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Hintikka Games for PCTL on Labeled Markov Chains

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Agenda

Hintikka Games

PCTL on Discrete Time Markov Chains

Game Semantics

Discussion

Hintikka Games

- Tarski's denotational semantics:
  truth as a predicate
- Hintikka's operational semantics:
  truth as winning strategy in two-player game
- Winning strategies as debugging information:
  witness truth or falsehood

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Hintikka Game for First Order Logic: an example play

	$\forall n \in \mathbb{N} \quad geq0(n) \wedge (odd(n) \vee even(n))$
Refuter chooses $4 \in \mathbb{N}$	$geq0(4) \wedge (odd(4) \vee even(4))$
Refuter chooses conjunct	$odd(4) \vee even(4)$
Verifier chooses disjunct	$even(4)$

Verifier wins iff $even(4)$ is true

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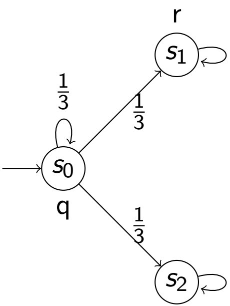
PCTL on Discrete Time Markov Chains

- Labeled Discrete Time Markov Chain (S, P)
  States $s \in S$
  Probability matrix $P \colon S \times S \longrightarrow [0, 1]$
  Labeling function $L \colon S \longrightarrow 2^{\mathrm{AP}}$



PCTL on Discrete Time Markov Chains

- Probabilistic Computational Tree Logic
  $\phi ::= \mathsf{q} \mid \neg\phi \mid \phi \vee \phi \mid [\alpha]_{\bowtie p}$
  $\alpha ::= \mathsf{X}\phi \mid \phi\mathsf{U}\phi \mid \phi\mathsf{W}\phi$
  $\mathsf{q} \in \mathsf{AP}$, $p \in [0,1]$, $\bowtie \in \{>, \geq\}$
- α defines path formulae, ϕ defines state formulae
- X, U, and W denote the "Next", "strong Until", "weak Until" path modalities state formulae
- Examples: $[\text{running U terminating}]_{\geq 0}$

Game Semantics

- Two players: Verifier (V) and Refuter (R)
- Configurations: $\langle s, \phi, C \rangle$
  - s a state
  - ϕ the current PCTL formula
  - C is Verifier or Refuter
- Game moves for player C regulated by rules, saying which configurations are reachable from a given configuration that C controls

Game Semantics

- Plays are finite or infinite sequences of configurations, resulting from game moves
- A strategy for player C is, informally, a specification of how to move from a configuration that C controls to another configuration
- A strategy for player C is winning for configuration $\langle s, \phi, C \rangle$: all plays from that configuration are won by player C if played according to that strategy

Game Moves

- As for First Order Logic for propositional logic connectives
- X is straight forward
- U and W are dual to each other

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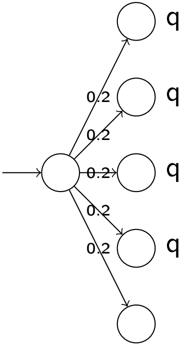

$[Xq]_{\geq 0.4}$

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Strong Until $\langle s, [\mathrm{qUr}]_{\geq p}, C \rangle$

- If $\bowtie = \geq$ then $!C$ chooses n and the threshold is changed to $> p - \frac{1}{n}$
- Finite approximation by unfolding

Strong Until $\langle s, [\mathrm{qUr}]_{>p}, C\rangle$

- C can claim r
- $!C$ can challenge q
- Else C chooses a sub-distribution $d(s')$ such that

$$\sum_{s' \in S} d(s') > p \qquad \text{and} \qquad d(s') \leq \mathbf{P}(s, s')$$

- $!C$ chooses s' with $d(s') > 0$
- Next configuration $\langle s', [\mathrm{qUr}]_{>d(s')\mathbf{P}(s,s')^{-1}}, C\rangle$

Winning conditions for plays

Finite plays ...

- ...that end in $\langle s, q, C \rangle$ are won by C
- ...that end in $\langle s, \neg q, C \rangle$ are won by $!C$

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- ... that end in $\langle s, \texttt{true}, C\rangle$ are won by C, where $\texttt{true}$ is any $[\alpha]_{\geq 0}$
- ... that end in $\langle s, \texttt{false}, C\rangle$ are won by $!C$, where $\texttt{false}$ is any $[\alpha]_{>1}$

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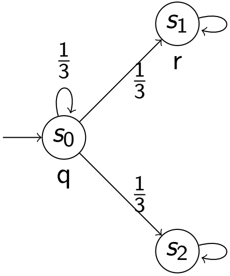
Infinite plays ...

- ... have a suffix of $\langle s_i, [\phi W \psi]_{\geq p_i}, C \rangle$ or $\langle s_i, [\phi U \psi]_{>p_i}, C \rangle$ configurations.
  - Infinite plays on W are won by C
  - Infinite plays on U are won by $!C$

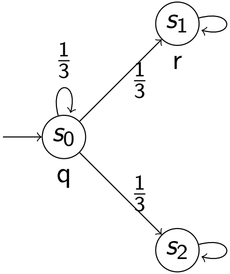
Example



$[qUr]_{\geq 0.5}$

Holds in the limit.
Winning would need infinite plays.

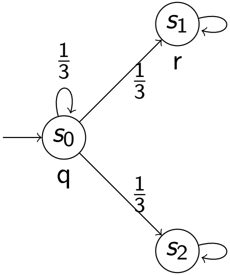
Example



$[qUr]_{>0.25}$

$d(s_0) = 0$
$d(s_1) = 0.26$
$d(s_2) = 0$

Example

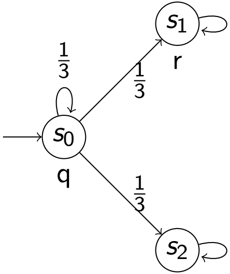


$[\mathsf{qUr}]_{>0.4}$

$d(s_0) = 0.067$
$d(s_1) = 0.3331$
$d(s_2) = 0$

$[\mathsf{qUr}]_{>0.201}$

Example



$[qUr]_{>0.6}$

Refuter stays in s_0 forever or until $d(s_2) > 0$

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Winning strategies that allow finite games

	$X_>$	$X_\geq$	$W_>$	$W_\geq$	$U_>$	$U_\geq$
Verifier	✓	✗ (✓)	✗	✗	✓	✗ (✓)
Refuter	✗	✗	✗ (✓)	✓	✗	✗

- ✗: infinite plays necessary
- ✓: finite plays enforceable
- (✓): infinity only because of ϵ-choice
- winning strategies as witnesses of finite abstractions
- Example: $\neg[aWb]_{\geq 0.7} \wedge [cUd]_{>0.5}$

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Summary

- We defined a Hintikka game which captures the semantics of PCTL on Discrete Time Markov Chain.
- For fragments of PCTL there exist winning strategies which guarantee finite plays, and therefore provide finite witnesses for the truth or falsehood of formulae.

Future Work

- Markov Decision Processes
- CSL on Continuous Time Markov Chains
- Finitary witnesses for all of PCTL