UNIVERSITY OF OXFORD

Verification and Refutation of Probabilistic Specifications via Games

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Games as abstractions [KNP06]

Currently:

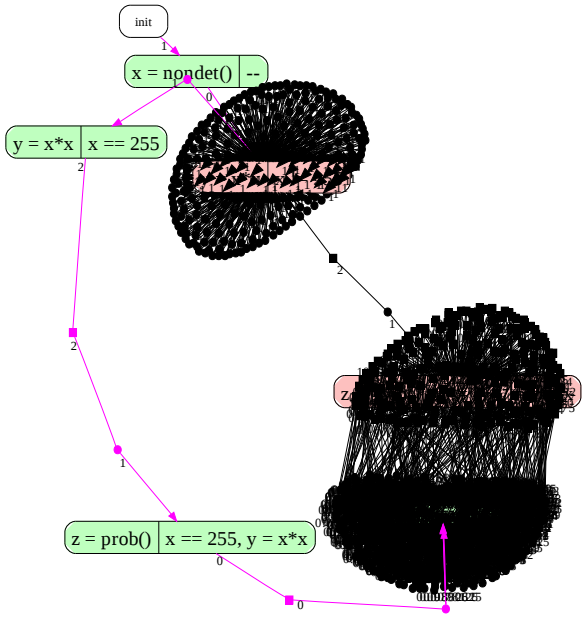
- game abstractions and abstraction refinement have yielded good results in verifying probabilistic systems such as: PRISM models [KKNP08], ANSI-C programs [KKNP09] and PTA [KNP09]
- for a fixed model & predicates [KNP09] only considers one game abstraction, which is typically expensive to construct/analyse

This paper:

- we develop a more fine-grained notion of abstraction for games via an abstraction relation over games
- this enables us to consider a "hierarchy of abstractions" with varying precisions (even for fixed predicates!)
- we develop a verification/refutation framework for arbitrary PCTL specifications (c.f. modal abstractions [Lar90])

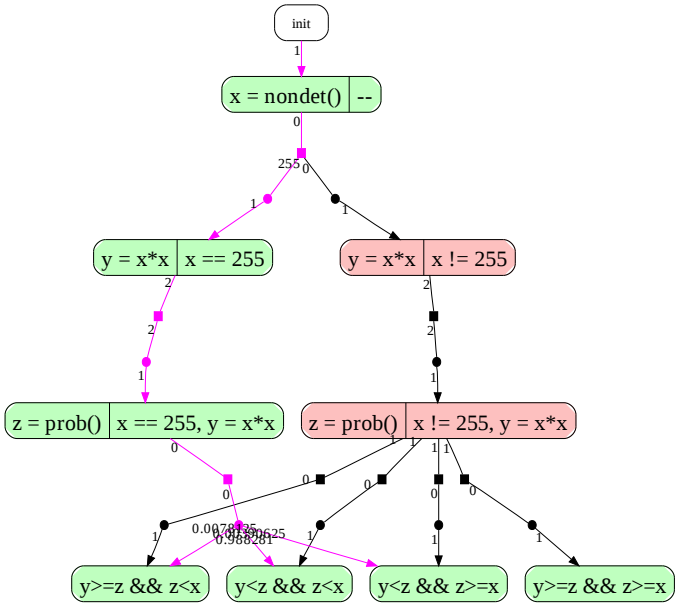
Games as abstractions (cont'd)

Currently:



– identifies one game abstraction

This paper:



– identifies many game abstractions

Overview (definitions follow later!)

Concrete:

- M are models (Markov Decision Processes)
- P are specifications (Probabilistic CTL, [HJ94])
- $\models$ is a satisfaction relation ([BdA95])

Abstract:

- G are abstract models (Games, [KNP06])
- *emb* : M $\rightarrow$ G is an embedding function (*this paper*)
- $\sqsubseteq_p \subseteq$ G $\times$ G is an abstraction relation (*this paper*)
- $\models^{\text{may}}, \models^{\text{must}} \subseteq$ G $\times$ P are PCTL semantics (*this paper*)

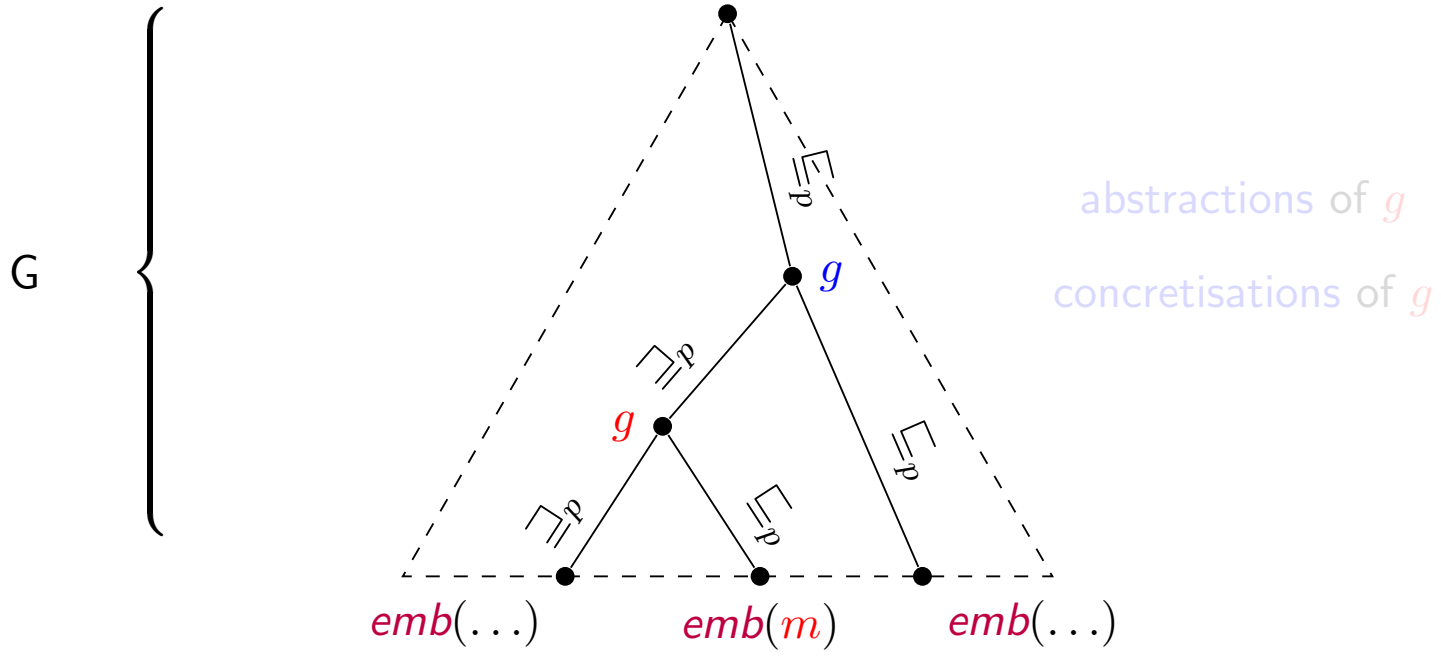
General idea:

- for $m \in$ M and $p \in$ P, verify or refute $m \models p$ via model checks on games $g \in$ G using properties of *emb*, $\sqsubseteq_p$ and $\models^{\text{may}}, \models^{\text{must}}$

Overview (definitions follow later!)

Intuition of emb, $\sqsubseteq_p$:

- $emb(m)$ is an exact representation of an MDP m in G
- $g \sqsubseteq_p g$ means g abstracts g
- $g \sqsubseteq_p emb(m)$ means the game g abstracts the MDP $emb(m)$



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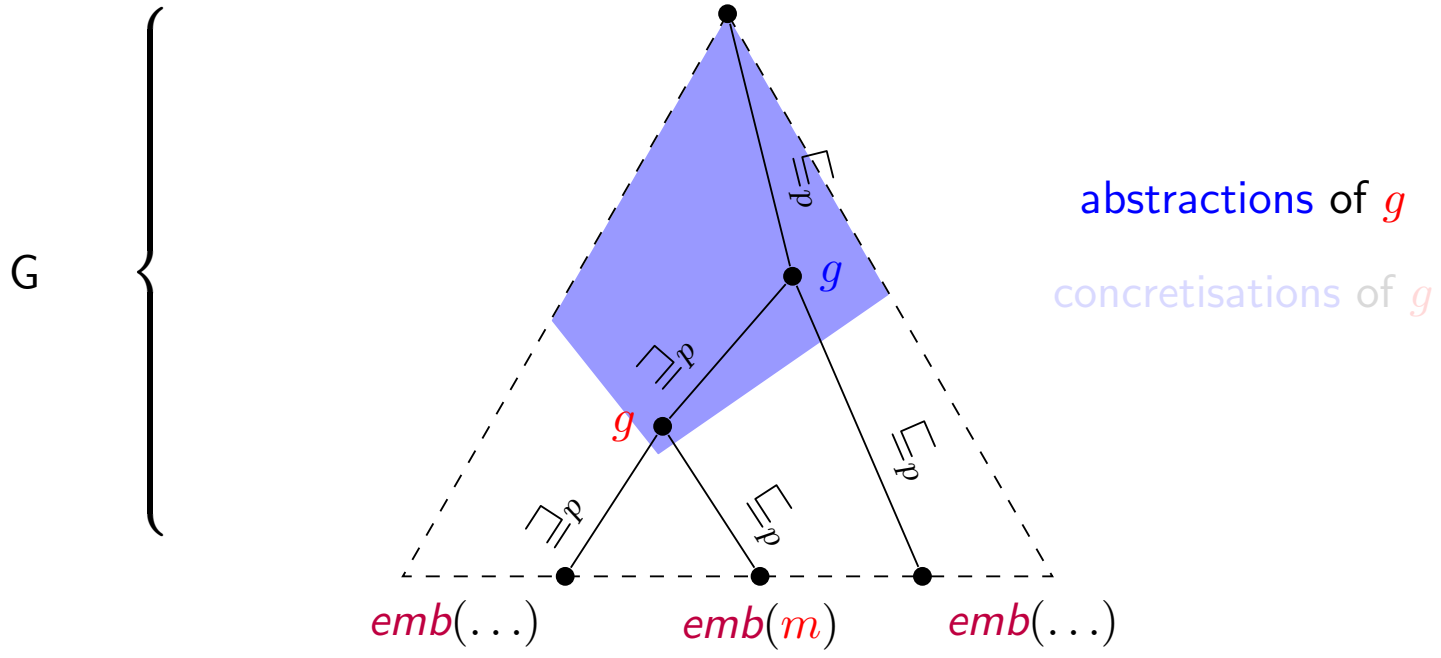

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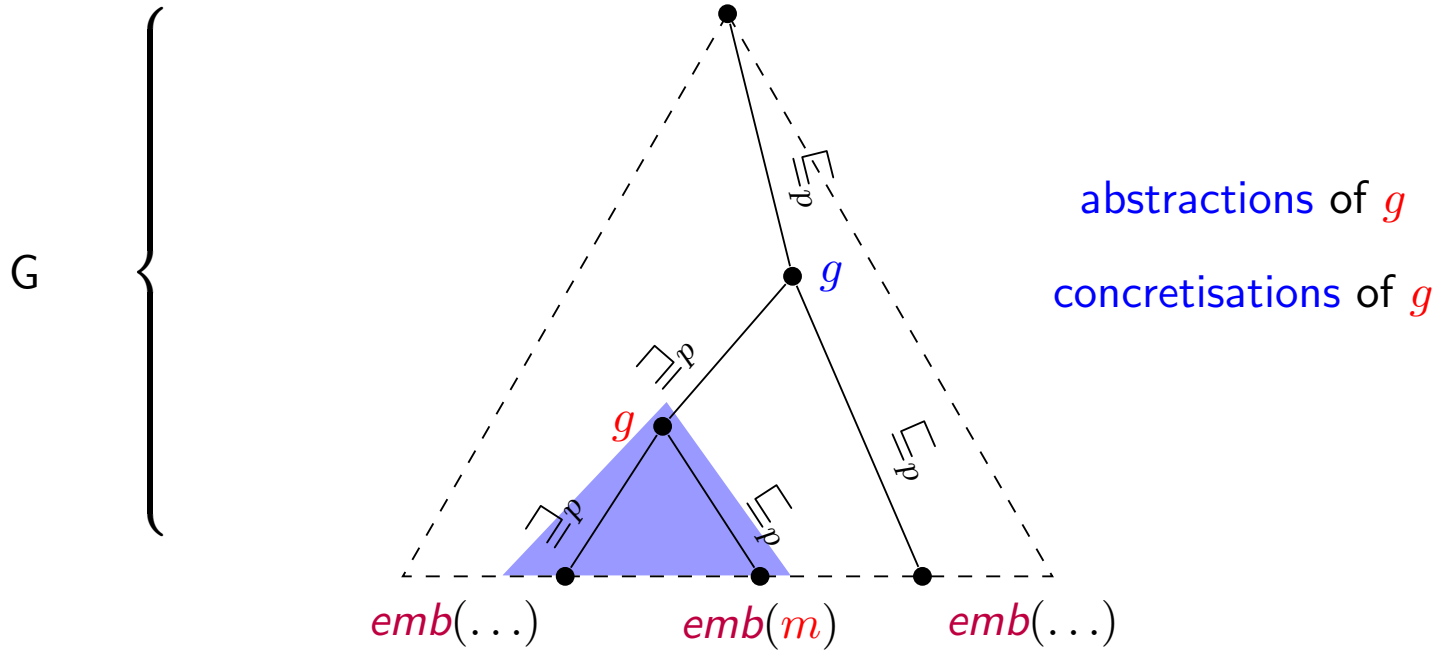

Overview (still no definitions!)

PCTL evaluations on games:

- $g \models^{\text{may}} p$ means concretisations of g possibly satisfies p
- $g \models^{\text{must}} p$ means concretisations of g definitely satisfies p

Soundness requirements:

- if $g \models^{\text{may}} p$ then all abstractions of g also may-satisfy p
- if $g \models^{\text{must}} p$ then all concretisations of g also must-satisfy p

Verification & refutation via games:

- to verify $m \models p$ find a game $g \in \mathsf{G}$ such that g abstracts m (i.e. $g \sqsubseteq_p \mathit{emb}(m)$) and g must-satisfies p (i.e. $g \models^{\text{must}} p$)
- to refute $m \models p$ find a game $g \in \mathsf{G}$ such that g abstracts m (i.e. $g \sqsubseteq_p \mathit{emb}(m)$) and g does not may-satisfy p (i.e. $g \not\models^{\text{may}} p$)

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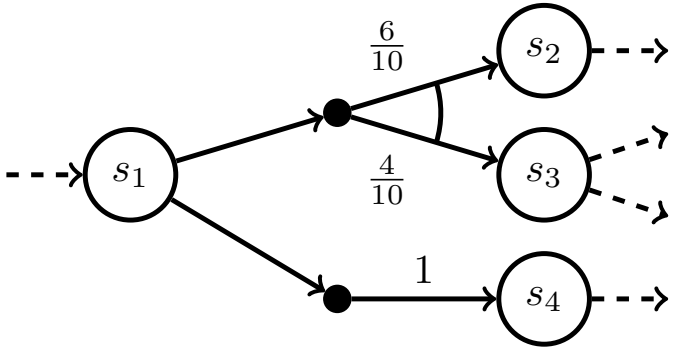
Models & specifications

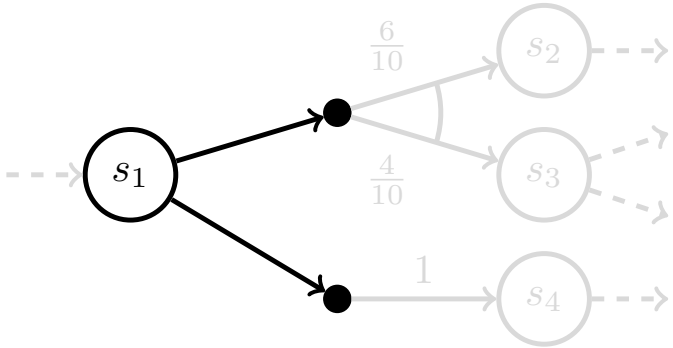
Markov decision processes



- non-deterministic choice
- probabilistic choice
- path
- strategy

Models & specifications

Markov decision processes



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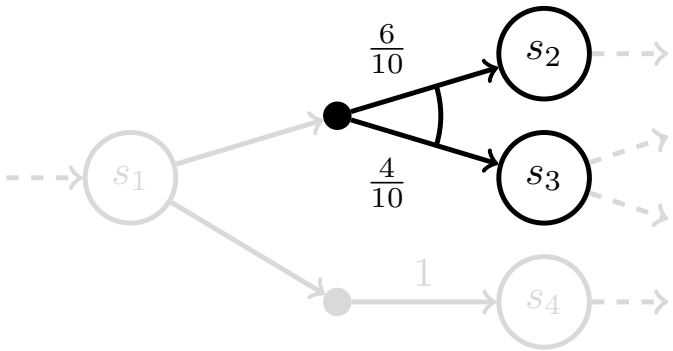

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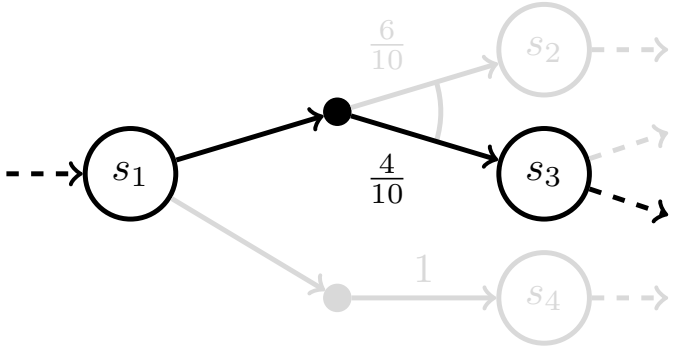

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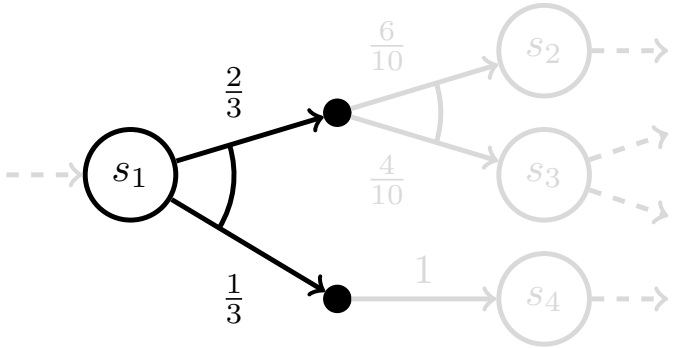

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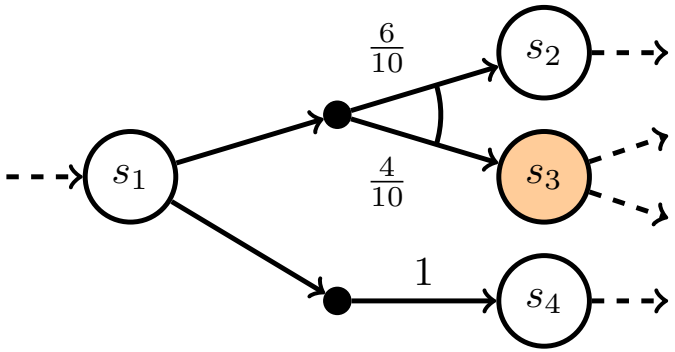

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PCTL formulas

- for all strategies the probability of reaching s_3 is ≤ 0.5
- for some strategy the probability of reaching s_3 is > 0.2

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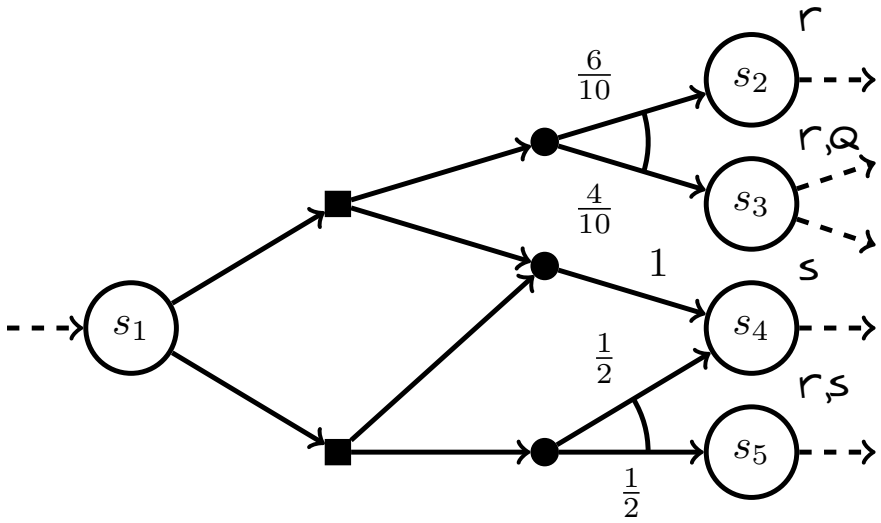
Abstraction: stochastic games (G)



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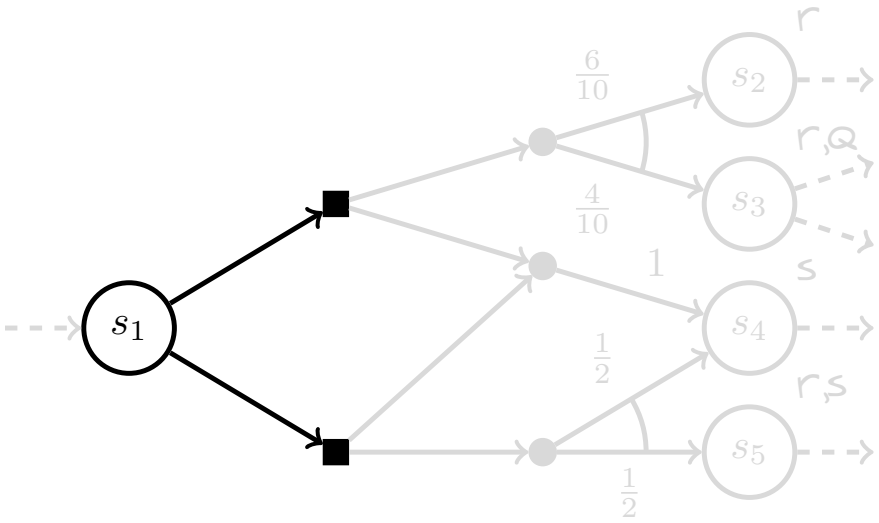

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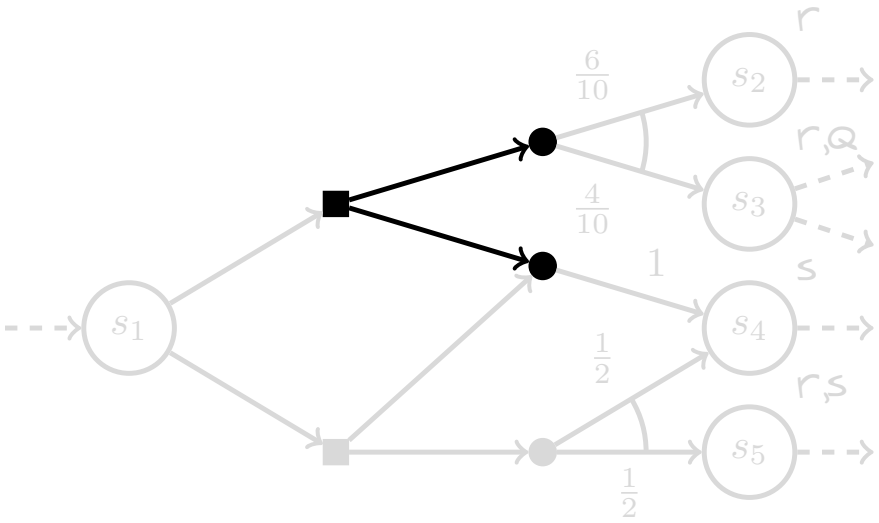

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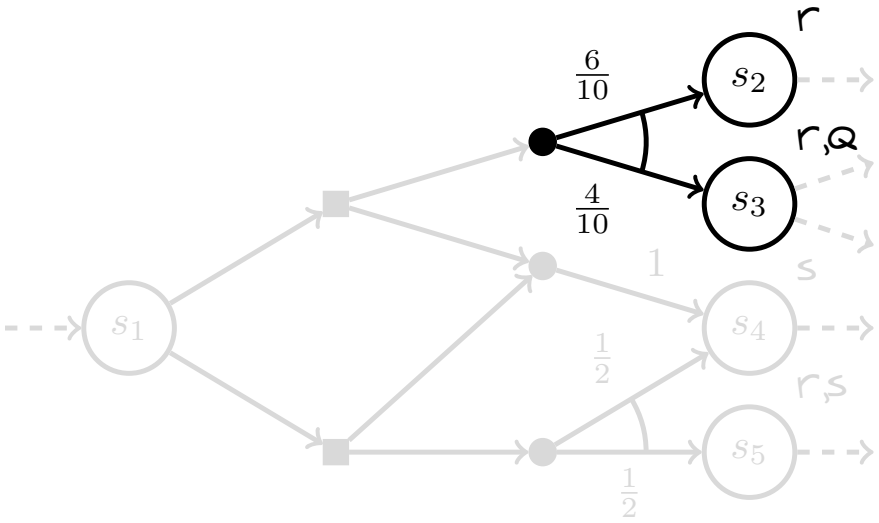

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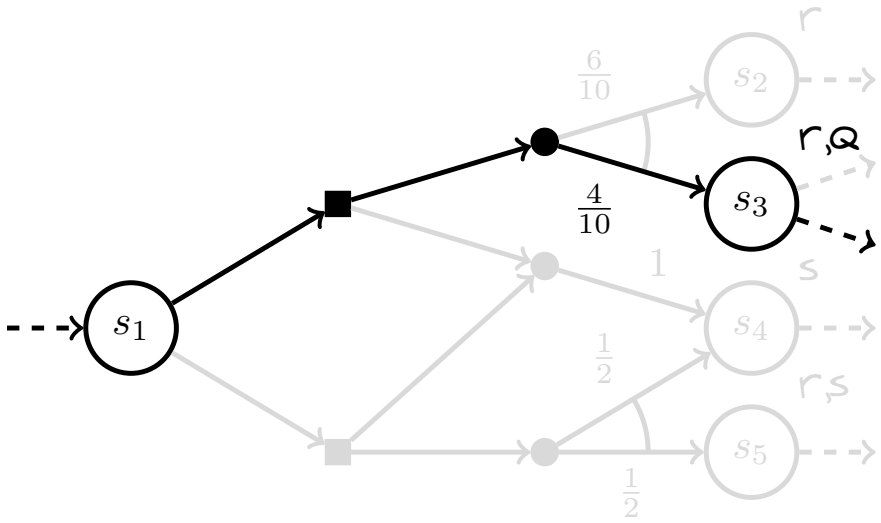

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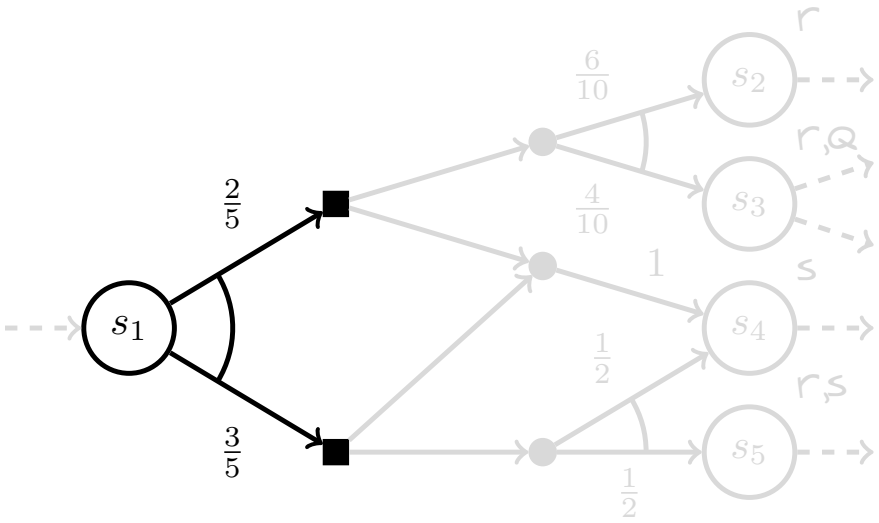

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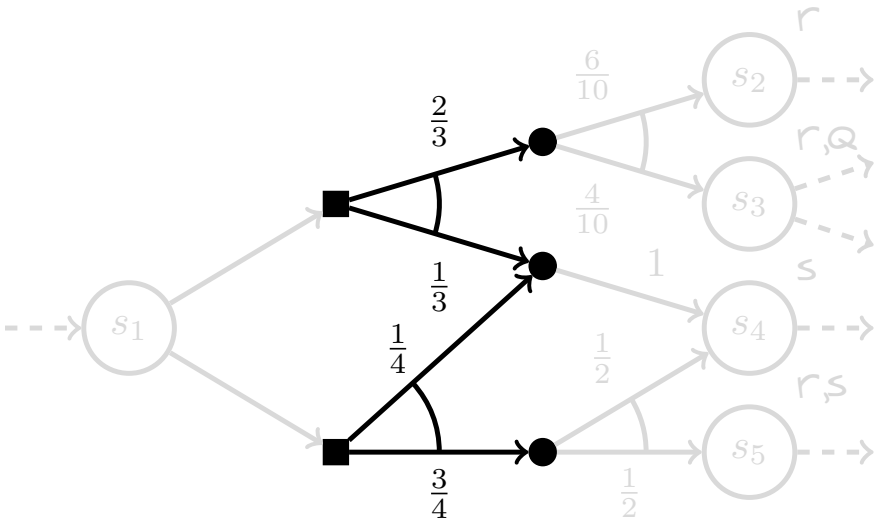

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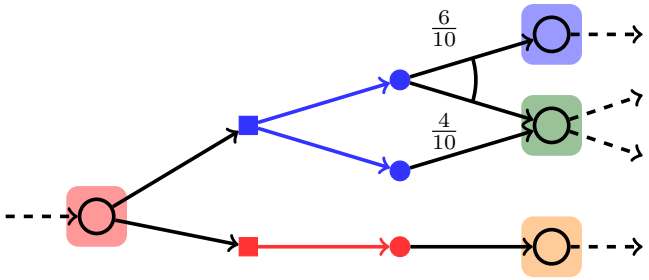
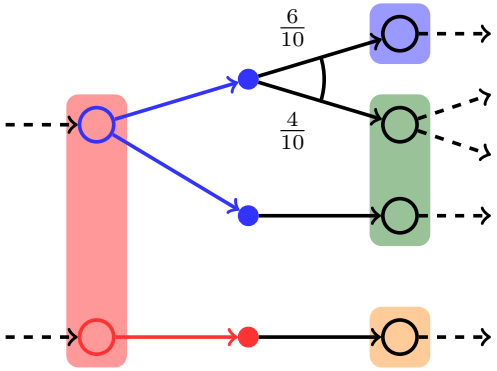
Games as abstractions of MDPs

Game



MDP



- player 1 "picks a concretisation"
- player 2 controls the non-determinism in the MDPs

Games as abstractions of MDPs

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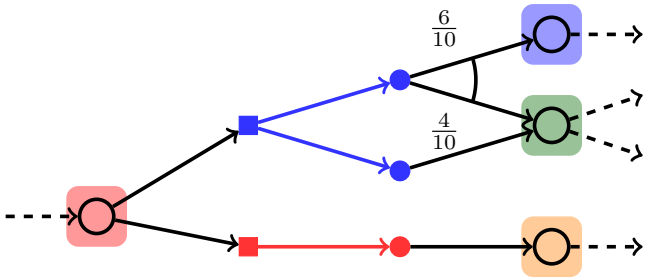

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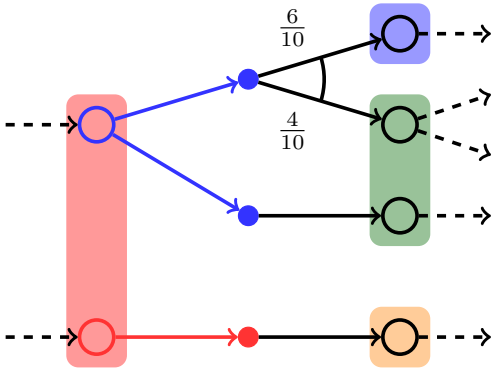

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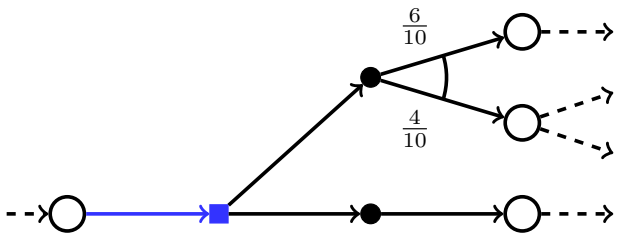
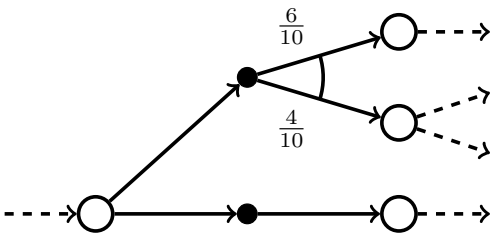
Embedding function emb : M → G

Game emb(m)



MDP m



– i.e. add a trivial player 1 transitions everywhere

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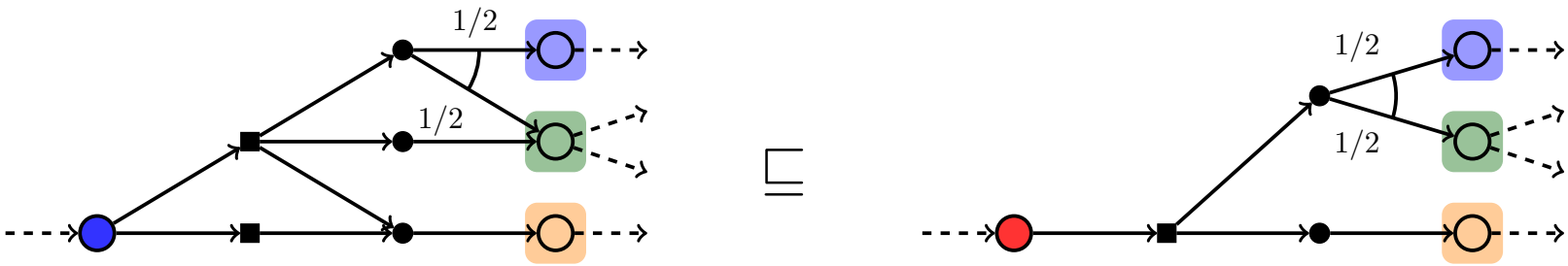
Abstraction relation $\sqsubseteq$

"Abstract" game **"Concrete" game**



iff

– player 1 in ● can over-approximate player 1 in ●:
$\forall$ ●→c : $\exists$ ●→a : $\forall$ c→c : $\exists$ a→a : a $\sqsubseteq$ c

– player 1 in ● can under-approximate player 1 in ●:
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(here $\sqsubseteq$ is lifted to distributions with standard methods [JL91])

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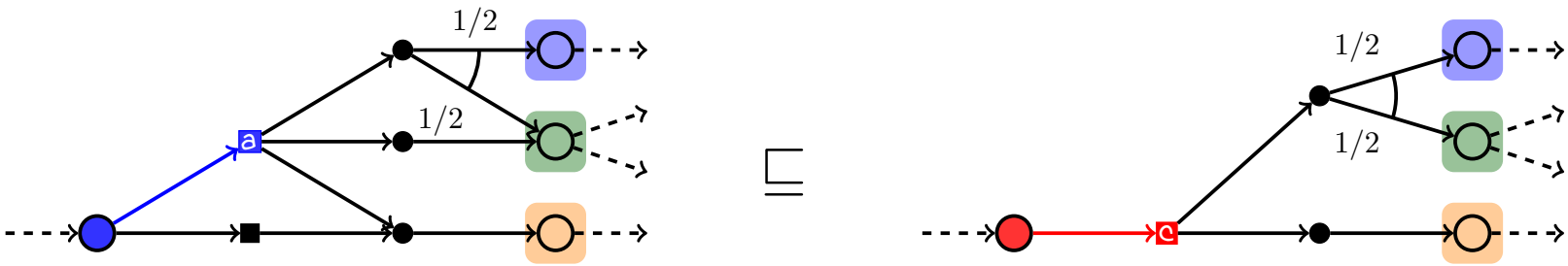

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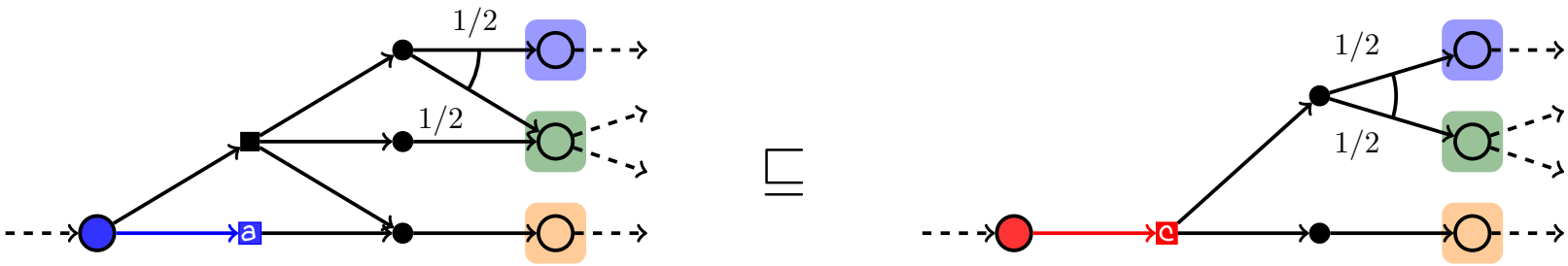

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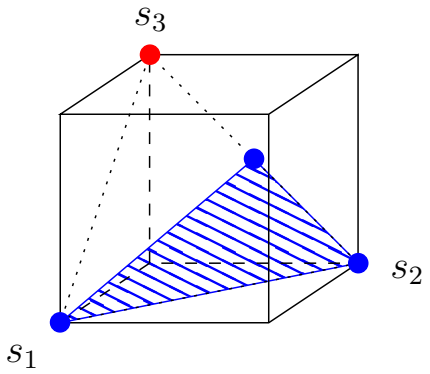
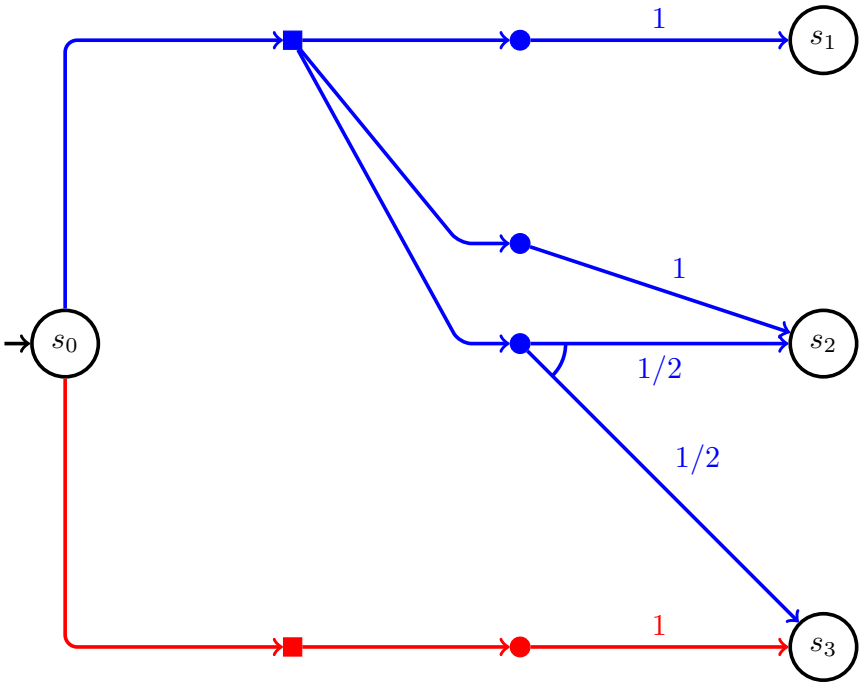
Combined transitions (c.f. [Seg95])





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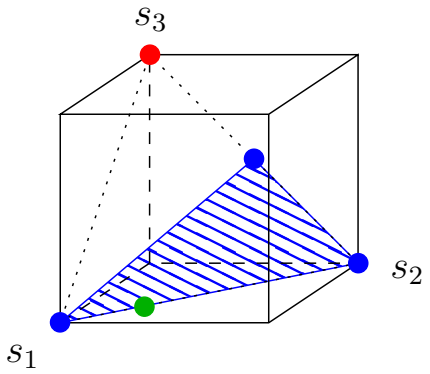

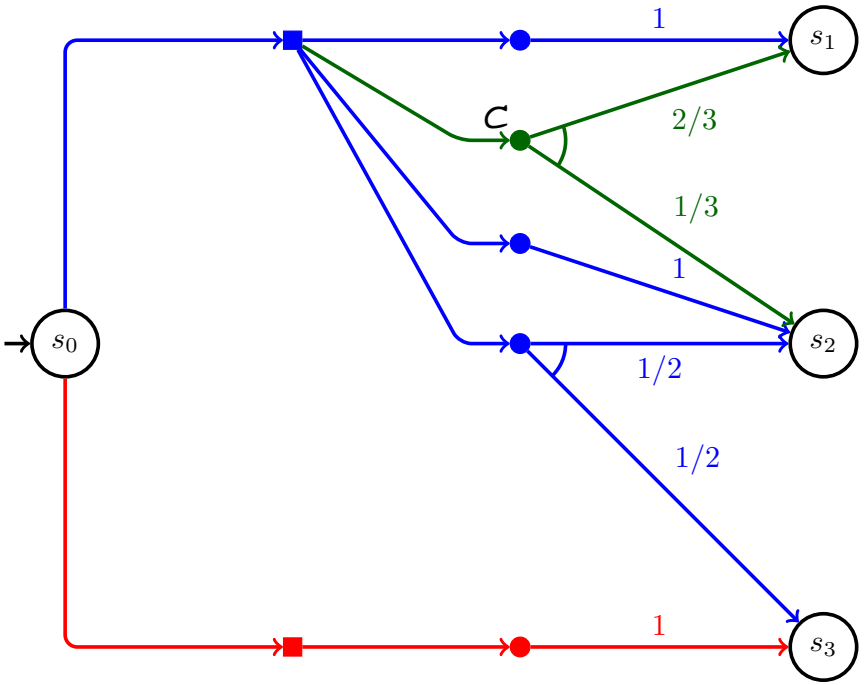

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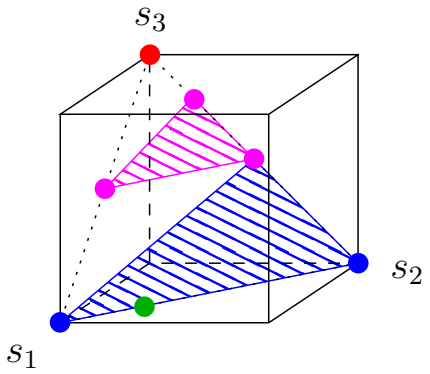

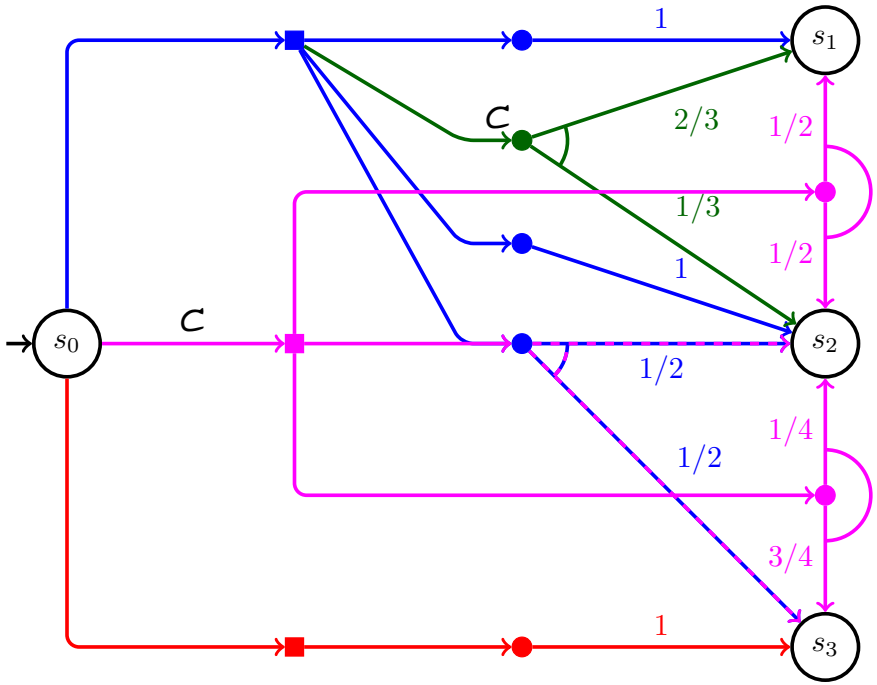

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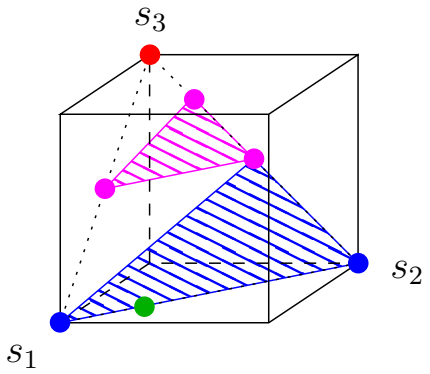

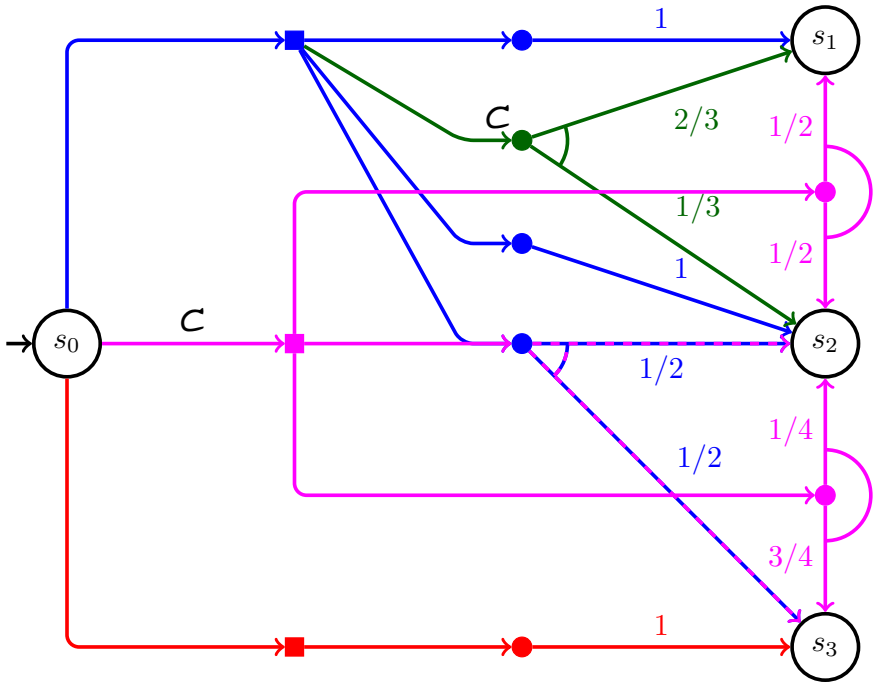

We now have ● $\sqsubseteq_p$ ● iff

– ∀ ● → c : ∃ ● —c→ a : ∀ c → c : ∃ a —c→ a : a ⊑ c

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PCTL semantics $\models^{may}$, $\models^{must}$ for games

Player 1 & 2 revisited:

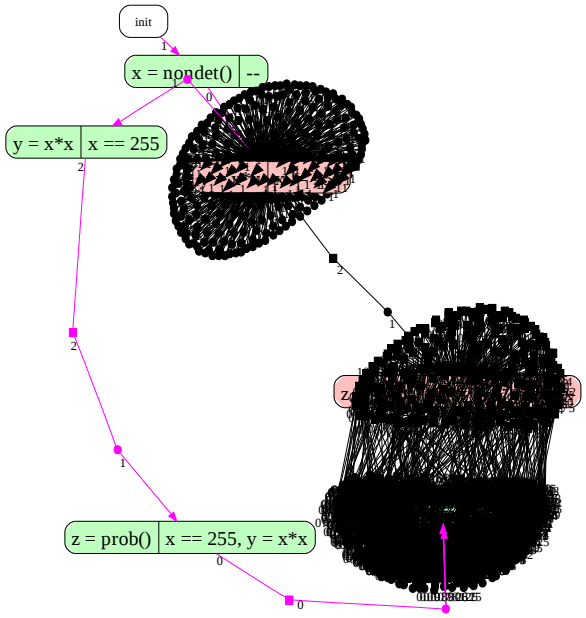
- player 1 strategies quantify over concretisations
- player 2 strategies correspond to MDP-strategies

Informal PCTL semantics:

Specification	must-satisfied	may-satisfied
for all strategies ...	for all player 1 strategies and for all player 2 strategies ...	for some player 1 strategies and for all player 2 strategies ...
for some strategy ...	for all player 1 strategies and for some player 2 strategy ...	for some player 1 strategies and for some player 2 strategy ...

Games as abstractions (revisited)

Currently:



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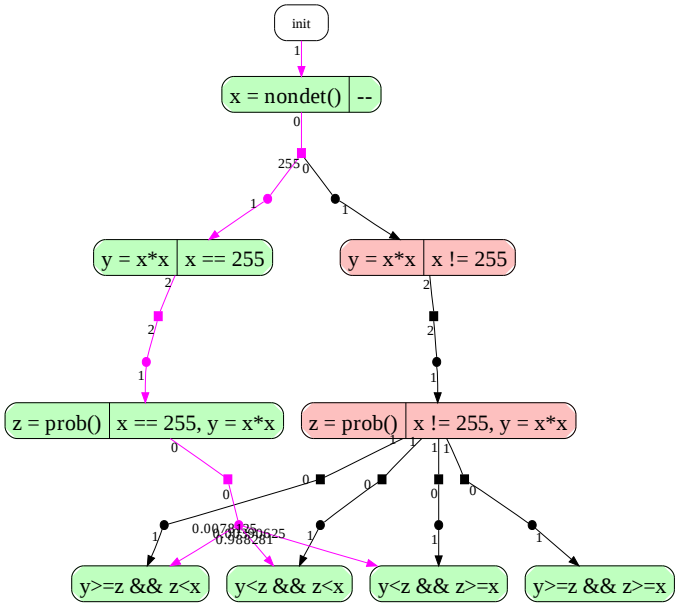

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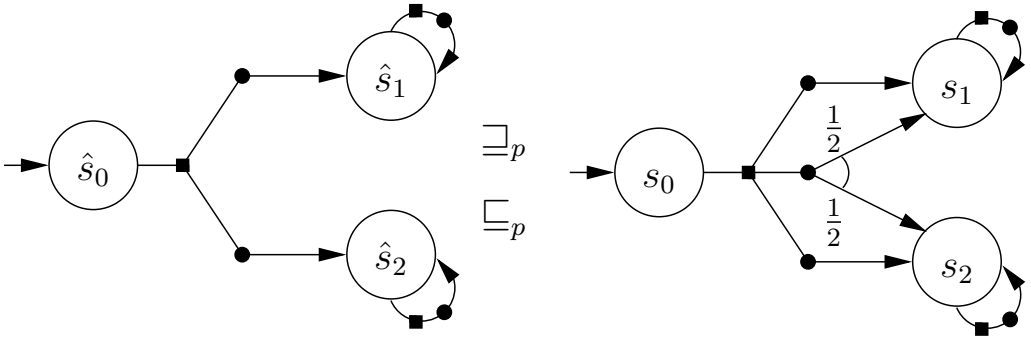
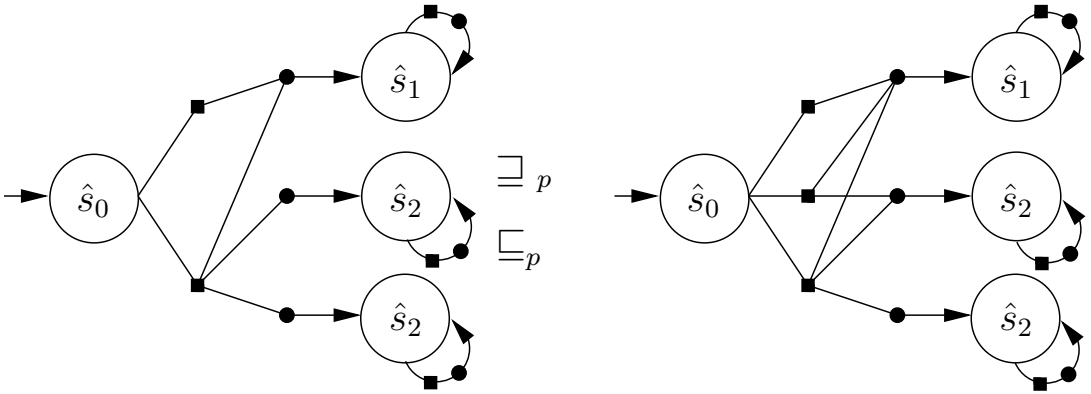
Finding better games (examples)

Without losing precision:

- remove non-extreme player 2 transitions

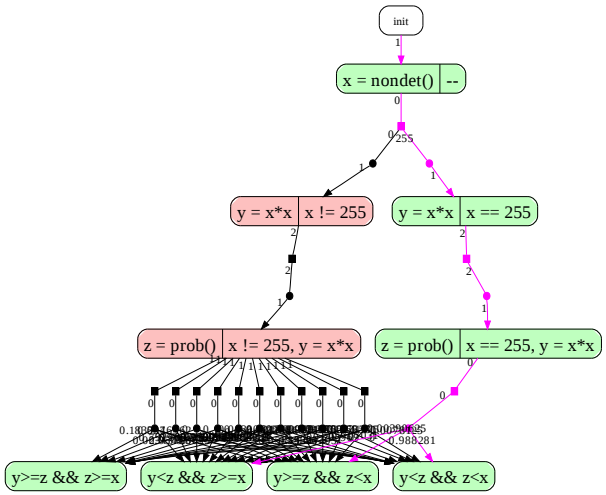


- remove non-extreme player 1 transitions



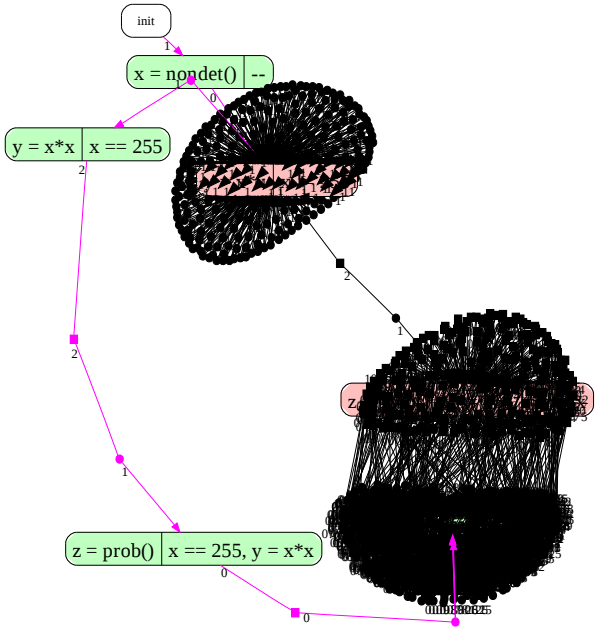
Finding better games (cont'd)

– a smaller abstraction is obtained by removing non-extreme transitions



$\sqsubseteq_p$

$\sqsupseteq_p$

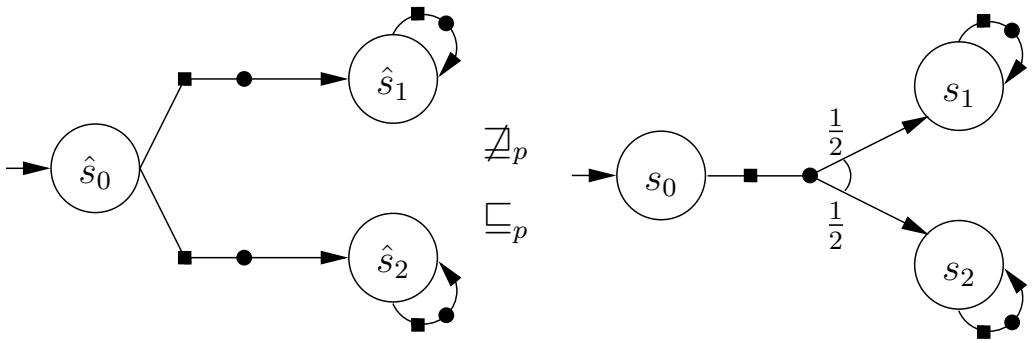


– both abstractions are equivalent!

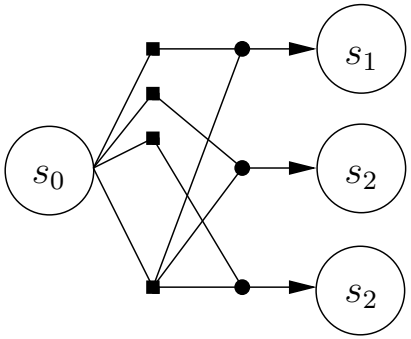
Finding better games (examples)

By losing precision:

- over-approximate probabilistic choice

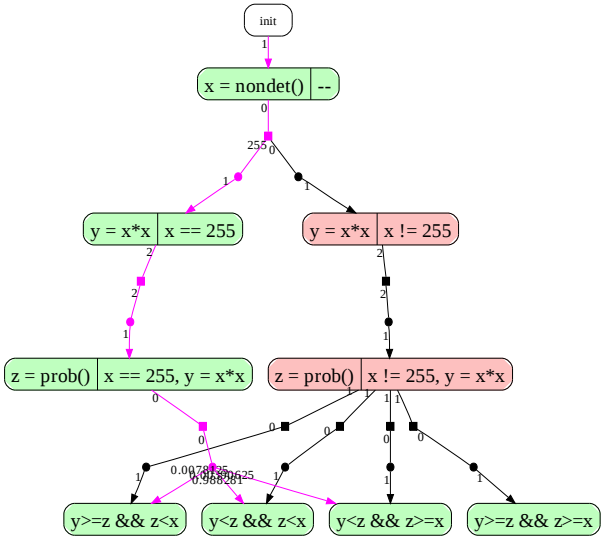


- use most abstract game over a partition



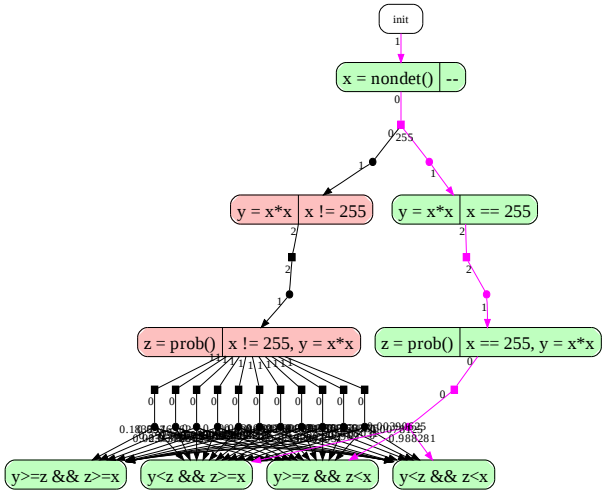
Finding better games (cont'd)

– a smaller abstraction is obtained by replacing probabilistic choice with player 1 non-determinism



$\sqsubseteq_p$

$\not\sqsubseteq_p$



– the left-hand game is more abstract (but still "good enough")

Summary

Summary:

- identified the need for a more fine-grained abstraction for games
- developed an abstraction framework (*emb*, $\sqsubseteq_p$ and $\models^{\text{may}}$, $\models^{\text{must}}$)
- enabled, via $\sqsubseteq_p$, a whole hierarchy of game abstractions with varying precisions (even for fixed predicates)
- developed a notion of combined player 1 and player 2 transitions for games
- developed 4-valued semantics for arbitrary pctl specifications over games

Future work

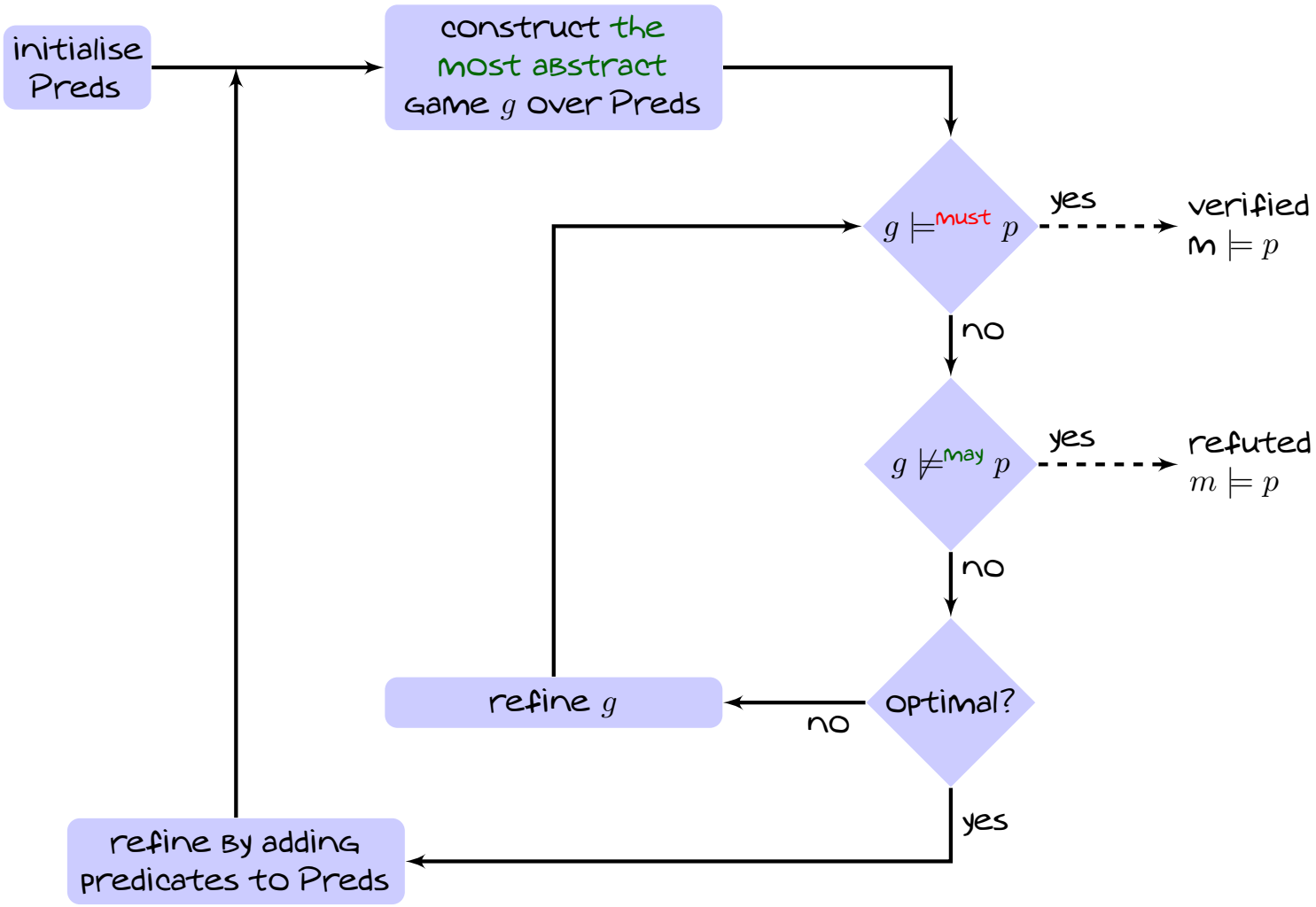
- automatically find good abstractions

Questions

Questions

- how can you find good games?
- what was the running example?

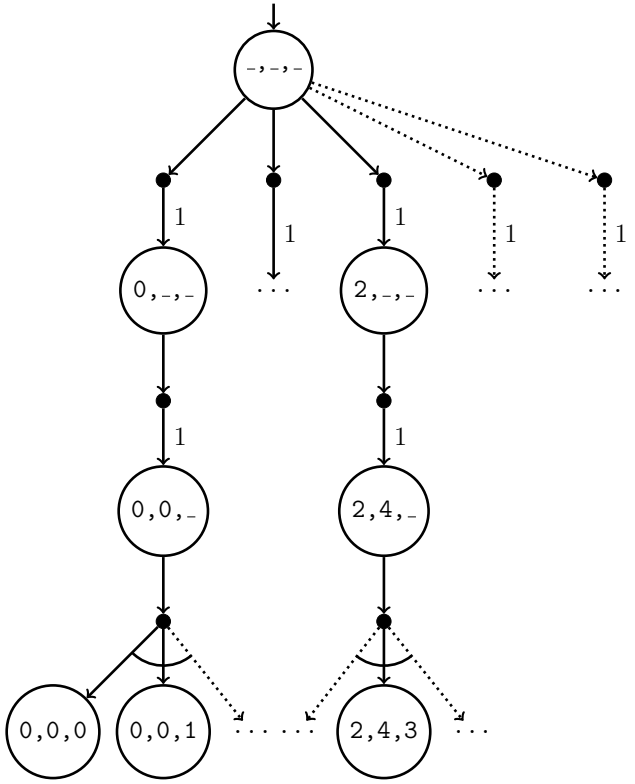
Abstraction refinement with games



– the machinery developed in this paper facilitates the inner refinement loop

Running example

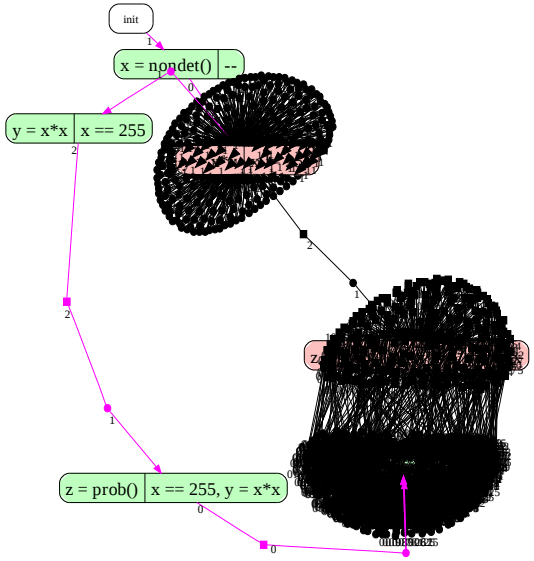
```
void main()
{
  uchar x,y,z;

  x = nondet();
  y = x*x;
  z = prob();

  if (y<z && z<x)
    TARGET;
}
```



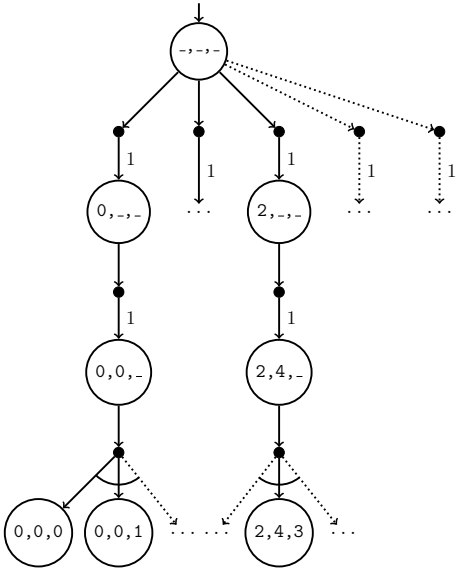
– does `main` satisfy $\mathsf{P}_{\leq 0.98}\langle \mathsf{F}\ \texttt{TARGET}\rangle$? (hint: take `x`=255)

Running example (cont'd)

refutation via emb, $\sqsubseteq_p$ and $\models^{\text{may}}$



$\sqsubseteq_p$ *emb*(



)

– the abstraction does not may-satisfy $\mathsf{P}_{\leq 0.98}\langle \mathsf{F}\ \mathtt{TARGET}\rangle$

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