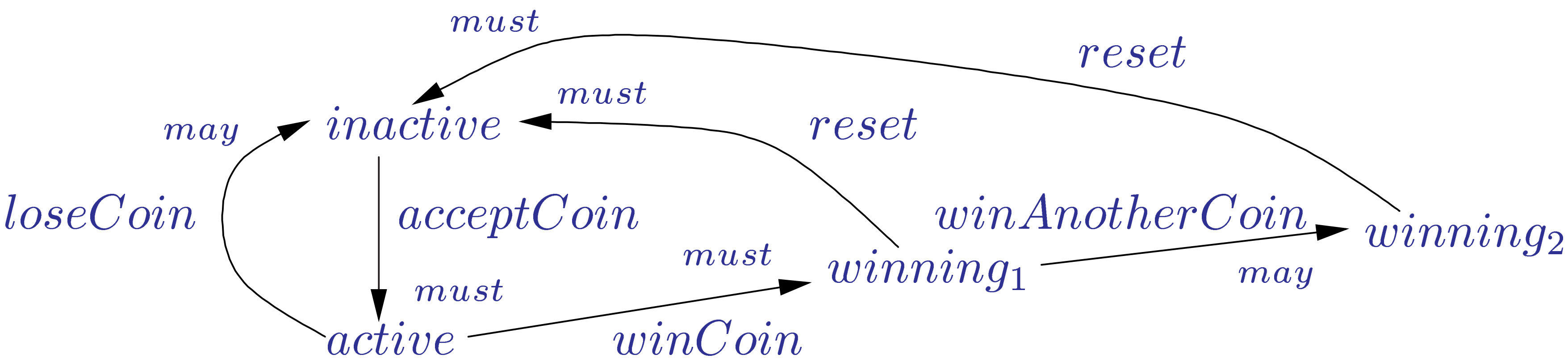

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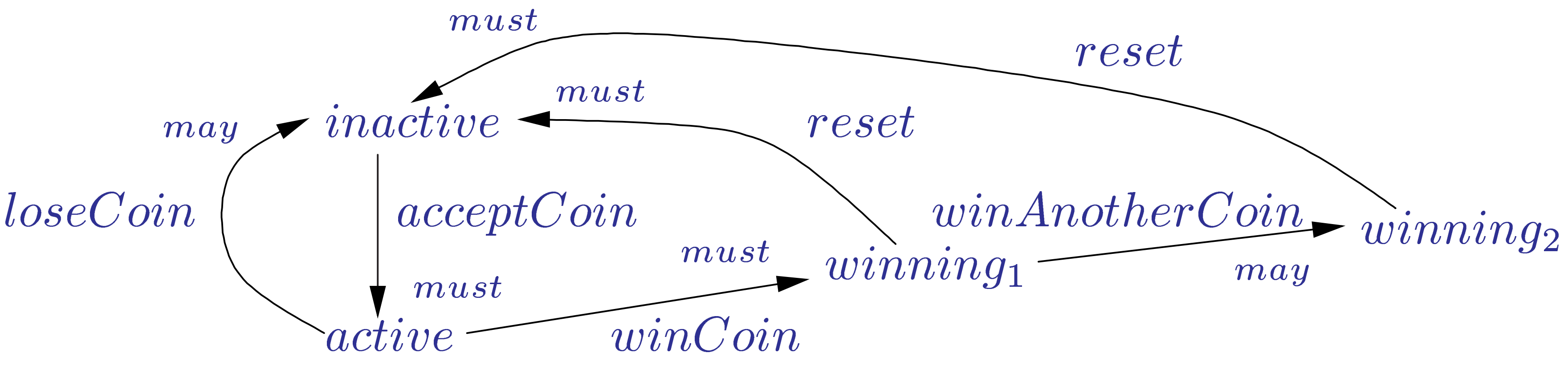

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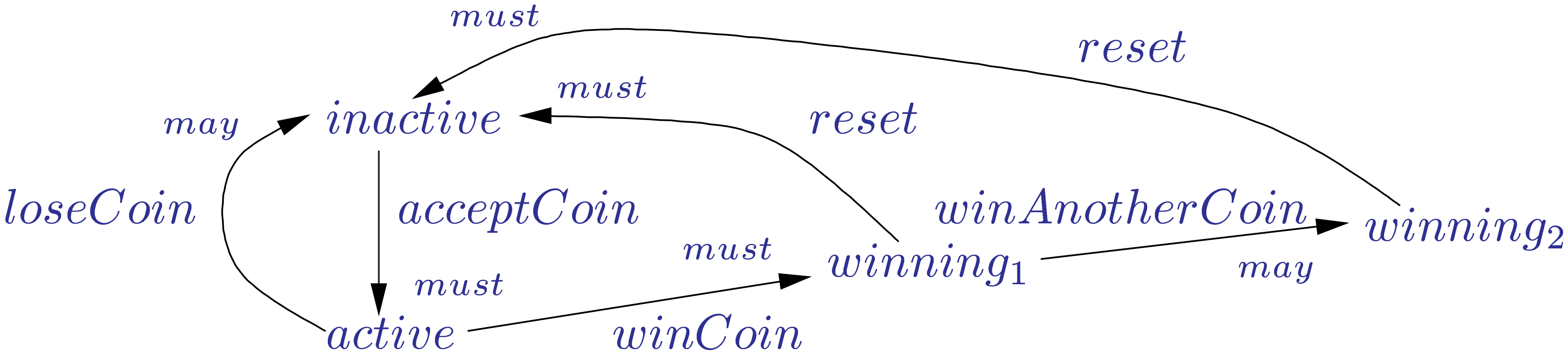
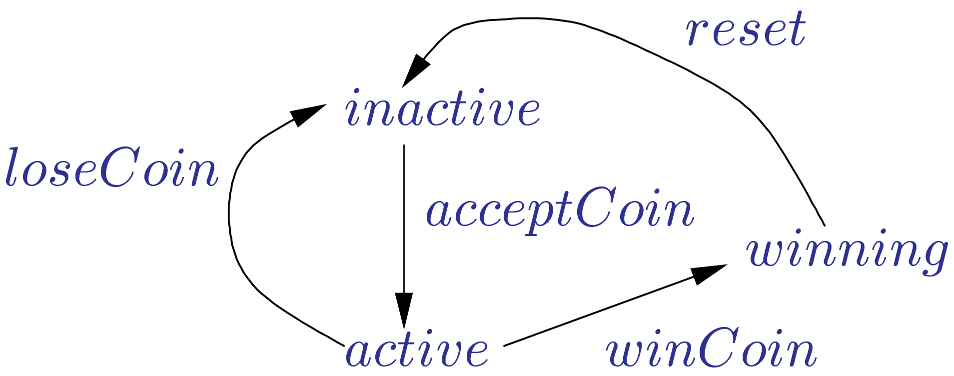


In implementations, all transitions are *must*-transitions:



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We write $\prec$ for the greatest such relation.

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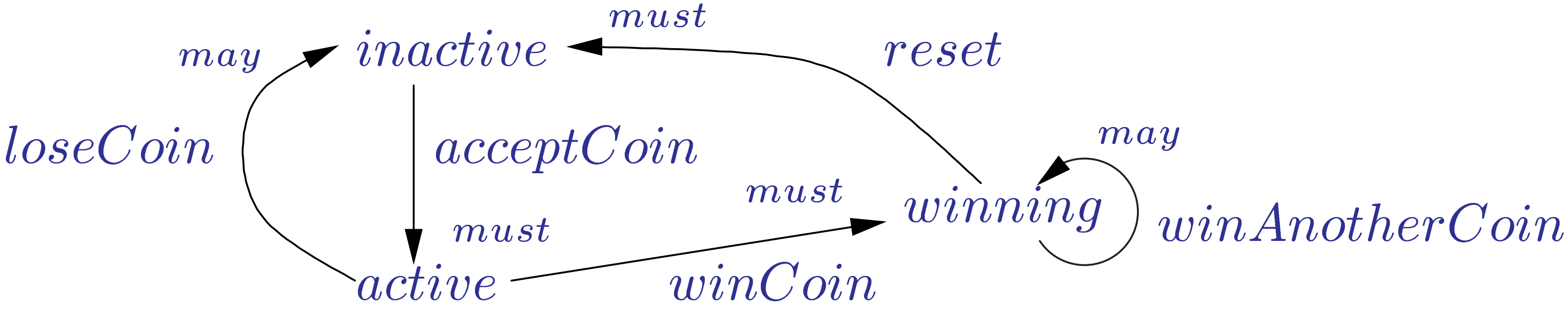

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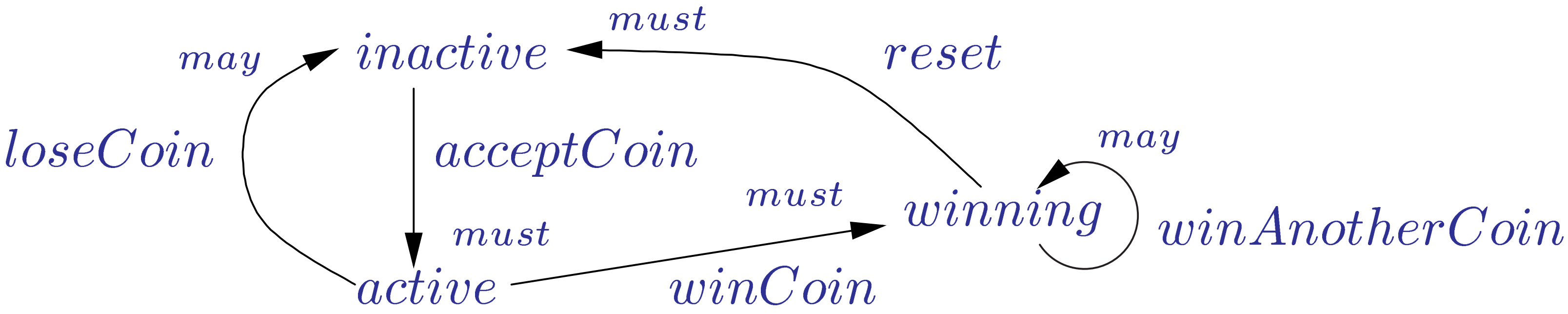

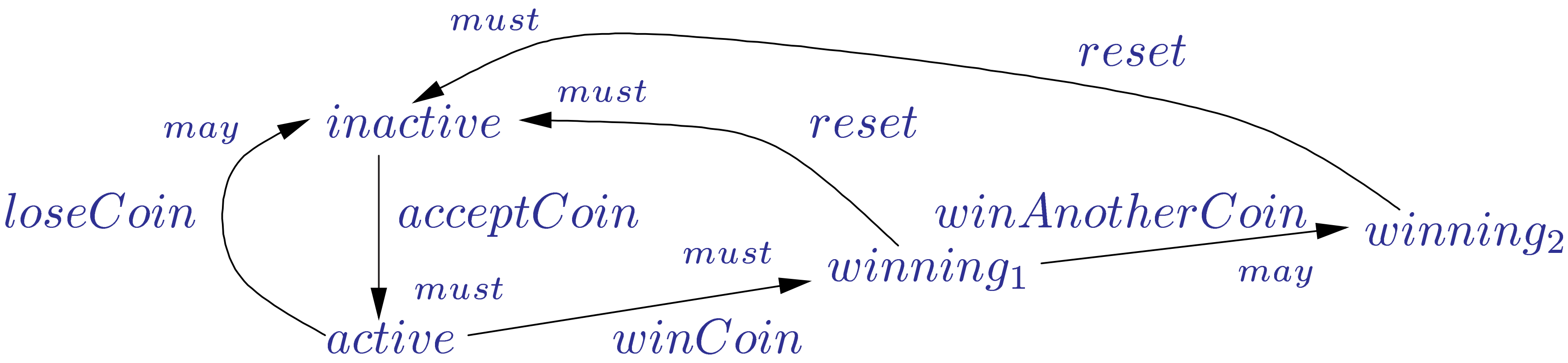

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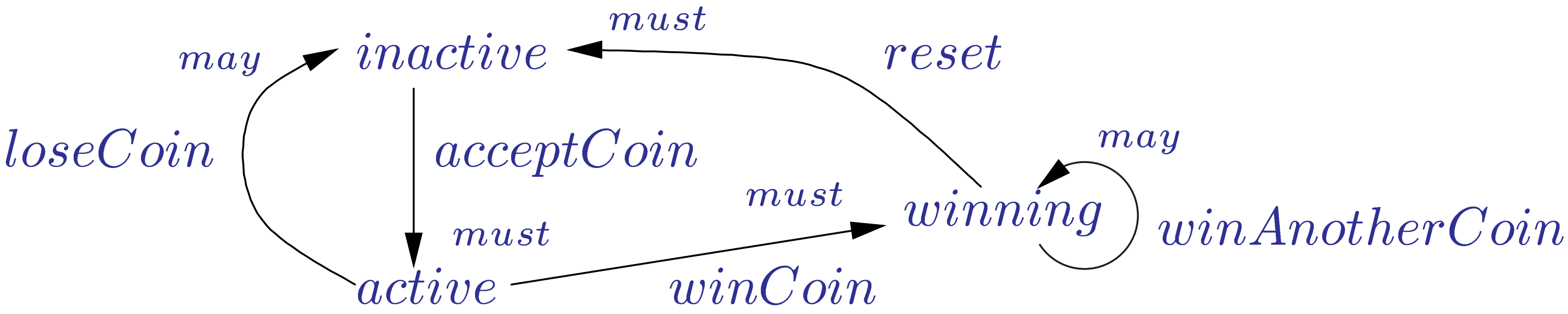

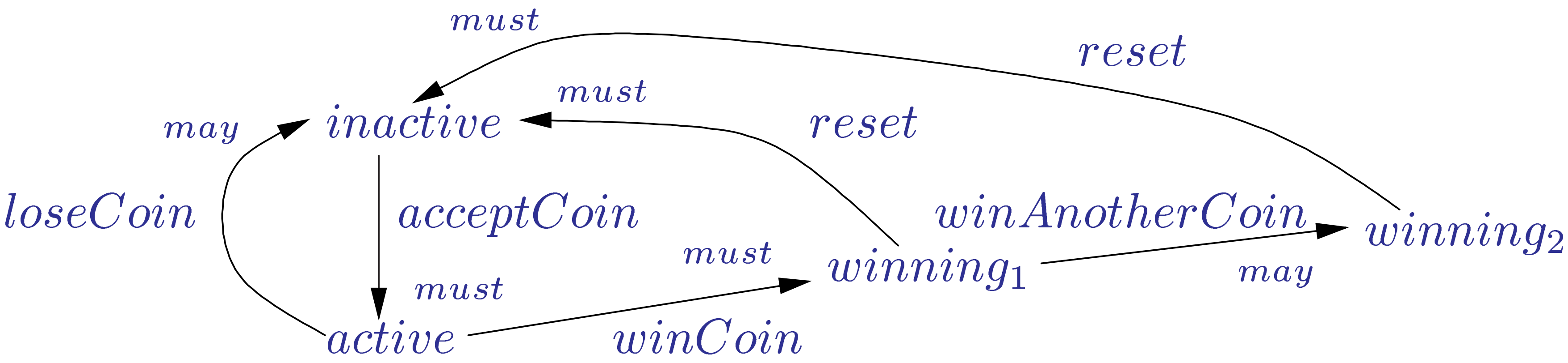

Refinement:

$\{(winning_1, winning), (winning_2, winning), (inactive, inactive), (active, active)\}$.

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Related work: Cousot & Cousot (2000) define a *universal* and an *existential* abstraction; the latter one, used *in isolation* restricts sound reasoning to *safety* properties.

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Kozen's modal mu-calculus (Kozen 1983) (subsumes tool logics):

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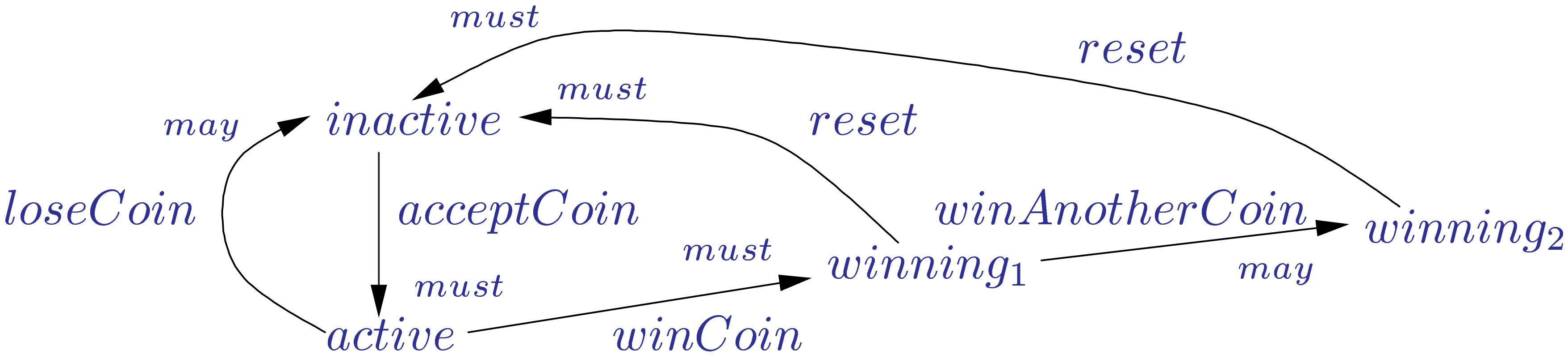
The definition of $\models^{\mathrm{pos}}$ is as for $\models^{\mathrm{nec}}$, but we swap $\mathrm{nec} \leftrightarrow \mathrm{pos}$ and $\mathrm{must} \leftrightarrow \mathrm{may}$.

E.g., $s \models^{\mathrm{pos}} \langle a \rangle\, \phi$ iff for some s', $s \rightarrow^{a}_{\mathrm{may}} s'$ and $s' \models^{\mathrm{pos}} \phi$.

Example refutation

We don't have $\text{inactive} \models^{\text{nec}} \texttt{EF}\,(\text{winning} \wedge \texttt{EX}\,\text{winning})$:



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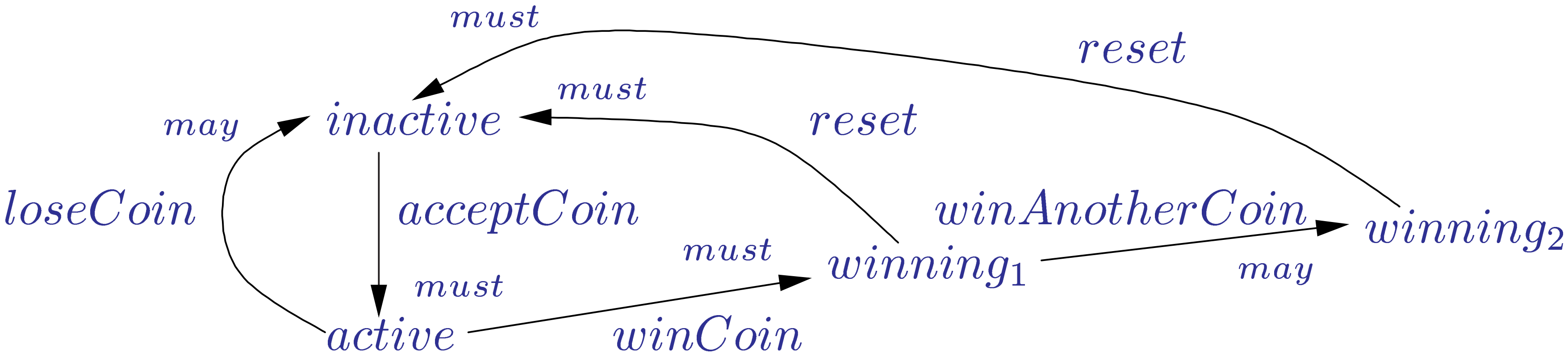

This makes sense as we have
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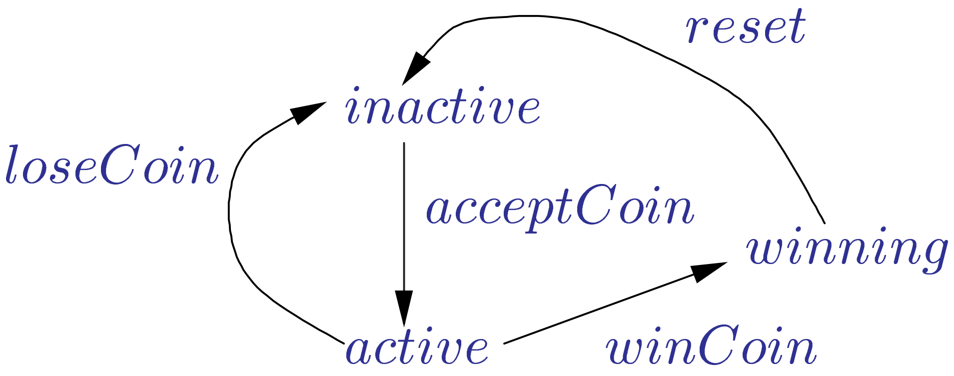

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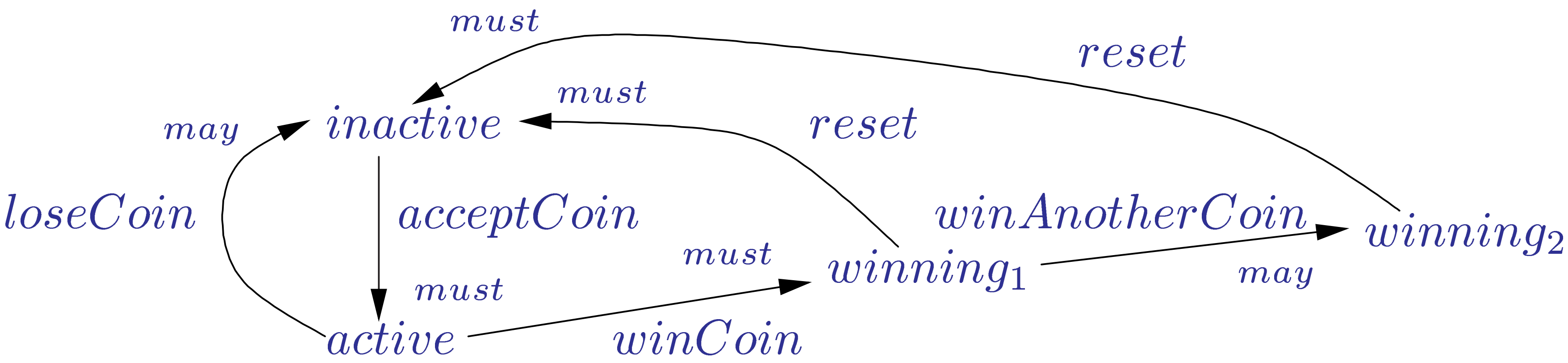

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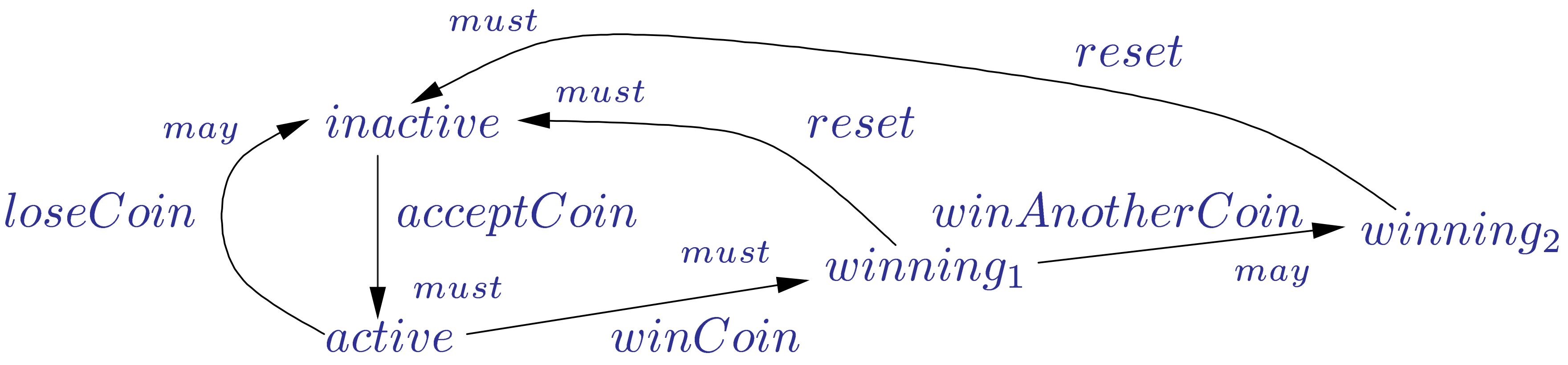
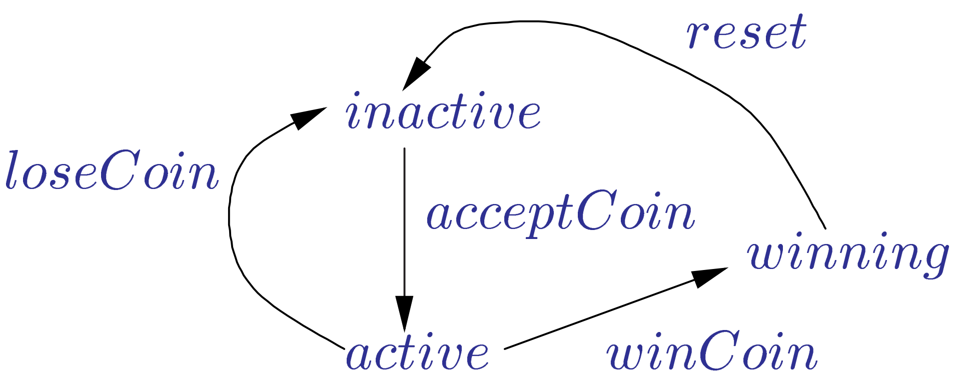


So this property holds for all implementations, in particular for



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means either refuting $t \models^{\text{pos}} \phi$ or verifying $t \models^{\text{nec}} \psi$.

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(c) modal shape graphs

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- example: $\Sigma_E \stackrel{\text{def}}{=} \{\mathtt{callp}, \mathtt{enterp}, \mathtt{runp}, \mathtt{exitp}\}$, with transitions $\mathtt{callp} \rightarrow^{\star} \mathtt{enterp}$, $\mathtt{enterp} \rightarrow^{\star} \mathtt{runp}$, $\mathtt{runp} \rightarrow^{\star} \mathtt{exitp}$, and $\mathtt{runp} \uparrow \star$.

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- $\mathcal{E}$ as a modal transition system $\mathcal{M}[\mathcal{E}]$:

$$s \rightarrow_a^{must} s' \quad \text{iff} \quad s \rightarrow_a s'$$
$$s \rightarrow_a^{may} s' \quad \text{iff} \quad s \rightarrow_a s' \text{ or } s \uparrow a.$$

Example requires more than one action label. Then, $\mathtt{runp} \uparrow a$ for all a, possibly except "$\mathtt{exitp}$", "$\mathtt{enterp}$" and "$\mathtt{callp}$".

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Conclusion: The models, abstraction, and temporal logic semantics for partial bisimulation are a special instance of the corresponding notions for modal transition systems.

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(b) Partial Kripke structures

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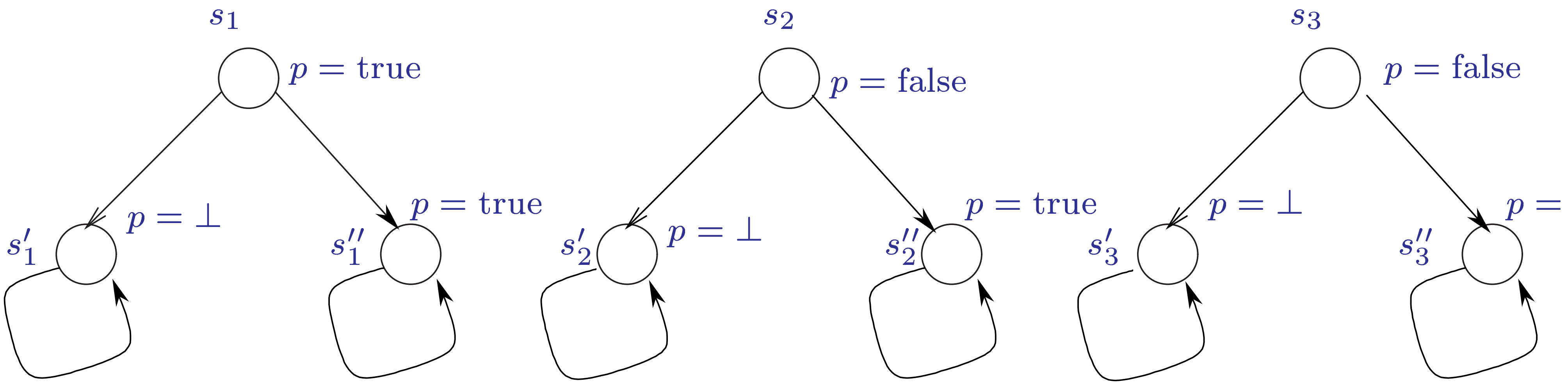

Conclusion

The order-isomorphism

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\begin{aligned}
\mathbf{M}[\Sigma] \quad &\cong \quad \Sigma \to \mathbf{3} \qquad\qquad (\Sigma \text{ discrete}) \\
(\{s \in \Sigma \mid f(s) = \text{true}\}, \{s \in \Sigma \mid f(s) = \perp \text{ or true}\}) \quad &\cong \quad f
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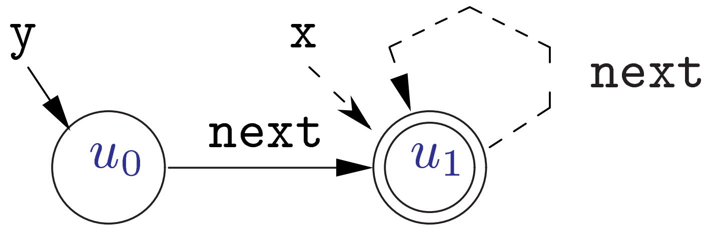
recaptures their models, abstraction relation ("completeness ordering" in Bruns (1999)), and CTL semantics as an important instance of the corresponding notions of modal transition systems.

(c) Modal shape graphs

are approximations of concrete contents of heap storage. E.g. the modal shape graph



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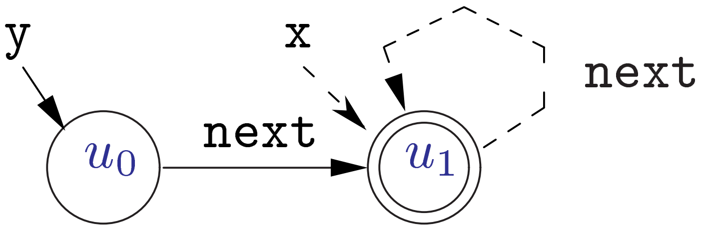

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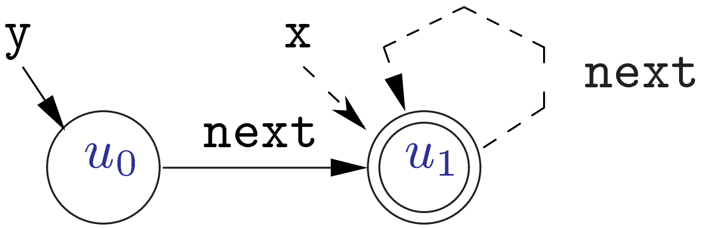

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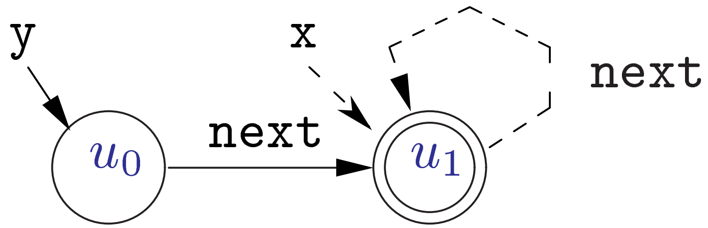

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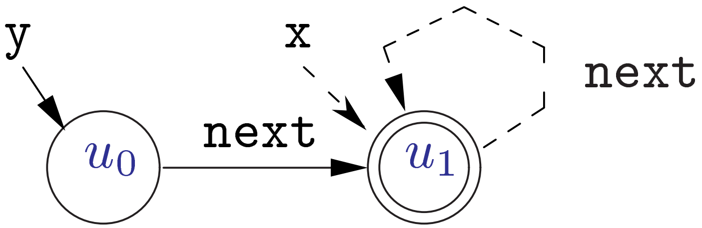
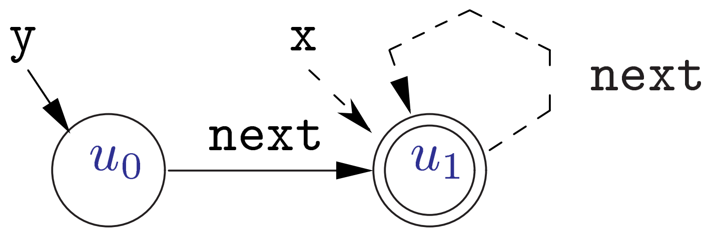
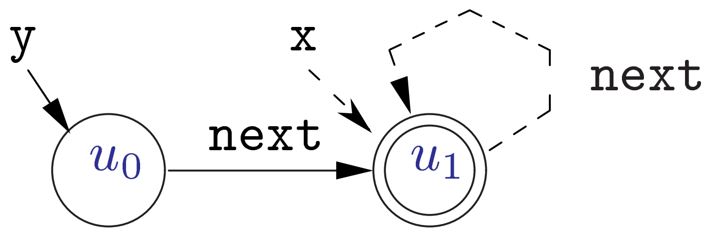


- approximates a singly linked list of length at least two.
- Local variables, `x` and `y`, point to objects.
- Objects have arcs to `next` fields; solid ones are must-, dotted ones are may-transitions.
- Double-circled objects are "summary nodes", possibly representing more than one object.

The shape graph



The shape graph and its coding via predicates



M. Sagiv, T. Reps, and R. Wilhelm (1999) use Kleene's ordering $0 < 1/2 < 1$: $\neg l = 1 - l$; $\wedge$ and $\vee$ are min and max (respectively).

$x(u_0) = 0$	$next(u_0, u_0) = 0$
$x(u_1) = 1/2$	$next(u_0, u_1) = 1$
$y(u_0) = 1$	$next(u_1, u_0) = 0$
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$sm(u_0) = 0$	proposition for summary nodes
$sm(u_1) = 1.$	

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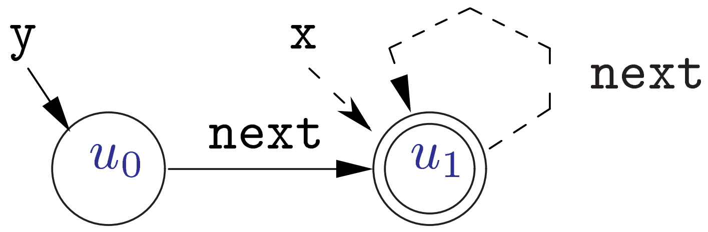
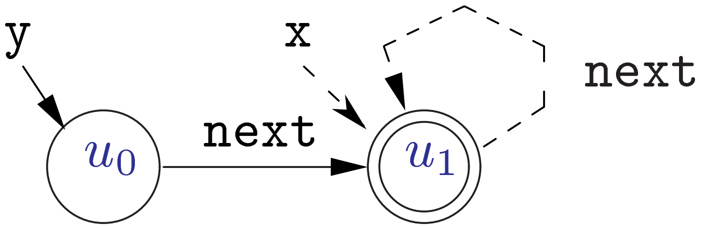
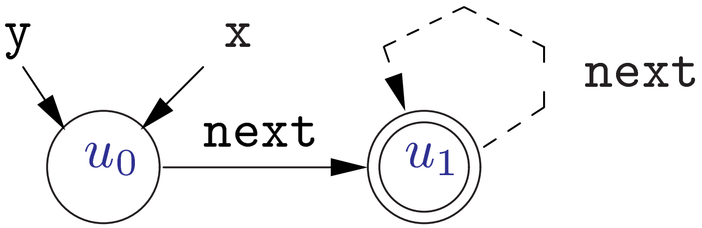


The same shape graph and the effect of `x = y`





Note how the may-pointer `x` turns into a must-pointer.

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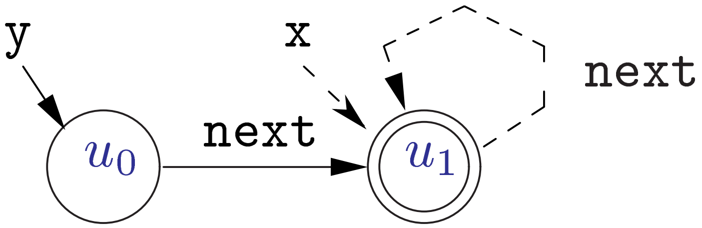

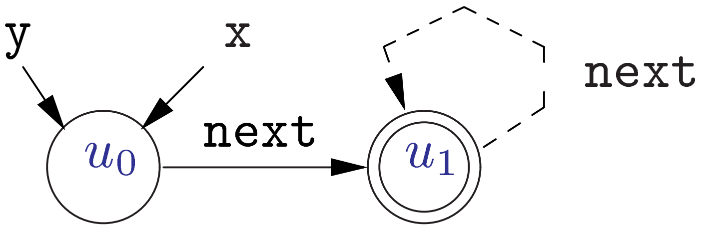

Note how the may-pointer `x` turns into a must-pointer. The general transfer function is:

$$T[\texttt{x = y}] : x'(v) = y(v);\ \ \text{all other predicates } p' = p.$$

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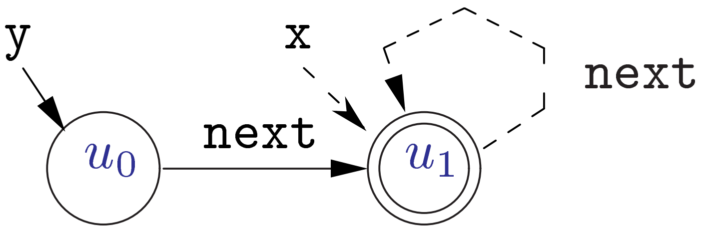

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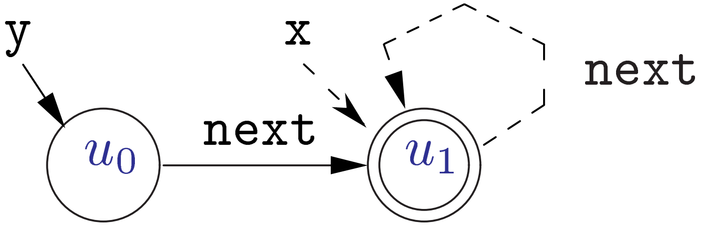

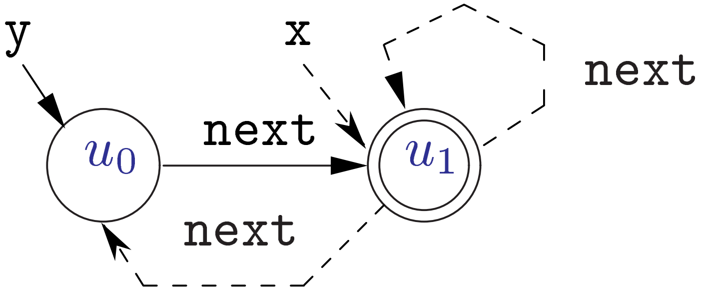

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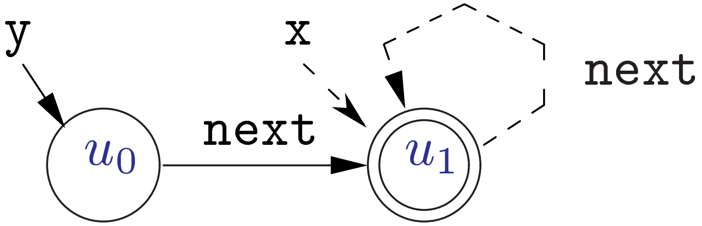

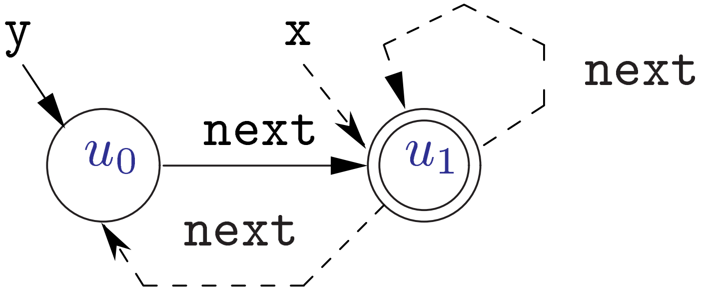

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The general transfer function is:

$$T[\texttt{x.next = y}] : next'(v_1, v_2) = next(v_1, v_2) \wedge (sm(v_1) \vee \neg x(v_1)) \vee (x(v_1) \wedge y(v_2)); \text{ all other predicates } p' = p.$$

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In the former case, the join operation is the pointwise three-valued $\wedge$ which weakens solid arcs into dotted ones when dissimilar graphs are joined.

In the latter case, a bounded set union *widening* operation (Cousot & Cousot 1999) is employed so that no infinite set of graphs is constructed.

Independent attribute analysis example

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y = Null;
while (x != Null) do
  t = y;
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For the program above, we wish to verify/refute:

> "If the input to the program above is an acyclic list without any aliasing, pointed to by `x`, then the list pointed to by `y` upon program termination is also acyclic and without any aliasing."

(Forward) data-flow analysis of shape

We now perform (*** in class ***) a forward data-flow analysis for the program

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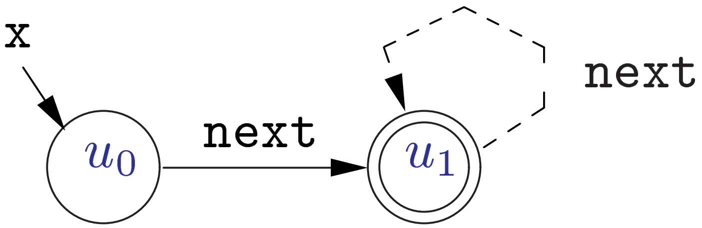

Validating/refuting direction relationships

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$$\mathtt{D}(\mathtt{x}, \mathtt{y}) \stackrel{\text{def}}{=} \mathtt{EF}_{\mathtt{next}}(\mathtt{x} \wedge \mathtt{EF}_{\mathtt{next}}\mathtt{y})$$

— an object, s, has atomic property `x` iff `x` points to s. Recall that $\mathtt{EF}_a\phi$ states, "there exists a path of a-labeled transitions such that, eventually, ϕ holds true."

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To validate that there is such a path from s, we check if $s \models^{\text{nec}} \mathtt{D}(\mathtt{x},\mathtt{y})$; to refute the existence of such a path, we check $s \models^{\text{nec}} \neg\mathtt{D}(\mathtt{x},\mathtt{y})$.

Validating/refuting interference relationships

The *interference relationship* (Ghiya & Hendren 1996) says that pointers `x` and `y` have access paths to a common heap node. It is encoded in the *inverse* transition system of $\rightarrow$:

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Similarly, we may want to try to validate that `x` and `y` definitely have access paths to a common heap node, s, by checking $s \models^{\text{nec}} \mathtt{I}(\mathtt{x},\mathtt{y})$.

Validating/refuting aliasing of pointers

Aliasing of pointers can be expressed: For

$$\mathtt{aliasing} \stackrel{\mathrm{def}}{=} \mathtt{EF}_{\mathtt{next}}(\bigvee_{\mathtt{x} \neq \mathtt{y}} \mathtt{x} \wedge \mathtt{y}),$$

the formulas

(a) $\mathtt{AG}_{\mathtt{next}} \neg \mathtt{aliasing}$,

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can then be used to check: (a) the absense of any kind of aliasing; (b) that `x` has no alias; and that (c) `x` and `y` never point to the same heap node. (Recall that $\texttt{AG}_a\phi$ states, "for all a-paths, it is globally true for all states along the path, that ϕ holds.")

Checking for cyclic data structures

We can check for possibly cyclic data structures. The predicate

$$\mathtt{cyclic} \stackrel{\mathrm{def}}{=} \bigvee_{\mathtt{x}\in\mathtt{AP}} \mathtt{x} \wedge \mathtt{EX}_{\mathtt{next}}\mathtt{EF}_{\mathtt{next}}\mathtt{x}$$

states that a heap node is pointed to by some `x` that has an access path to, presumably the same, heap node pointed to by `x`. (Recall that $\mathtt{EX}_a\phi$ says, "there exists an a-transition to a next state where ϕ holds.")

Recasting remaining parts of three-valued shape analysis

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- their embedding theorem expresses *soundness* of the $\models^{\text{nec}}$ semantics for refinement;
- their `focus` operation, `instrumentation predicates`, and `coerce` operation have connections to assume-guarantee reasoning and generalized model checking (Bruns & Godefroid 2000).

5 Instrumented model checking

Overview:

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Model checking partial Kripke structures

- A partial Kripke structure $\mathcal{M} = (\Sigma_M, \rightarrow, L^{must}, L^{may})$ determines two ordinary Kripke structures with the same state set and transition relation;
  - a "pessimistic" one, $\mathcal{M}_{\mathrm{p}} \stackrel{\text{def}}{=} (\Sigma_M, \rightarrow, L^{must})$, and
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- If ϕ does not contain any negations (or implications), we have
  - $(\mathcal{M}, s) \models^{\text{nec}} \phi$ if $(\mathcal{M}_{\mathrm{p}}, s) \models \phi$;
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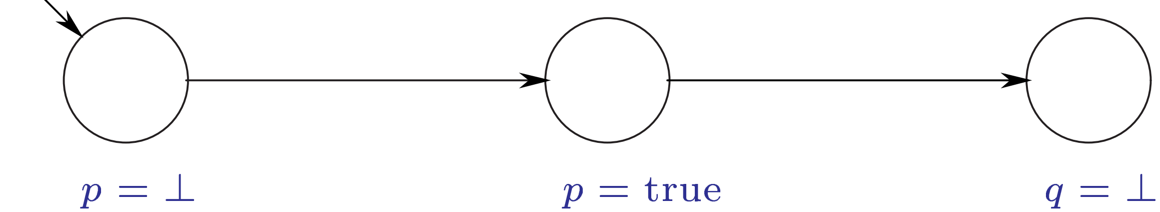
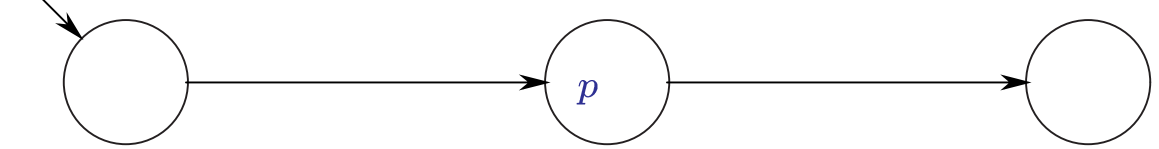
- Finally, (i) translate *any* ϕ into an equivalent negation normal form $T(\phi)$, and (ii) apply previous item to $T(\phi)$.

Example

partial Kripke structure



pessimistic Kripke structure



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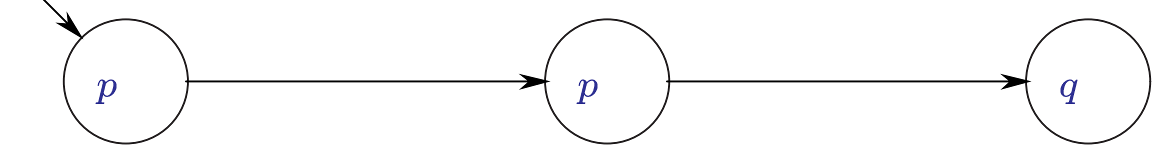

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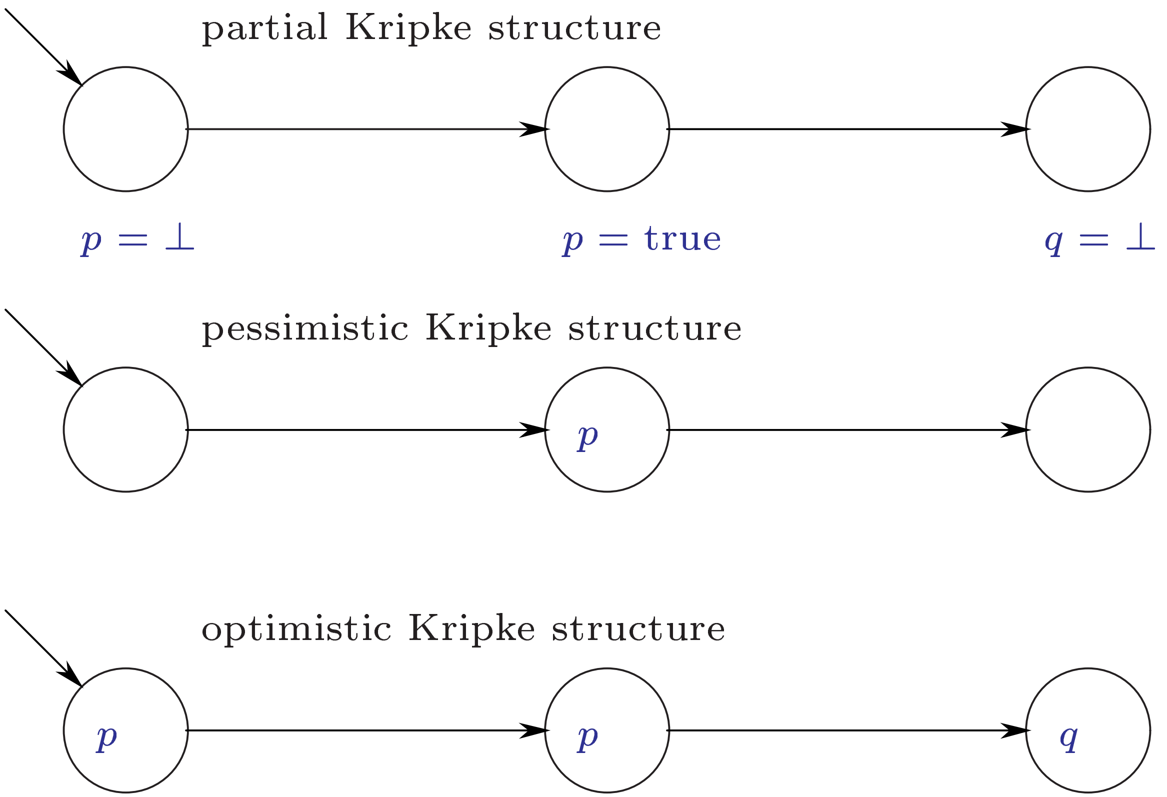

We don't have $\mathcal{M} \models^{\text{nec}} \mathtt{A}[p\,\mathtt{U}\,q]$ as we don't have $\mathcal{M}_{\text{p}} \models \mathtt{A}[p\,\mathtt{U}\,q]$.

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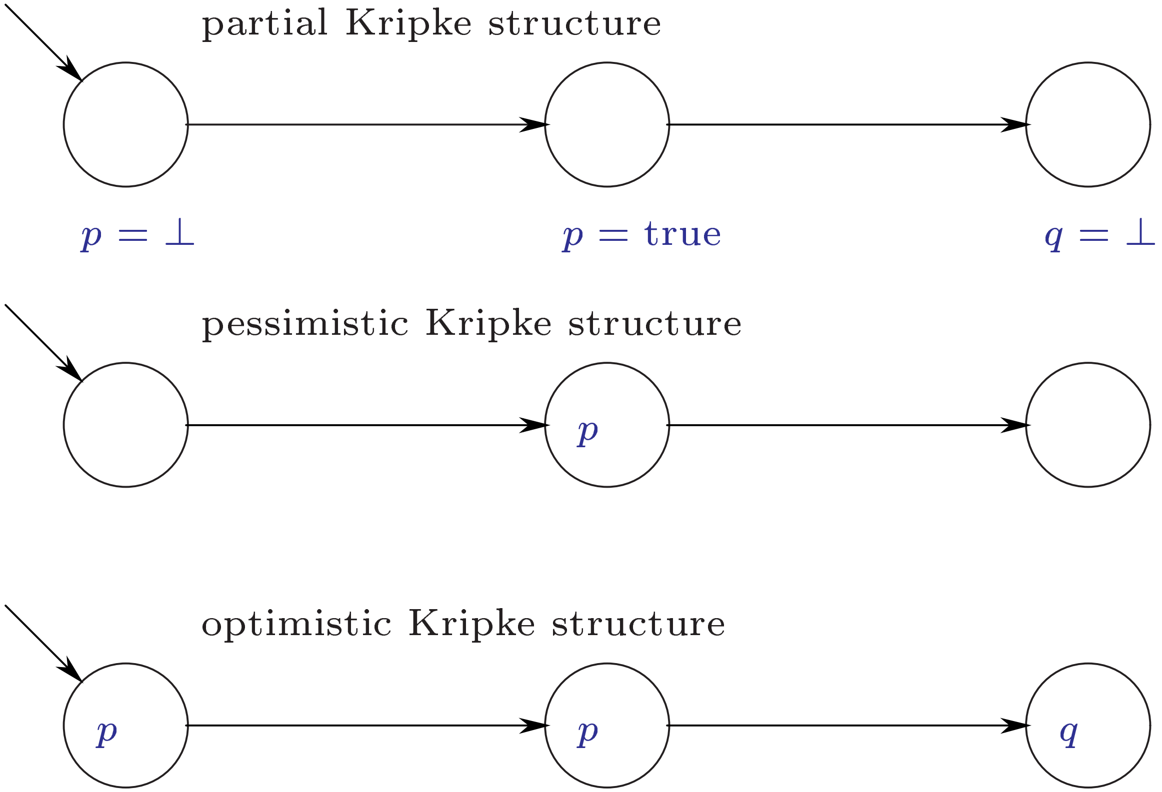

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2. Translate ϕ into a negation normal form $T(\phi)$ over a complement-closed set of actions: $\beta \in \mathtt{Act} \cup \{\bar{\alpha} \mid \alpha \in \mathtt{Act}\}$.

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2. Translate ϕ into a negation normal form $T(\phi)$ over a complement-closed set of actions: $\beta \in \texttt{Act} \cup \{\bar{\alpha} \mid \alpha \in \texttt{Act}\}$.

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3. Extend the may-component and must-component of the given modal transition system to make them complement-closed.

4. Model check $T(\phi)$ on the extended must-component for $\models^{\mathrm{nec}}$ and on the extended may-component for $\models^{\mathrm{pos}}$.

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$$\phi ::= \top \mid Z \mid \neg\phi \mid \phi \wedge \phi \mid \langle\alpha\rangle\,\phi \mid \mu Z.\phi$$

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The translation $\phi \mapsto T(\phi)$ should preserve important fragments of that logic, such as CTL and LTL.

Step 2: part one

Given ϕ generated by

$$\phi ::= \top \mid Z \mid \neg\phi \mid \phi \wedge \phi \mid \langle\alpha\rangle\, \phi \mid \mu Z.\phi$$

the negation normal form $T(\phi)$ is defined over

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where β ranges over $\mathtt{Act} \cup \{\bar{\alpha} \mid \alpha \in \mathtt{Act}\}$.

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The translation T leaves $\top$, Z, and $\neg Z$ unchanged. Further,
$T(\neg\top) \stackrel{\text{def}}{=} \bot$, $T(\neg\neg\phi) \stackrel{\text{def}}{=} T(\phi)$, $T(\neg(\phi_1 \wedge \phi_2)) \stackrel{\text{def}}{=} T(\neg\phi_1) \vee T(\neg\phi_2)$,
$T(\neg\langle\alpha\rangle\,\phi) \stackrel{\text{def}}{=} [\alpha]T(\neg\phi)$, $T(\neg\mu Z.\phi) \stackrel{\text{def}}{=} \nu Z.T(\neg\phi)$,
$T(\phi_1 \wedge \phi_2) \stackrel{\text{def}}{=} T(\phi_1) \wedge T(\phi_2)$, $T(\langle\alpha\rangle\,\phi) \stackrel{\text{def}}{=} \langle\alpha\rangle\,T(\phi)$, and
$T(\mu Z.\phi) \stackrel{\text{def}}{=} \mu Z.T(\phi)$.

Step 2: part two

We defined $\phi \mapsto T(\phi)$ by structural induction on ϕ. On the resulting formula, we transform all matching subformulas according to

$$T([\alpha]\top) \stackrel{\text{def}}{=} \top \text{ and } T([\alpha]\bot) \stackrel{\text{def}}{=} \langle\overline{\alpha}\rangle\,\top$$

and denote the result by $T(\phi)$ again.

- Note that $[\alpha]\top$ (all α-next states satisfy "true") and $\top$ ("true") are semantically equivalent.
- Note that $s \models^{\text{nec}} [\alpha]\bot$ (all possible α-next states satisfy "false") holds iff there are no α-may-transitions with s as source.
- Consequence: all minimal existential subformulas of $T(\phi)$ are thus of the form $\langle\alpha\rangle\,\phi$ or $\langle\overline{\alpha}\rangle\,\phi$, where ϕ is $\top$ or some Z.

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A modal transition system $\mathcal{A}$ consists of

- a must-component $\mathcal{A}^{\mathrm{must}} = (\Sigma_A, \mathtt{Act}, \rightarrow_{\mathrm{must}})$ and
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We define $\overline{\mathcal{A}} = (\overline{\mathcal{A}}^{\mathrm{must}}, \overline{\mathcal{A}}^{\mathrm{may}})$ with set of states

$$\Sigma_{\overline{A}} \stackrel{\mathrm{def}}{=} \Sigma_A \cup \{s_\perp\} \qquad (s_\perp \notin \Sigma_A)$$

and set of action labels $\mathtt{Act} \cup \{\overline{\alpha} \mid \alpha \in \mathtt{Act}\}$. This MTS retains all $\mathtt{may}$-transitions and $\mathtt{must}$-transitions of $\mathcal{A}$, and *adds* new transitions:

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and set of action labels $\mathtt{Act} \cup \{\overline{\alpha} \mid \alpha \in \mathtt{Act}\}$. This MTS retains all `may`-transitions and `must`-transitions of $\mathcal{A}$, and *adds* new transitions:

$$\begin{array}{rcll} (s, \overline{\alpha}, s_\perp) \in \ \rightarrow_{\mathrm{must}} & \text{if} & \text{for all } s' \in \Sigma_A,\ (s, \alpha, s') \notin \ \rightarrow_{\mathrm{may}} & (1) \\ (s, \overline{\alpha}, s') \in \ \rightarrow_{\mathrm{may}} & \text{if} & (s, \alpha, s') \in \ \rightarrow_{\mathrm{may}} \setminus \rightarrow_{\mathrm{must}} . & (2) \end{array}$$

Computing the extended must-component

Computing the component $\overline{\mathcal{A}}^{\text{must}}$, we have to

- prove/compute the must-transitions of $\mathcal{A}^{must}$, and
- prove the *absence* of may-transitions in $\mathcal{A}^{may}$ to establish the must-transitions into $s_\perp$:

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We can decide $(\mathcal{A}, s) \models^{\text{nec}} \phi$ by model checking $T(\phi)$ on the pointed *labeled transition system* $(\overline{\mathcal{A}}^{\text{must}}, s)$.

Step 4

Theorem 4.2.1 (Correctness of model checking algorithm) *Given a MTS $\mathcal{A}$ and a (closed) modal mu-calculus formula ϕ, let $T(\phi)$ and $\overline{\mathcal{A}}$ be the formula and MTS (respectively) as defined above. For any state $s \in \Sigma_A$, we then have*

1. $(\mathcal{A}, s) \models^{\mathrm{nec}} \phi$ *iff* $(\overline{\mathcal{A}}^{\mathrm{must}}, s) \models T(\phi)$;
2. $(\mathcal{A}, s) \models^{\mathrm{pos}} \phi$ *iff* $(\overline{\mathcal{A}}^{\mathrm{may}}, s) \models T(\phi)$; *and*
3. *not* $(\mathcal{A}, s) \models^{\mathrm{pos}} \phi$ *iff* $(\overline{\mathcal{A}}^{\mathrm{may}}, s) \not\models T(\phi)$.

Note that the righthand sides can be decided by conventional model checks of pointed labeled transition systems.

Losing precision

(Bruns & Godefroid 2000) Ideally, we want $s \models^{\mathrm{nec}} \phi$ iff for all refinements s' of s in *concrete/complete* systems, the conventional $s' \models \phi$ holds.

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- We don't have $s \models^{\text{nec}} \phi$, because we don't have $s \models^{\text{nec}} p \rightarrow p$: we cannot refute p (i.e. not $s \models^{\text{nec}} \neg p$) and we cannot verify p (i.e. not $s \models^{\text{nec}} p$).

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Our semantics for $\models^{\text{nec}}$ only loses precision for disjunctions ($\vee$).
Our semantics for $\models^{\text{pos}}$ only loses precision for conjunctions ($\wedge$).

Generalized model checking

Definition 4.2.2 [Generalized model checking problem (Bruns & Godefroid 2000)] Given a state s of a partial Kripke structure $\mathcal{M}$ and some ϕ, is there a state s' of a *Kripke* structure $\mathcal{M}'$ such that (i) $(\mathcal{M}', s')$ refines $(\mathcal{M}, s)$ and (ii) $(\mathcal{M}', s') \models \phi$?

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We write

$$\mathrm{GMC}(\mathcal{M}, s, \phi)$$

for the decision problem above. E.g. $\mathrm{GMC}(\mathcal{M}, s, \phi) = \mathrm{true}$ means that there is a pointed Kripke structure $(\mathcal{M}', s')$ which refines $(\mathcal{M}, s)$ such that $(\mathcal{M}', s') \models \phi$.

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Completeness means: not $\mathrm{GMC}(\mathcal{M}, s, \neg\phi)$ implies $(\mathcal{M}, s) \models^{\mathrm{nec}} \phi$.

Instrumented use of $\mathrm{GMC}(\mathcal{M}, s, \phi)$

We can compute $\models^{\mathrm{nec}}$ *without loss of precision* by computing one instance of the generalized model checking problem:

$$s \models^{\mathrm{nec}} \phi \qquad \stackrel{\mathrm{def}}{=} \qquad \text{not } \mathrm{GMC}(\mathcal{M}, s, \neg\phi).$$

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Dually, we can compute $\models^{\mathrm{pos}}$ without loss of precision by setting:

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Generalized model checking may be defined similarly over other classes of partial models, e.g. modal transition systems.

Alternatively, there are linear translations of MTSs into partial Kripke structures that preserve refinement.

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- Thus, ϕ is satisfiable iff $\mathrm{GMC}(\mathcal{U}, s_U, \phi)$ holds.
- Consequence: worst-case complexity of GMC is never less than the one for satisfiability.

Model checking and GMC

Consider pointed Kripke structures $(\mathcal{M}_1, s_1)$ and $(\mathcal{M}_2, s_2)$.

- Then these models are essentially *complete/concrete*, pointed partial Kripke structure.

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  - $(\mathcal{M}_1, s_1) \models \phi$ iff $\mathrm{GMC}(\mathcal{M}_1, s_1, \phi)$.
- Consequence: Generalized model checking and model checking coincide (as problems) on complete structures; in that case, you are well advised to run the algorithms for model checking. :-))

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- Lesson: GMC has two extremal points: satisfiability (universal model $\mathcal{U}$) and model checking (Kripke structures).

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- Complexity: For LTL and CTL, the generalized model checking problem is EXPTIME-complete.
- Consequence: For CTL, satisfiability checking and generalized model checking are equally hard. For LTL, however, satisfiability checking is much simpler (PSPACE-complete) than generalized model checking!

6 Computing abstract modal transition systems

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- Relational specification of abstract modal transition systems.

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- Abstraction refinement for modal transition systems.

Relational specification of abstract MTSs

Definition 4.2.3 [Relational abstraction (Dams 1996)] Let $\mathcal{A}_1 = (\texttt{Act}, \mathcal{A}_1^{\mathrm{must}}, \mathcal{A}_1^{\mathrm{may}})$ be a (abstract or concrete) MTS. Given a set of (designated) abstract states, Σ_{A_2}, and a total relation $\rho\colon \Sigma_{A_1} \times \Sigma_{A_2}$, we define $\mathcal{A}_2 = (\Sigma_{A_2}, \texttt{Act}, \rightarrow_{\mathrm{must}}, \rightarrow_{\mathrm{may}})$:

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- $a_2 \rightarrow^{\alpha}_{\mathrm{must}} a_2'$ iff for all $a_1 \in \Sigma_{A_1}$ with $a_1 \,\rho\, a_2$ there exists $a_1' \in \Sigma_{A_1}$ such that $a_1' \,\rho\, a_2'$ and $a_1 \rightarrow^{\alpha}_{\mathrm{must}} a_1'$;

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Lemma 4.2.4 *Given $\mathcal{A}_1$ and $\mathcal{A}_2$ as above, $\mathcal{A}_2$ is a MTS and ρ is a refinement.*

Proof: Instructive Exercise. ;-))

Example

Let $\mathcal{A}_1$ be a complete MTS with set of stores for three integer variables `x`, `y`, and `z` as (infinite) state space. The assignment statement `x = z` induces all state transitions, e.g. one from $c = \{\mathtt{x} \mapsto i, \mathtt{y} \mapsto j, \mathtt{z} \mapsto k\}$ to $c' = \{\mathtt{x} \mapsto k, \mathtt{y} \mapsto j, \mathtt{z} \mapsto k\}$.

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The predicates $\phi_1 \stackrel{\text{def}}{=} \mathtt{odd}(\mathtt{x})$, $\phi_2 \stackrel{\text{def}}{=} (\mathtt{y} > \mathtt{0})$, and $\phi_3 \stackrel{\text{def}}{=} (\mathtt{z} < \mathtt{0})$ induce an equivalence relation on Σ_{A_1}. Let Σ_{A_2} be the set of all *equivalence-closed subsets* of the states of $\mathcal{A}_1$.

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- there is *no* `must`-transition from $\phi_1 \wedge \phi_2 \wedge \phi_3$ to any monomial, e.g. $\phi_1 \wedge \phi_2 \wedge \phi_3$ or $\neg\phi_1 \wedge \phi_2 \wedge \phi_3$.

Pre and post

Notation. For any predicate η on a set of states Σ_S, the post operator and weakest precondition are defined as

$$\begin{aligned}
\texttt{post}(\eta) &\stackrel{\text{def}}{=} \{s' \in \Sigma_S \mid \exists s \in \Sigma_S : s \models \eta, s \to s'\} \\
\tilde{\texttt{pre}}(\eta) &\stackrel{\text{def}}{=} \{s \in \Sigma_S \mid \forall s' \in \Sigma_S : s \to s' \text{ implies } s' \models \eta\}.
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Needed property. For any predicates η, ψ on states, satisfiability witnesses s' for $\texttt{post}(\psi) \wedge \eta$ correspond to satisfiability witnesses s for $\psi \wedge \neg\tilde{\texttt{pre}}(\neg\eta)$ via $s \to s'$ (even in the presence of deadlock!).

Proof: Instructive Exercise. ;-))

Methodological assumptions

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- Thus, we may focus on the abstraction of a *single* program statement, *allowing us to consider MTSs without explicit action labels.*
- For such program statements and a quantifier-free formula η, we assume that $\tilde{\texttt{pre}}(\eta)$ is *quantifier-free* as well.
- The latter is the case for usual programming language constructs and enables the use of decision procedures as realized in tools such as SVC or PVS.

Boolean abstraction: specification

Let $\mathcal{S}$ be a LTS. Given a finite set of quantifier-free formulas of first-order logic,

$$\Phi = \{\phi_1, \phi_2, \ldots, \phi_n\},$$

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$$s\, \rho_{\mathrm{b}}^{\Phi}\, b \qquad \text{iff} \qquad s \models \langle b, \Phi\rangle.$$

We make ρ_{b}^{Φ} total by setting $\Sigma_{B_\Phi} \stackrel{\text{def}}{=} \{b \in \{0,1\}^n \mid \exists s \in \Sigma_S \colon s\, \rho_{\mathrm{b}}^{\Phi}\, b\}$.

Boolean abstraction: implementation of may

The set of `may`-successors of a state set ψ is

$$\mathtt{next}(\psi)^{\mathrm{may}}_{\mathrm{b}} \stackrel{\mathrm{def}}{=} \{b' \in \Sigma_B \mid \psi \wedge \neg\tilde{\mathtt{pre}}(\neg\langle b', \Phi\rangle) \text{ is satisfiable } \}.$$

Recall

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If ψ is a boolean combination of $\{\phi_i \mid m \leq i \leq n\}$, we compute $\mathtt{next}(\psi)^{\mathrm{may}}_{\mathrm{b}}$ as a BDD $H^{may}(\psi, \mathrm{true}, m)$:

$$H^{may}(\psi, \eta, i) \stackrel{\mathrm{def}}{=} \begin{cases} (x_i \wedge H^{may}(\psi, \eta \wedge \phi_i, i+1)) & \\ \vee(\neg x_i \wedge H^{may}(\psi, \eta \wedge \neg\phi_i, i+1) & \text{if } 0 < i \leq n, \\ 1 \qquad \text{if } i = n+1 \text{ and } \psi \wedge \neg\tilde{\mathtt{pre}}(\neg\eta) \text{ is satisfiable,} & \\ 0 \qquad \text{if } i = n+1 \text{ and } \psi \wedge \neg\tilde{\mathtt{pre}}(\neg\eta) \text{ is unsatisfiable.} & \end{cases}$$

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- If $H^{must}(\langle b, \Phi \rangle, \mathrm{true}, m)$ applied to b' renders 1, then $b \rightarrow_{\mathrm{must}} b'$ in $\mathcal{B}_\Phi$ (converse true if theorem prover has a good day :-)).
- The overall computation makes $O(2^n)$ calls to the theorem prover: the *same cost* as computing the may-component only!

Boolean/cartesian abstraction: motivation

Revisiting our example with $\phi_1 \stackrel{\text{def}}{=} \texttt{odd}(\texttt{x})$, $\phi_2 \stackrel{\text{def}}{=} (\texttt{y} > \texttt{0})$, $\phi_3 \stackrel{\text{def}}{=} (\texttt{z} < \texttt{0})$ and the assignment `x = z`;

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Calculating `must`-transitions whose targets can be abstract states of the above kind is generally too expensive: given n predicates, there are a possible 2^{2^n} such states. Therefore, we use cartesian abstraction (bka "independent attribute analysis").

Boolean/cartesian abstraction: specification

Given $\mathcal{S}$ and Φ as before, we specify a finite-state abstract MTS $\mathcal{C}_\Phi$ ($\mathcal{A}_1$ is $\mathcal{S}$ and $\mathcal{A}_2$ is $\mathcal{C}_\Phi$):

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Boolean/cartesian abstraction: implementation of may

Now, we use a *ternary decision diagram* (TDD):

$$G^{may}(\psi,\eta,i) \stackrel{\text{def}}{=} \begin{cases} ([1/x_i] \wedge G^{may}(\psi,\eta\wedge\phi_i,i+1)) \vee ([0/x_i] \wedge G^{may}(\psi,\eta\wedge\neg\phi_i,i+1)) \\ \vee([\star/x_i] \wedge G^{may}(\psi,\eta,i+1)) \qquad \text{if } 0 < i \leq n, \\ 1 \qquad \text{if } i = n+1 \text{ and } \psi \wedge \neg\tilde{\mathtt{pre}}(\neg\eta) \text{ is satisfiable,} \\ 0 \qquad \text{if } i = n+1 \text{ and } \psi \wedge \neg\tilde{\mathtt{pre}}(\neg\eta) \text{ is unsatisfiable.} \end{cases}$$

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$$G^{may}(\psi,\eta,i) \stackrel{\text{def}}{=} \begin{cases} ([1/x_i] \wedge G^{may}(\psi,\eta \wedge \phi_i, i+1)) \vee ([0/x_i] \wedge G^{may}(\psi,\eta \wedge \neg\phi_i, i+1)) \\ \vee([\star/x_i] \wedge G^{may}(\psi,\eta,i+1)) \qquad \text{if } 0 < i \leq n, \\ 1 \qquad \text{if } i = n+1 \text{ and } \psi \wedge \neg\tilde{\texttt{pre}}(\neg\eta) \text{ is satisfiable,} \\ 0 \qquad \text{if } i = n+1 \text{ and } \psi \wedge \neg\tilde{\texttt{pre}}(\neg\eta) \text{ is unsatisfiable.} \end{cases}$$

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The function G^{may} essentially computes the abstract post-operator of SLAM. The TDD $G^{may}(\psi, \mathrm{true}, m)$ represents the set of `may`-successors of ψ:

$$\mathtt{next}(\psi)_{\mathrm{c}}^{\mathrm{may}} \stackrel{\text{def}}{=} \{c' \in \Sigma_C \mid \psi \wedge \neg \tilde{\mathtt{pre}}(\neg\langle c', \Phi\rangle) \text{ is satisfiable } \}.$$

Boolean/cartesian abstraction: implementation of must

The TDD $G^{must}(\psi, \text{true}, m)$ represents the set of `must`-successors of ψ:

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Concretely,

$$\texttt{next}(\psi)_{\mathrm{c}}^{\mathrm{must}} \stackrel{\text{def}}{=} \{c' \in \Sigma_C \mid \psi \wedge \tilde{\texttt{pre}}(\neg\langle c', \Phi\rangle) \text{ is unsatisfiable } \}.$$

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- If $G^{must}(\langle c, \Phi\rangle, \mathrm{true}, m)$ applied to c' renders 1, then $c \rightarrow_{\mathrm{must}} c'$ in $\mathcal{C}_\Phi$ (converse true if theorem prover has a good day :-)).
- The overall computation makes $O(3^n)$ calls to the theorem prover: the *same cost* as computing the may-component only!

Abstraction refinement

- In the boolean case, abstraction refinement works for the may-component only! Recall our example of adding ϕ_1 to $\{\phi_2, \phi_3\}$.
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Theorem 4.2.5 (Incremental refinement) *For $\Phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$ and $\Psi = \Phi \cup \{\phi_{n+1}, \phi_{n+2}, \ldots, \phi_{n+m}\}$, the MTS $\mathcal{C}_\Psi$ is a refinement of the MTS $\mathcal{C}_\Phi$.*

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Proof. The refinement $\rho \subseteq \Sigma_{C_\Psi} \times \Sigma_{C_\Phi}$ is given by $\{(c', c) \mid c$ is a prefix of $c'\}$. $\square$

7 Assume-guarantee model checking

*** to be completed ***

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4. await novel applications, e.g., compositional pointer analysis (Whaley & Rinard 1999).